

SOME RESULTS FOR ANALYTIC FUNCTIONS RELATED TO BERNOULLI NUMBERS

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Abstract

This paper explores the interesting relationship between Bernoulli numbers and the Schwarz lemma, two seemingly disparate concepts from number theory and complex analysis. Bernoulli numbers, a sequence of rational numbers, play a crucial role in various mathematical contexts, including power series expansions, the Riemann zeta function, and the Euler–Maclaurin formula. The Schwarz lemma, a fundamental result in complex analysis, provides constraints on analytic functions mapping the unit disc to itself. We investigate how these two areas intersect in the study of analytic functions and power series, highlighting how the properties of Bernoulli numbers influence the coefficients and behaviour of functions subject to the Schwarz lemma. By examining specific examples and deriving relevant identities, we uncover the subtle yet significant connections between these mathematical constructs, offering new insights into their combined application in modern mathematical research.

Key words: Bernoulli numbers, Schwarz lemma, angular derivative, Julia–Wolff lemma

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1. Introduction. The fields of number theory and complex analysis are enriched by numerous significant concepts and results, among which Bernoulli numbers and the Schwarz lemma stand out for their profound implications and diverse applications. Bernoulli numbers, a sequence of rational numbers with roots in the early work of Jacob Bernoulli, are pivotal in the study of power

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series, special functions, and numerous summation formulas, such as the Euler–Maclaurin formula. Their recursive definition and appearance in the expansions of functions make them a cornerstone of analytic number theory.

On the other hand, the Schwarz lemma is a fundamental theorem in complex analysis, providing key insights into the behaviour of analytic functions mapping the unit disc to itself. This lemma imposes constraints on such functions, particularly concerning their modulus and derivatives at the origin, and has far-reaching consequences in the study of conformal mappings and the geometry of analytic functions.

Despite their origins in different branches of mathematics, Bernoulli numbers and the Schwarz lemma intersect intriguingly in the context of analytic functions and power series. Bernoulli numbers frequently appear as coefficients in the expansions of certain analytic functions, which are subject to the constraints imposed by the Schwarz lemma. This intersection allows for the exploration of how the properties of Bernoulli numbers can influence the behaviour of these functions, providing deeper insights into their structure and properties.

This paper aims to delve into this relationship by examining specific instances where Bernoulli numbers and the Schwarz lemma converge, illustrating the impact of Bernoulli numbers on the coefficients of power series expansions of analytic functions and analyzing the resulting identities and inequalities. Through this exploration, we aim to uncover new connections and applications that enrich both number theory and complex analysis.

We now introduce the function that establishes the connection between the Bernoulli numbers and the Schwarz lemma.

Let $D = \{z: |z| < 1\}$ be the unit disc and let $g(z)$ be analytic in D . The Bernoulli numbers are introduced via the classical generating function given by the power series expansion of $\frac{z}{e^z - 1}$:

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \cdots, \quad z \in D,$$

where B_{2n} are the Bernoulli numbers [1, 2].

Let $\mathcal{A}$ represent the class of functions in the form $g(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ which are analytic in the unit disc D . Also, let $\mathcal{P}$ be the subclass of $\mathcal{A}$ satisfying the condition

$$g'(z) \prec \frac{z}{e^z - 1},$$

where the symbol “ $\prec$ ” indicates the subordinate principle [3]. Determining the upper bound of Taylor coefficients has been crucial in the study of geometric properties, providing valuable insights into various subclasses of $\mathcal{P}$. In this section, we will establish an upper bound for a_2 , one of the coefficients of the function $g(z)$. For this we will need to use the Schwarz lemma. Additionally, in the next section,

we will evaluate the modulus of the first derivative of the function $g(z)$ from below, necessitating the use of the boundary version of the Schwarz lemma.

The Schwarz lemma asserts that for any analytic function $w(z)$ mapping the unit disc D into itself and satisfying $w(0) = 0$, the magnitude of $w(z)$ is bounded by $|z|$ within the unit disc and $|w'(0)| \leq 1$. In simpler terms, it asserts that if a function maps the unit disc into itself and maps the origin to the origin, then the function cannot increase the distances between points inside the disc by more than 1 [3].

