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ON WEAK PROJECTION INVARIANT EXTENDING MODULES

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Abstract

We introduce the notion of weak π-extending modules which is a proper generalization of π-extending and weak CS-modules. Several characterizations and connections between weak π-extending modules and related concepts are obtained. Direct sums and direct summand properties are also provided. Moreover, we investigate when the former class has an indecomposable decomposition and exchange properties.

Key words: extending module, projection invariant submodule, exchange property

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1. Introduction. A module M is called *extending* [4], if every submodule of M is essential in a direct summand of M. Recall that a submodule A of M is *fully invariant* (resp., *projection invariant*), if $g(A) \subseteq A$ for each $g \in End(M_R)$ (resp., $g = g^2 \in End(M_R)$). There are miscellaneous generalizations of extending condition involving the following classes: (1) A module M is *FI-extending* [3], if every fully invariant submodule of M is essential in a direct summand of M. (2) M is called *π-extending* [2], if every projection invariant submodule of M is essential in a direct summand of M. (3) M is said to be a *weak CS*-module [10], if every semisimple module of M is essential in a direct summand of M. Following the idea in [10], a module M is called *weak FI-extending* [13], if every

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semisimple fully invariant submodule is essential in a direct summand of M. The weak FI-extending modules properly contain both weak CS-modules and FI-extending modules. Observe that extending condition $\Rightarrow$ π-extending condition $\Rightarrow$ FI-extending condition [2]. However, weak CS condition $\Rightarrow$ weak FI-extending condition [13].

The goal of this paper is to investigate the "weak" version of π-extending property. Thus, we say a module M is *weak projection invariant extending* (denoted by, *weak π-extending*), if every semisimple projection invariant submodule is essential in a direct summand of M. A ring R is *right weak π-extending*, if R_R is a weak π-extending module. This condition settles between weak CS and weak FI-extending conditions. Moreover, the class of weak π-extending modules properly contains the class of π-extending modules. In Section 2, we make connections between the former class of modules and related notions. We provide that when the module has an essential socle, then weak π-extending and π-extending conditions agree. As a consequence of this result, we characterize torsion Abelian groups with weak π-extending condition. Further, we explore under what conditions the classes of weak π-extending, weak CS and weak FI-extending modules coincide. Additionally, we explain how the aforementioned property carries up to intrinsic ring extensions. It is known from [14] that the direct sums of weak CS-modules is not weak CS. As opposed to weak CS-modules, we prove that weak π-extending property is closed under direct sums. In Section 3, we explore when the direct summand of weak π-extending modules enjoys the property. Further, we prove that if M is a weak π-extending module which satisfies suitable conditions, then the quotient ring of its endomorphism ring with Jacobson radical is a (von Neumann) regular ring. Moreover, we show that under some module theoretical conditions weak π-extending modules have an indecomposable decomposition. We apply our former results to obtain the exchange properties.

R and M stand by an associative ring with unity and unital right R-modules, respectively. The notations $A \leq M$, $A \trianglelefteq_p M$, $A \trianglelefteq M$, $A \leq^{ess} M$, $A \leq^{\oplus} M$, $SocM$, $End(M_R)$, $\mathbf{I}(R)$ and $M_n(R)$ mean that A is a right R-submodule of M, A is a projection invariant right R-submodule of M, A is a fully invariant submodule of M, A is an essential submodule of M and A is a direct summand of M, the socle submodule of M, the endomorphism ring of M, the subring of R generated by the idempotents of R, and the n-by-n matrix ring over R, respectively. Recall that a module P has C_2, if for each $P_1 \leq P$ such that $P_1 \cong P_2 \leq^{\oplus} P$, $P_1 \leq^{\oplus} P$. A module Q has C_3, if for each $Q_1, Q_2 \leq^{\oplus} Q$ such that $Q_1 \cap Q_2 = 0$, $Q_1 \oplus Q_2 \leq^{\oplus} Q$. A ring R is *Abelian*, if every idempotent of R is central. Note that a module F has *(finite) exchange property*, if for any (finite) index set χ, whenever $F \oplus H = \bigoplus_{\kappa \in \chi} K_\kappa$ for modules H and K_κ, $F \oplus H = F \oplus \left(\bigoplus_{\kappa \in \chi} Q_\kappa \right)$ for $Q_\kappa \leq K_\kappa$. For unknown notation and terminology, we refer to [1,9,12].