Extending the Schwarz lemma to the boundary of the unit disc provides valuable insights into the behaviour of analytic functions near this boundary, making it particularly useful for analyzing functions as they approach the edge of the disc. This paper explores the application and implications of this important theorem across different classes of functions. The boundary version of the Schwarz lemma implies that $|w'(1)| \geq 1$. OSSERMAN [4] and UNKELBACH [5] showed that in this case we have in fact

$$|w'(1)| \geq 1 + \frac{1 - |w'(0)|}{1 + |w'(0)|} = \frac{2}{1 + |w'(0)|},$$

where w satisfies the conditions of the Schwarz lemma and w extends continuously to the boundary point $1 \in \partial D = \{z: |z| = 1\}$, $|w(1)| = 1$ and $w'(1)$ exists. These inequalities are sharp. In mathematical literature, these inequalities and their extensions are subjects of ongoing debate and are highly significant in the geometric theory of functions [6–10]. In [11, 12], above and below evaluations were made for the coefficients for functions belonging to different classes.

If we use the subordinate principle for the class we defined above, there exists a Schwarz function $w(z)$ such that $g'(z) = \frac{w(z)}{e^{w(z)} - 1}$.

Here, the function $w(z)$ meets the criteria of the Schwarz lemma [3]. Therefore, applying the Schwarz lemma, we derive

$$1 + 2a_2z + 3a_3z^2 + \dots = \frac{w(z)}{e^{w(z)} - 1} = 1 - \frac{w(z)}{2} + \frac{w(z)^2}{12} - \dots$$

and

$$2a_2 + 3a_3z + \dots = -\frac{w(z)}{2z} + \frac{w(z)^2}{12z} - \dots.$$

If we take a limit as z approaches zero in the last equation, we take

$$2a_2 = -\frac{1}{2}w'(0) + \frac{1}{24}w'(0)w(0) + \dots \quad \text{and} \quad |a_2| \leq \frac{1}{4}.$$

We will now demonstrate that the final inequality is sharp. Let

$$(1.1) \quad g(z) = z - \frac{z^2}{4} + \frac{z^3}{36} - \frac{z^5}{3600} + \dots.$$

Then

$$g'(z) = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots, \quad 1 + 2a_2z + 3a_3z^2 + \cdots = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots$$

and $|a_2| = \frac{1}{4}$.

Lemma 1. *If $g \in \mathcal{P}$, then $|a_2| \leq \frac{1}{4}$. The sharpness of this result is confirmed by the extremal function (1.1).*

The subsequent lemma, referred to as the Julia–Wolff lemma, is required for the following discussion (see, [13])

Lemma 2 (Julia–Wolff lemma). *If w is an analytic function in the unit disc D with $w(0) = 0$ and $w(D) \subset D$, and additionally, w has an angular limit $w(1)$ at $1 \in \partial D$, where $|w(1)| = 1$, then the angular derivative $w'(1)$ exists and $1 \leq |w'(1)| \leq \infty$.*

2. Main results. In this section, we will examine the second derivative of the analytic function $g(z)$. By considering the coefficients in the Taylor expansion of $g(z)$, we will derive stronger inequalities. Furthermore, we will present an inequality that demonstrates the relationship between these coefficients.

Theorem 1. *Let $g(z) \in \mathcal{P}$. Suppose that, for $1 \in \partial D$, g has an angular limit $g(1)$ at 1, $g'(1) = \frac{1}{e-1}$. Then we have the inequality*

$$(2.1) \quad |g''(1)| \geq \frac{1}{(e-1)^2}.$$

This result is sharp, with equality for the function

$$g(z) = \int_0^z \frac{t}{e^t - 1} dt.$$

Proof. Let

$$g'(z) = \frac{w(z)}{e^{w(z)} - 1}.$$

Then

$$g''(z) = \frac{w'(z)(e^{w(z)} - 1) - w'(z)e^{w(z)}w(z)}{(e^{w(z)} - 1)^2}$$

and

$$g''(1) = \frac{w'(1)(e-1) - w'(1)e}{(e-1)^2} = -\frac{w'(1)}{(e-1)^2}.$$

Since the function $w(z)$ satisfies the conditions of the Schwarz lemma at the boundary, we obtain

$$1 \leq |w'(1)| = (e-1)^2 |g''(1)| \quad \text{and} \quad |g''(1)| \geq \frac{1}{(e-1)^2}.$$

Now, we will prove that inequality (2.1) is sharp. Let

$$g(z) = \int_0^z \frac{t}{e^t - 1} dt.$$

Then

$$g'(z) = \frac{z}{e^z - 1} \quad \text{and} \quad g''(1) = \frac{e - 1 - e}{(e - 1)^2} = \frac{-1}{(e - 1)^2}. \quad \square$$

Theorem 2. Assuming the same conditions as in Theorem 1, we obtain

$$(2.2) \quad |g''(1)| \geq \frac{1}{(e - 1)^2} \frac{2}{1 + 2|a_2|}.$$

The bound in inequality (2.2) is sharp, and equality holds for the function

$$g(z) = \int_0^z \frac{t}{e^t - 1} dt.$$

Proof. Let $w(z)$ be as defined in the proof of Theorem 1. Thus, from the Schwarz lemma on the boundary,

$$\frac{2}{1 + |w'(0)|} \leq |w'(1)| = (e - 1)^2 |g'(1)|.$$

Since $|w'(0)| = 4|a_2|$, we take

$$\frac{2}{1 + 4|a_2|} \leq (e - 1)^2 |g''(1)| \quad \text{and} \quad |g''(1)| \geq \frac{1}{(e - 1)^2} \frac{2}{1 + 4|a_2|}.$$

Next, we will demonstrate that inequality (2.2) is sharp. Consider the function

$$g(z) = \int_0^z \frac{t}{e^t - 1} dt.$$

Then, we have $|g''(1)| = \frac{1}{(e-1)^2}$. However, we also have

$$1 + 2a_2z + 3a_3z^2 + \cdots = \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{z^2}{12} - \cdots$$

and

$$2a_2 + 3a_3z + \cdots = \frac{-1}{2} + \frac{1}{12}z + \cdots.$$

Upon taking the limit as z approaches 0 in the final equation, we find that $a_2 = -\frac{1}{4}$. Consequently, this yields

$$\frac{1}{(e - 1)^2} \frac{2}{1 + 4|a_1|} = \frac{1}{(e - 1)^2}. \quad \square$$

Theorem 3. Under the conditions of Theorem 1, we obtain

$$(2.3) \quad |g''(1)| \geq \frac{1}{(e-1)^2} \left(1 + \frac{2(1-4|a_2|)^2}{1-16|a_2|^2+|4a_2^2-6a_3|} \right).$$

The bound is sharp with the extremal function given by

$$g(z) = \int_0^z \frac{t^2}{e^{t^2}-1} dt.$$

Proof. Let $w(z)$ denote the same function as in the proof of Theorem 1, and let $d(z) = z$. According to the maximum principle, for every $z \in D$, it follows that $|w(z)| \leq |d(z)|$. Therefore, $\psi(z) = \frac{w(z)}{d(z)}$ is an analytic function and $|\psi(z)| < 1$ for $|z| < 1$. Using Taylor expansion of the function $w(z)$, we have $w(z) = b_1z + b_2z^2 + b_3z^3 + \dots$. Thus, we take

$$\psi(z) = \frac{w(z)}{d(z)} = \frac{b_1z + b_2z^2 + b_3z^3 + \dots}{z} = b_1 + b_2z + b_3z^2 + \dots,$$

where $|\psi(0)| = |b_1|$ and $|\psi'(0)| = |b_2|$. Through straightforward computations, we obtain