2. Characterizations. We present useful characterizations and the relations between weak π-extending modules and related notions in this section.

Lemma 2.1 ([6], p. 50)**.** *Let $\mathcal{B}$ be a right R-module. Then*

(i) *Suppose $\{\mathcal{B}_\gamma : \gamma \in \Gamma\}$ is the family of projection invariant submodules of $\mathcal{B}$. Then $\bigcap_{\gamma\in\Gamma} \mathcal{B}_\gamma \trianglelefteq_p \mathcal{B}$ and $\sum_{\gamma\in\Gamma} \mathcal{B}_\gamma \trianglelefteq_p \mathcal{B}$.*

(ii) *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be submodules of $\mathcal{B}$ such that $\mathcal{B}_1 \leq \mathcal{B}_2$. If $\mathcal{B}_1 \trianglelefteq_p \mathcal{B}_2$ and $\mathcal{B}_2 \trianglelefteq_p \mathcal{B}$, then $\mathcal{B}_1 \trianglelefteq_p \mathcal{B}$.*

(iii) *Let $\mathcal{B} = \bigoplus_{\gamma\in\Gamma} \mathcal{B}_\gamma$ and $\mathcal{Y} \trianglelefteq_p \mathcal{B}$. Then $\mathcal{Y} = \bigoplus_{\gamma\in\Gamma} (\mathcal{Y}\cap\mathcal{B}_\gamma)$ such that $\mathcal{Y}\cap\mathcal{B}_\gamma \trianglelefteq_p \mathcal{B}_\gamma$ for all $\gamma \in \Gamma$.*

Lemma 2.2. (i) *M is weak π-extending if and only if every semisimple projection invariant submodule of M has a complement which is a direct summand of M.*

(ii) *If M is weak π-extending, then $M = M_1 \oplus M_2$ such that $SocM_1 \leq^{ess} M_1$ and $SocM_2 = 0$.*

Proof. (i) Assume B is a semisimple projection invariant submodule of M. Then there exists a $K \leq^{\oplus} M$ such that $B \leq^{ess} K$. Observe that $B \cap K' = 0$ and $B \oplus K' \leq^{ess} M$, where $M = K \oplus K'$ for some $K' \leq M$. Thus, K' is the desired direct summand. Conversely, let Y be a semisimple projection invariant submodule of M. Then there exists a $P \leq^{\oplus} M$ such that $Y \cap P = 0$ and $Y \oplus P \leq^{ess} M$. Since $Y \trianglelefteq_p M$, $Y = (Y \cap P) \oplus (Y \cap P')$ by Lemma 2.1. Thus $Y = P' \cap (Y \oplus P) \leq^{ess} P'$, so M is weak π-extending.

(ii) Note that $SocM \trianglelefteq_p M$. Thence $SocM \leq^{ess} M_1$ for some $M_1 \leq^{\oplus} M$. Hence $M = M_1 \oplus M_2$ for some $M_2 \leq M$. Thus $SocM_2 = M_2 \cap SocM \leq^{ess} M_1 \cap M_2$, so $SocM_2 = 0$. Hence $SocM_1 \leq^{ess} M_1$. □

Example 2.3. (i) Let $M_{\mathbb{Z}} = \prod_{i=1}^{\infty} \mathbb{Z}$ be the Specker group. Observe that $Soc(M_{\mathbb{Z}}) = 0$, so clearly $M_{\mathbb{Z}}$ is weak π-extending. However, $M_{\mathbb{Z}}$ is not π-extending by [6].

(ii) Let S be a simple domain which is not a division ring and $A = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$. Then A_A is not FI-extending by ([3], Example 4.11). It follows from ([2], Proposition 3.7) that A_A is not π-extending. Since $Soc(A_A) = 0$, A_A is weak π-extending.