$$\begin{aligned} 1 + 2a_2z + 3a_3z^2 + \dots &= \frac{w(z)}{e^{w(z)}-1} = 1 - \frac{1}{2}w(z) + \frac{1}{12}w(z)^2 + \dots, \\ 2a_2z + 3a_3z^2 + \dots &= -\frac{1}{2}(b_1z + b_2z^2 + b_3z^3 + \dots) + \frac{1}{12}(b_1z + b_2z^2 + b_3z^3 + \dots)^2 + \dots \\ &= -\frac{1}{2}z(b_1 + b_2z + b_3z^2 + \dots) + \frac{1}{12}z^2(b_1 + b_2z + b_3z^2 + \dots)^2 + \dots, \\ 2a_2 + 3a_3z + \dots &= -\frac{1}{2}(b_1 + b_2z + b_3z^2 + \dots) + \frac{1}{12}z(b_1 + b_2z + b_3z^2 + \dots)^2 + \dots, \\ a_2 &= -\frac{1}{4}b_1 \text{ and } a_3 = -\frac{b_2}{6} + \frac{1}{24}b_1^2. \text{ Thus, based on the expression for } \psi(z), \text{ we have} \\ |\psi(0)| &= 4|a_2| \text{ and } |\psi'(0)| = |4a_2^2 - 6a_3|. \end{aligned}$$

The combined function

$$\Upsilon(z) = \frac{\psi(z) - \psi(0)}{1 - \overline{\psi(0)}\psi(z)}$$

is analytic in the unit disc D , $\Upsilon(0) = 0$, $|\Upsilon(z)| < 1$ for $z \in D$ and $|\Upsilon(1)| = 1$ for $1 \in \partial D$. By the Schwarz lemma on the boundary, we obtain

$$\frac{2}{1 + |\Upsilon'(0)|} \leq |\Upsilon'(1)| = \frac{1 - |\psi(0)|^2}{|1 - \overline{\psi(0)}\psi(1)|^2} |\psi'(1)| \leq \frac{1 + |\psi(0)|}{1 - |\psi(0)|} (|w'(1)| - |d'(1)|).$$

Since $|\Upsilon'(0)| = \frac{|\psi'(0)|}{1 - |\psi(0)|^2} = \frac{|4a_2^2 - 6a_3|}{1 - 16|a_2|^2}$, we take

$$\frac{2}{1 + \frac{|4a_2^2 - 6a_3|}{1 - 16|a_2|^2}} \leq \frac{1 + 4|a_2|}{1 - 4|a_2|} \left((e-1)^2 |g''(1)| - 1 \right)$$

and

$$|g''(1)| \geq \frac{1}{(e-1)^2} \left(1 + \frac{2(1-4|a_2|)^2}{1-16|a_2|^2+|4a_2^2-6a_3|} \right).$$

We will now demonstrate that the inequality (2.3) achieves equality. Consider

$$g(z) = \int_0^z \frac{t^2}{e^{t^2}-1} dt.$$

Then

$$g'(z) = \frac{z^2}{e^{z^2}-1}, \quad g''(z) = \frac{2z(e^{z^2}-1) - 2ze^{z^2}z^2}{(e^{z^2}-1)^2} \quad \text{and} \quad |g''(1)| = \frac{2}{(e-1)^2}.$$

On the other hand, we have

$$\begin{aligned} 1 + 2a_2z + 3a_3z^2 + \dots &= \frac{z^2}{e^{z^2}-1} = 1 - \frac{1}{2}z^2 + \frac{1}{12}z^4 - \dots, \\ 2a_2z + 3a_3z^2 + \dots &= -\frac{1}{2}z^2 + \frac{1}{12}z^4 - \dots \end{aligned}$$

and

$$2a_2 + 3a_3z + \dots = -\frac{1}{2}z + \frac{1}{12}z^3 - \dots.$$

Passing to limit ($z \rightarrow 0$) in the last equality yields $a_2 = 0$. Similarly, using straightforward calculations, we take $a_3 = -\frac{1}{6}$. Therefore, we obtain

$$\frac{1}{(e-1)^2} \left(1 + \frac{2(1-4|a_2|)^2}{1-16|a_2|^2+|4a_2^2-6a_3|} \right) = \frac{2}{(e-1)^2}. \quad \square$$

Based on the findings from [6], this theorem derives the modulus of the function's derivative at point 1 by considering its Taylor expansions around two points.