(iii) Let $\mathbb{R}$ be the real field and $T = \mathbb{R}[x_1, x_2, \cdots, x_n]$ for any odd integer $n \geq 3$. Then the ring $R = T/tT$, where $t = \left(\sum_{i\in 1}^{n} x_i^2\right) - 1$, is a commutative domain. Say $M_R = \bigoplus_{i=1}^{n} R_i$, $R_i \cong R$. Then M_R has a direct summand B, is not π-extending by ([2], Example 5.5). However, B is weak π-extending.

(iv) Assume R is a principal ideal domain which is not a complete discrete valuation ring. Then by ([7], Theorem 19), there is an indecomposable torsion-free module A_R of rank 2. Since $Soc(A_R) = 0$, A satisfies the weak π-extending condition. However, the uniform dimension of A_R is 2. So, by ([2], Proposition 3.8), A_R is not π-extending.

Proposition 2.4. *Let M be a weak π-extending module with $SocM \leq^{ess} M$. Then M is π-extending.*

Proof. Let $X \trianglelefteq_p M$. If $X = 0$, the proof is clear. So, assume $0 \neq X \trianglelefteq_p M$. Since $SocX \trianglelefteq_p X$ and $X \trianglelefteq_p M$, $SocX \trianglelefteq_p M$ by Lemma 2.1. Then there exists a $K \leq^{\oplus} M$ such that $SocX \leq^{ess} K$. Since $SocM \leq^{ess} M$, $SocX \leq^{ess} X$. Consequently, $SocX \cap K' = 0$, where $M = K \oplus K'$ for some $K' \leq M$. Then $X \cap K' = 0$. Moreover, $SocX \oplus K' \leq^{ess} M$ which gives that $X \oplus K' \leq^{ess} M$. Therefore Lemma 2.2(i) yields the result. □

Corollary 2.5. *If any of the following conditions hold, then M is π-extending if and only if M is weak π-extending.*

(a) *M is an Artinian module.* (b) *M is a finitely cogenerated module.*

(c) *R is a semi-artinian ring.*

Proof. It is a conclusion of Proposition 2.4. □

Corollary 2.6. *Let Q be a torsion Abelian group.*

(i) *Q is weak π-extending if and only if Q is a direct sum of a divisible group and separable p-groups.*

(ii) *Q is weak π-extending if and only if Q is π-extending.*

Proof. It is clear from Proposition 2.4 and ([2], Lemma 4.17). □

Example 2.7. Let $R = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ & \diagdown \\ 0 & \mathbb{Z}_4 \end{bmatrix} = \left\{ \begin{bmatrix} x & y \\ & \diagdown \\ 0 & x \end{bmatrix} : x \in \mathbb{Z}_4, y \in 2\mathbb{Z}_4 \right\}$, $A_1 = 2\mathbb{Z}_4 \oplus 0$ and $B_1 = 0 \oplus 2\mathbb{Z}_4$. Consider the R-module $M = A \oplus B$ such that $A = R/A_1$ and $B = R/B_1$. Since A and B are uniform, M_R is weak π-extending by Theorem 2.16. However, M_R is not weak CS-module by [14].

In the following result, we provide when a weak π-extending module behaves like a weak CS-module.

Proposition 2.8. *If any of the following statements hold, then M is weak CS if and only if M is weak π-extending.*

(a) *M is an indecomposable module.*

(b) *M is a multiplication module.*

(c) *M is a duo module.*

(d) *M is a distributive module.*

(e) *$M = R$ and R is Abelian.*

Proof. Definitions and ([2], Proposition 2.1) give the result. □

Lemma 2.9. *Assume T_R is an indecomposable module such that $SocT \neq 0$. Then T is weak π-extending if and only if T is uniform.*

Proof. Assume T_R is weak π-extending and $0 \neq X \leq T$. Thence $SocT \leq^{ess} T$. Since T_R is weak π-extending, $SocX \leq^{ess} D \leq^{\oplus} T$ for some $D \leq T$. It follows that $SocX \leq^{ess} T$. Hence $X \leq^{ess} T$, so T_R is uniform. The converse is clear. □

Example 2.10. Let S be any domain that is not right Ore. Hence S_S is FI-extending by ([2], Proposition 3.7). It follows that S_S is weak FI-extending. Since S_S is indecomposable but not uniform, S_S is neither π-extending nor weak π-extending by Lemma 2.9.

Recall that M is called *complement (strongly) bounded* [1], if every nonzero complement (submodule) contains a nonzero fully invariant submodule of M.

Proposition 2.11. (i) *Assume M_R is a complement bounded such that $SocM \leq^{ess} M$. Then M_R is weak π-extending if and only if M_R is weak FI-extending.*

(ii) *Suppose $End(M_R) = \mathbf{I}(End(M_R))$. Then M_R is weak π-extending if and only if M_R is weak FI-extending.*

(iii) *Assume M_R is strongly bounded. Then M_R is weak CS if and only if M_R is weak π-extending if and only if M_R is weak FI-extending.*

Proof. (i) Suppose M_R is complement bounded such that $SocM \leq^{ess} M$ and M_R is weak FI-extending. Let Y be semisimple projection invariant submodule of M and K a complement of Y in M. Then $K \cap Y = 0$ and $K \oplus Y \leq^{ess} M$. If $K = 0$, the proof is done. Assume $K \neq 0$. Then there is a nonzero fully invariant submodule X of M such that X is the sum of all fully invariant submodules of M contained in K. Thus, $X \leq^{ess} K$. Since $SocX \trianglelefteq X$ and $X \trianglelefteq M$, $SocX \trianglelefteq M$ by Lemma 2.1. Thus $SocX \leq^{ess} eM$ for some $e = e^2 \in End(M_R)$, as M is weak FI-extending. Notice that $SocX \cap Y = 0$, hence $Y \cap eM = 0$. Since $Y \trianglelefteq_p M$ and $Y \cap eM = 0$, it can be checked that $Y \subseteq (1-e)M$. Moreover, $X \oplus Y \leq^{ess} M$ and $SocX \cap (1-e)M = 0$. Since $SocM \leq^{ess} M$, clearly $SocX \leq^{ess} X$ which gives $X \cap (1-e)M = 0$. It follows that $Y = (1-e)M \cap (X \oplus Y) = Y \oplus (X \cap (1-e)M) \leq^{ess} (1-e)M$. Therefore $Y \oplus eM \leq^{ess} M$. Thus M is weak π-extending by Lemma 2.2 (i).

(ii) Any projection invariant submodule is fully invariant when $End(M_R) = \mathbf{I}(End(M_R))$.

(iii) By ([12], Lemma 5.109), each semisimple module is fully invariant Hence weak FI-extending condition implies weak CS condition. □

Corollary 2.12. (i) *Assume $R = \mathbf{I}(R)$. Then R_R is weak π-extending if and only if R_R is weak FI-extending.*

(ii) *For any integer $n > 1$, $M_n(R)$ is right weak π-extending if and only if $M_n(R)$ is right weak FI-extending.*

(iii) *Assume M_R is a free module with a finite rank. Then M_R is weak π-extending if and only if M_R is weak FI-extending.*

Proof. (i) It is integrated by Proposition 2.11 (ii).

(ii) Since $M_n(R) = \mathbf{I}(M_n(R))$, the proof is a consequence of part (i).

(iii) Let $M_R = \bigoplus_{t=1}^{n} R_t$ where $R_t \cong R$. Then $End(M_R) \cong M_n(R)$. Therefore Proposition 2.11 (ii) invokes the result. $\square$

Recall from [5] that a ring extension T of R is said to be *right intrinsic over* R, if $X \cap R \neq 0$ for each nonzero right ideal X of T, denoted by $R \leq_r^{int} T$.

Proposition 2.13. *If $R \leq_r^{int} S$ and R_R is weak π-extending, then S_S is weak π-extending.*

Proof. Let Y be a semisimple projection invariant submodule of S and $X = R \cap Y$. Then X is a semisimple projection invariant submodule R_R. Thus there is $e = e^2 \in R$ such that $X_R \leq^{ess} eR_R$. Let $y \in Y$. Then $y = ey + (1-e)y$. If $(1-e)y \neq 0$, then there exists $r \in R$ such that $0 \neq (1-e)yr \in R \cap Y = X \subseteq eR$, a contradiction. Therefore $(1-e)y = 0$. So $Y \leq eS$. Let $0 \neq es \in eS$. Then $0 \neq esr_1 \in R \cap eS \leq eR$, for some $r_1 \in R$. Hence $R \cap Y \leq R \cap eS \leq eR$. It follows that $0 \neq es(r_1 r_2) \in R \cap Y \leq Y$, for some $r_2 \in R$. Thus $Y_S \leq^{ess} eS_S$ which gives that S_S is weak π-extending. $\square$