Theorem 4. *Let $g \in \mathcal{P}$ and $g'(\beta_0) = 1$ for $0 < |\beta_0| < 1$. Suppose that, for $1 \in \partial D$, g has an angular limit $g(1)$ at 1, $g'(1) = \frac{1}{e-1}$. Then we have the inequality*

$$\begin{aligned} (2.4) \quad |g''(1)| &\geq \frac{1}{(e-1)^2} \left(1 + \frac{1-|\beta_0|^2}{|1-\beta_0|^2} + \frac{|\beta_0|-4|a_2|}{|\beta_0|+4|a_2|} \right. \\ &\quad \times \left. \left[1 + \frac{|\beta_0|^2+2|g''(0)|2|g''(\beta_0)|(1-|\beta_0|^2)-2|g''(\beta_0)|(1-|\beta_0|^2)-2|g''(0)|}{|\beta_0|^2+2|g''(0)|2|g''(\beta_0)|(1-|\beta_0|^2)+2|g''(\beta_0)|(1-|\beta_0|^2)+2|g''(0)|} \frac{1-|\beta_0|^2}{|1-\beta_0|^2} \right] \right). \end{aligned}$$

Inequality (2.4) is sharp, with equality for each possible value of $|g''(0)|$ and $|g''(\beta_0)|$.

Proof. According to the Schwarz–Pick Lemma [3], we have

$$\left| \frac{p(z) - p(\beta_0)}{1 - \overline{p(\beta_0)}p(z)} \right| \leq \left| \frac{z - \beta_0}{1 - \overline{\beta_0}z} \right| = |\lambda(z)|$$

and

$$(2.5) \quad |p(z)| \leq \frac{|p(\beta_0)| + |\lambda(z)|}{1 + |p(\beta_0)||\lambda(z)|},$$

where $p: D \rightarrow D$ is an analytic function and $\beta_0 \in D$. If $G: D \rightarrow D$ is analytic and $0 < |\beta_0| < 1$, letting $p(z) = \frac{G(z) - G(0)}{z(1 - \overline{G(0)}G(z))}$ in (2.5), we obtain

$$\left| \frac{G(z) - G(0)}{z(1 - \overline{G(0)}G(z))} \right| \leq \frac{\left| \frac{G(\beta_0) - G(0)}{\beta_0(1 - \overline{G(0)}G(\beta_0))} \right| + |\lambda(z)|}{1 + \left| \frac{G(\beta_0) - G(0)}{\beta_0(1 - \overline{G(0)}G(\beta_0))} \right| |\lambda(z)|}$$

and

$$|G(z)| \leq \frac{|G(0)| + |z| \frac{|A| + |\lambda(z)|}{1 + |A||\lambda(z)|}}{1 + |G(0)||z| \frac{|A| + |\lambda(z)|}{1 + |A||\lambda(z)|}}, \quad \text{where } A = \frac{G(\beta_0) - G(0)}{\beta_0(1 - \overline{G(0)}G(\beta_0))}.$$

If we take $G(z) = \frac{w(z)}{z \frac{z - \beta_0}{1 - \overline{\beta_0}z}}$, then we have

$$G(0) = \frac{w'(0)}{-\beta_0}, \quad G(\beta_0) = \frac{w'(\beta_0)(1 - |\beta_0|^2)}{\beta_0} \quad \text{and} \quad A = \frac{\frac{w'(\beta_0)(1 - |\beta_0|^2)}{\beta_0} + \frac{w'(0)}{\beta_0}}{\beta_0 \left(1 + \frac{w'(0)}{\beta_0} \frac{w'(\beta_0)(1 - |\beta_0|^2)}{\beta_0} \right)},$$

where $|A| \leq 1$.