Corollary 2.14. *Let $\boldsymbol{I}(R) \leq_r^{int} R$ and $\boldsymbol{I}(R)$ is right weak FI-extending. Then $\boldsymbol{I}(R)$ and R are right weak π-extending.*

Proof. It is clear from Corollary 2.12 (i) and Proposition 2.13. $\square$

Proposition 2.15. *Assume M is weak π-extending and $X \trianglelefteq_p M$. Then X is weak π-extending.*

Proof. Suppose Y is a semisimple projection invariant submodule of X. Then Y is a semisimple projection invariant submodule of M by Lemma 2.1. Hence there exists a $T \leq^{\oplus} M$ such that $Y \leq^{ess} T$. Thus $M = T \oplus T'$ for some $T' \leq M$. Since $X \trianglelefteq_p M$, $X = (X \cap T) \oplus (X \cap T')$ by Lemma 2.1. Therefore $Y \leq^{ess} X \cap T$, where $X \cap T \leq^{\oplus} X$. So X is weak π-extending. $\square$

Opposed to weak CS-modules (see, Example 2.7), we deduce the following direct sums result.

Theorem 2.16. *Assume $M = \bigoplus_{\kappa \in K} M_\kappa$, where M_κ is weak π-extending for all $\kappa \in K$. Then M is weak π-extending.*

Proof. Let B be a semisimple projection invariant submodule of M. Thus, by Lemma 2.1, $B = \bigoplus_{\kappa \in K} (B \cap M_\kappa)$, where $B \cap M_\kappa \trianglelefteq_p M_\kappa$ for all $\kappa \in K$. Since B is semisimple, $B \cap M_\kappa$ is a semisimple submodule of M_κ. Then there exists a $K_\kappa \leq^{\oplus} M_\kappa$ such that $B \cap M_\kappa \leq^{ess} K_\kappa$. Therefore $B = \bigoplus_{\kappa \in K} (B \cap M_\kappa) \leq^{ess} \bigoplus K_\kappa \leq^{\oplus} M$. Thus M is weak π-extending. $\square$

Corollary 2.17. *If M is a direct sum of π-extending (resp., uniform or extending) modules, then M is weak π-extending.*

Proof. It is an application of Theorem 2.16. $\square$

3. Decompositions. We explore decomposition results for weak π-extending modules in this section.

Proposition 3.1. *Let T_T be weak π-extending and every direct summand of a weak π-extending T-module is weak π-extending. Then every indecomposable projective right T-module which has a nonzero socle is uniform.*

Proof. Suppose Y_T is indecomposable projective with $SocY \neq 0$. Then $F = Y \oplus Z$ for some $Z \leq F$, where F is a free T-module. By Theorem 2.16, F_T is weak π-extending. Hence, by hypothesis, Y_T is weak π-extending. It follows from Lemma 2.9 that Y_T is uniform. □

Lemma 3.2. *Suppose $B = B_1 \oplus B_2$ for some $B_1, B_2 \leq B$. Then B_1 is weak π-extending if and only if for all semisimple projection invariant submodule A_1 of B_1, there is a $K \leq^{\oplus} B$ such that $B_2 \subseteq K$, $K \cap A_1 = 0$ and $K \oplus A_1 \leq^{ess} B$.*

Proof. Let B_1 be weak π-extending and A_1 a semisimple projection invariant submodule of B_1. Then there is a $C_1 \leq^{\oplus} B_1$ such that $A_1 \cap C_1 = 0$ and $A_1 \oplus C_1 \leq^{ess} B_1$ by Lemma 2.2(i). It is clear that $C_1 \oplus B_2 \leq^{\oplus} B$ and $B_2 \subseteq C_1 \oplus B_2$ and $(C_1 \oplus B_2) \cap A_1 = 0$, and $C_1 \oplus B_2 \oplus A_1 \leq^{ess} B$. Conversely, assume B_1 holds the stated property and A_1 is a semisimple projection invariant submodule of B_1. Thus there is a $K \leq^{\oplus} B$ such that $B_2 \subseteq K$, $K \cap A_1 = 0$ and $K \oplus A_1 \leq^{ess} B$. So, $K = K \cap (B_1 \oplus B_2) = B_2 \oplus (K \cap B_1)$ hence $K \cap B_1 \leq^{\oplus} B_1$. Observe that $A_1 \cap (K \cap B_1) = 0$ and $(K \cap B_1) \oplus A_1 \leq^{ess} B_1$. Therefore, B_1 is weak π-extending by Lemma 2.2(i). □

Theorem 3.3. *Let $B = B_1 \oplus B_2$ be weak π-extending such that $SocB_2 \leq^{ess} B_2$ and for all $A \leq^{\oplus} B$ with $A \cap B_2 = 0$ and $A \oplus B_2 \leq^{\oplus} M$. Then B_1 is weak π-extending.*

Proof. Let N_1 be a semisimple projection invariant submodule of B_1. Then $N_1 \oplus SocB_2$ is a semisimple projection invariant submodule of B. Then there exists a $K \leq^{\oplus} B$ such that $K \cap (N_1 \oplus SocB_2) = 0$ and $K \oplus N_1 \oplus SocB_2 \leq^{ess} B$ by Lemma 2.2(i). Since $SocB_2 \leq^{ess} B_2$, $K \cap (N_1 \oplus B_2) = 0$. Hence $K \cap B_2 = 0$. By hypothesis, $K \oplus B_2 \leq^{\oplus} B$. Moreover, $N_1 \oplus K \oplus B_2 \leq^{ess} B$. Therefore Lemma 3.2 completes the proof. □

Corollary 3.4. *Assume $B = B_1 \oplus B_2$ is a weak π-extending module such that $SocB_2 \leq^{ess} B_2$ and B/B_1 is B_1-injective. Then B_1 is weak π-extending.*

Proof. Clearly, B_2 is B_1-injective. Let $C_1 \leq^{\oplus} B$ such that $C_1 \cap B_2 = 0$. Then there exists a $A \leq B$ such that $A \cap B_2 = 0$, $B = A \oplus B_2$ and $C_1 \subseteq A$ by ([4], Lemma 7.5). Hence $C_1 \leq^{\oplus} A$ and $C_1 \oplus B_2 \leq^{\oplus} B$. Hence B_1 is weak π-extending by Theorem 3.3. □

Combining Theorem 2.16 with Corollary 3.4, we have the following result.

Corollary 3.5. *Let $B = B_1 \oplus B_2$ such that $SocB_2 \leq^{ess} B_2$ and B_2 is injective. Then B is weak π-extending if and only if B_1 is weak π-extending.*

A module M has the *summand intersection property*, SIP, provided that the intersection of any two direct summands is again a direct summand of M.

Theorem 3.6. *Assume M is weak π-extending with SIP. Then any direct summand of M is weak π-extending.*

Proof. Let $M_1 \leq^{\oplus} M$ and N_1 be a semisimple projection invariant submodule of M_1. Thus $N_1 \oplus SocM_2$ is a semisimple projection invariant submodule of M. Then there is a $K \leq^{\oplus} M$ such that $N_1 \oplus SocM_2 \leq^{ess} K$. Observe that $N_1 = M_1 \cap (N_1 \oplus SocM_2)$ by modular law. Hence $N_1 \leq^{ess} K \cap M_1$. Since M has SIP, $K \cap M_1 \leq^{\oplus} M$. Thus $M_1 = (K \cap M_1) \oplus (M_1 \cap X)$. Hence $N_1 \leq^{ess} K \cap M_1 \leq^{\oplus} M_1$. □

Proposition 3.7. *Suppose $End(M_R)$ is Abelian for a weak π-extending module M_R. Then any direct summand of M_R is weak π-extending.*