Let $|G(0)| = \gamma$ and $A_1 = \frac{\left| \frac{w'(\beta_0)(1 - |\beta_0|^2)}{\beta_0} \right| + \left| \frac{w'(0)}{\beta_0} \right|}{|\beta_0| \left(1 + \left| \frac{w'(0)}{\beta_0} \right| \left| \frac{w'(\beta_0)(1 - |\beta_0|^2)}{\beta_0} \right| \right)}$. From the inequality $G(z)$, we take

$$|w(z)| \leq |z| |\lambda(z)| \frac{\gamma + |z| \frac{A_1 + |\lambda(z)|}{1 + A_1|\lambda(z)|}}{1 + \gamma |z| \frac{A_1 + |\lambda(z)|}{1 + A_1|\lambda(z)|}}$$

and

$$\frac{1 - |w(z)|}{1 - |z|} \geq \frac{1 + \gamma |z| \frac{A_1 + |\lambda(z)|}{1 + A_1|\lambda(z)|} - \gamma |z| |\lambda(z)| - |z|^2 |\lambda(z)| \frac{A_1 + |\lambda(z)|}{1 + A_1|\lambda(z)|}}{(1 - |z|) \left(1 + \gamma |z| \frac{A_1 + |\lambda(z)|}{1 + A_1|\lambda(z)|} \right)} = I.$$

Let $u(z) = 1 + \gamma |z| \frac{A_1 + |\lambda(z)|}{1 + A_1 |\lambda(z)|}$ and $r(z) = 1 + A_1 |\lambda(z)|$. Considering the functions $u(z)$ and $r(z)$ in the earlier inequality, we obtain

$$I = \frac{1}{u(z)r(z)} \left\{ \frac{1 - |z|^2 |\lambda(z)|}{1 - |z|} + A_1 |\lambda(z)| \frac{1 - |z|^2}{1 - |z|} + \gamma A_1 |z| \frac{1 - |\lambda(z)|^2}{1 - |z|} \right\}.$$

Since

$$\begin{aligned} \lim_{z \rightarrow 1} u(z) &= \lim_{z \rightarrow 1} \left(1 + \gamma |z| \frac{A_1 + |\lambda(z)|}{1 + A_1 |\lambda(z)|} \right) = 1 + \gamma, \\ \lim_{z \rightarrow 1} b(z) &= \lim_{z \rightarrow 1} (1 + A_1 |\lambda(z)|) = 1 + A_1 \end{aligned}$$

and

$$1 - |\lambda(z)|^2 = 1 - \left| \frac{z - \beta_0}{1 - \overline{\beta_0} z} \right|^2 = \frac{(1 - |\beta_0|^2)(1 - |z|^2)}{|1 - \overline{\beta_0} z|^2},$$

passing to the angular limit in I gives

$$|w'(1)| \geq 1 + \frac{1 - |\beta_0|^2}{|1 - \beta_0|^2} + \frac{1 - \gamma}{1 + \gamma} \left[1 + \frac{1 - A_1}{1 + A_1} \frac{1 - |\beta_0|^2}{|1 - \beta_0|^2} \right].$$

Moreover, since

$$\frac{1 - \gamma}{1 + \gamma} = \frac{1 - |G(0)|}{1 + |G(0)|} = \frac{1 - \left| \frac{w'(0)}{\beta_0} \right|}{1 + \left| \frac{w'(0)}{\beta_0} \right|} = \frac{|\beta_0| - |w'(0)|}{|\beta_0| + |w'(0)|} = \frac{|\beta_0| - 4|a_2|}{|\beta_0| + 4|a_2|}$$

and

$$\frac{1 - A_1}{1 + A_1} = \frac{|\beta_0|^2 + 2|g''(0)|2|g''(\beta_0)|(1 - |\beta_0|^2) - 2|g''(\beta_0)|(1 - |\beta_0|^2) - 2|g''(0)|}{|\beta_0|^2 + 2|g''(0)|2|g''(\beta_0)|(1 - |\beta_0|^2) + 2|g''(\beta_0)|(1 - |\beta_0|^2) + 2|g''(0)|},$$

we obtain

$$\begin{aligned} |\varphi'(1)| &\geq 1 + \frac{1 - |\beta_0|^2}{|1 - \beta_0|^2} + \frac{|\beta_0| - 4|a_2|}{|\beta_0| + 4|a_2|} \\ &\quad \times \left[1 + \frac{|\beta_0|^2 + 2|g''(0)|2|g''(\beta_0)|(1 - |\beta_0|^2) - 2|g''(\beta_0)|(1 - |\beta_0|^2) - 2|g''(0)|}{|\beta_0|^2 + 2|g''(0)|2|g''(\beta_0)|(1 - |\beta_0|^2) + 2|g''(\beta_0)|(1 - |\beta_0|^2) + 2|g''(0)|} \frac{1 - |\beta_0|^2}{|1 - \beta_0|^2} \right]. \end{aligned}$$

From the definition of $w(z)$, we have $w'(1) = (e - 1)^2 |g''(1)|$. We thus take inequality (2.4).

Let β_0 be any real number in the interval $(-1, 0)$, and let c_1 and c_2 be arbitrary real numbers such that $0 < c_1 = |w'(0)| < |\beta_0|^2$, $0 < c_2 = |w'(\beta_0)| < \frac{|\beta_0|^2}{(1 - |\beta_0|^2)^2}$ to show that inequality (2.4) is sharp. Let

$$\mathbb{F} = \frac{\frac{c_1}{\beta_0} + \frac{c_2(1 - |\beta_0|^2)}{\beta_0}}{\beta_0 \left(1 + c_1 c_2 \frac{1 - |\beta_0|^2}{\beta_0^2} \right)} = \frac{1}{\beta_0^2} \frac{c_2 (1 - |\beta_0|^2) + c_1}{1 + c_1 c_2 \frac{1 - |\beta_0|^2}{\beta_0^2}}.$$

Consider the function

$$(2.6) \quad w(z) = z \frac{z - \beta_0}{1 - \beta_0 z} \frac{\frac{-c_1}{\beta_0} + z \frac{\mathbb{F} + \lambda(z)}{1 + \lambda(z)\mathbb{F}}}{1 - \frac{c_1}{\beta_0} z \frac{\mathbb{F} + \lambda(z)}{1 + \lambda(z)\mathbb{F}}}.$$

From equation (2.6), after performing straightforward calculations, we derive $w'(0) = c_1$ and $w'(\beta_0) = c_2$. Therefore, we obtain

$$w'(1) = 1 + \frac{1 - \beta_0^2}{(1 - \beta_0)^2} + \frac{\beta_0 + c_1}{\beta_0 - c_1} \left(1 + \frac{\beta_0^2 + c_1 c_2 (1 - |\beta_0|^2) - c_2 (1 - |\beta_0|^2) - c_1}{\beta_0^2 + c_1 c_2 (1 - |\beta_0|^2) + c_2 (1 - |\beta_0|^2) + c_1} \frac{1 - \beta_0^2}{(1 - \beta_0)^2} \right).$$

By selecting appropriate signs for the numbers β_0 , c_1 and c_2 , we can infer from the final equation that inequality (2.4) is sharp. $\square$

3. Conclusion. We studied a class of analytic functions whose derivatives are subordinated to the generating function of the Bernoulli numbers and established several sharp inequalities describing their coefficient structure and boundary behaviour. By applying the Schwarz lemma and its boundary refinements, we obtained explicit and sharp bounds for Taylor coefficients and for the second derivative at a boundary point. The extremal functions achieving equality were given in integral form and illustrate the direct influence of Bernoulli numbers on the geometric properties of the associated analytic functions.

These results reveal a clear connection between Bernoulli numbers and boundary versions of the Schwarz lemma, providing new sharp estimates in geometric function theory. The approach developed here suggests that similar methods can be used to study other subclasses of analytic functions generated by special number sequences, offering a useful framework for further investigations at the interface of number theory and complex analysis.

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