Proof. Assume $End(M_R)$ is Abelian and $Q \leq^{\oplus} M$. Then $Q \trianglelefteq_p M_R$. Let X be a semisimple projection invariant submodule of Q. Thus $X \trianglelefteq_p M$ by Lemma 2.1. Hence there exists a $P \leq^{\oplus} M$ such that $X \leq^{ess} P \leq^{\oplus} M$. Thus $M = P \oplus P'$ for some $P, P' \leq M$. Since $Q \trianglelefteq_p M$, $Q = (Q \cap P) \oplus (Q \cap P')$. Note that $X = X \cap Q \leq^{ess} Q \cap P \leq^{\oplus} Q$. Hence Q is weak π-extending. □

Proposition 3.8. *Assume B_1 is a semisimple projection invariant submodule of B such that $B = B_1 \oplus B_2$ is weak π-extending. Then B_1 and B_2 are weak π-extending.*

Proof. It is clear from Proposition 2.15 that B_1 is weak π-extending. Let X_2 be a semisimple projection invariant submodule of B_2. Then $B_1 \oplus X_2 \trianglelefteq_p B$ by ([2], Lemma 4.13). Also, $B_1 \oplus X_2$ is a semisimple submodule of B. Thus there exists a $Y \leq^{\oplus} B$ such that $B_1 \oplus X_2 \leq^{ess} Y$. It follows from modular law that $X_2 = B_2 \cap (B_1 \oplus X_2) \leq^{ess} Y \cap B_2$. Observe that $Y = B_1 \oplus (Y \cap B_2)$, so $Y \cap B_2 \leq^{\oplus} B_2$. Then B_2 is weak π-extending. □

Theorem 3.9. (i) *Let M_R be a module with finite uniform dimension and R a Dedekind domain. Then every direct summand of M_R is weak π-extending.*

(ii) *Let $M = U_1 \oplus U_2$, where U_1 and U_2 are uniform modules. Then every direct summand of M_R is weak π-extending.*

(iii) *Let M be a $\mathbb{Z}$-module such that M is a direct sum of uniform modules. Then every direct summand of M is weak π-extending.*

Proof. (i) Let $M = M_1 \oplus M_2$ for some submodules M_1 and M_2 of M. If M_1 is torsion-free, then $SocM_1 = 0$. Clearly, M_1 is weak π-extending. If M_1 is not torsion-free, then $M_1 = X_1 \oplus X_2 \oplus X_3$ such that X_1 is injective, X_2 is finitely generated, and X_3 is torsion-free by ([8], Theorem 9). It follows from ([12], Theorem 4.12) that X_2 is weak CS, so is weak π-extending. Therefore Theorem 2.16 implies that M_1 is weak π-extending.

(ii) Let $0 \neq P$ be a direct summand of M. If $P = M$, then P is weak π-extending by Theorem 2.16. If $P \neq M$, then P is uniform, so it is weak π-extending.

(iii) Let $N \leq^{\oplus} M$. Then N is the direct sum of uniform modules ([11], Theorem 5.5). Now, Theorem 2.16 yields N is weak π-extending. □

S, Δ, and $J(S)$ will show the endomorphism ring of M, $\{f \in S : \ker f \leq^{ess} M\}$ and the Jacobson radical of S, respectively. It was pointed out in ([9], Proposition 3.5) that for a continuous module M, S/Δ is von Neumann regular and $\Delta = J(S)$. It is natural to consider whether ([9], Proposition 3.5) can be carried up to weak π-extending modules with C_2. However, ([12], Example 3.102) eliminates this possibility. Accordingly, we investigate under what conditions weak π-extending modules with C_2 satisfies the extended version of ([9], Proposition 3.5).

Theorem 3.10. *Let M be a weak π-extending module with an essential socle. If M has C_2, then S/Δ is a regular ring and $\Delta = J(S)$.*

Proof. Let $f \in S$ and $K = \ker f$. By ([12], Lemma 5.109), M is strongly bounded. Hence every semisimple submodule of M is fully invariant in M by ([12], Lemma 5.109), so is projection invariant in M. Thus $SocK \trianglelefteq_p M$. Then there exists $D \leq^{\oplus} M$ such that $SocK \cap D = 0$ and $SocK \oplus D \leq^{ess} M$. Since $SocM \leq^{ess} M$, $SocK \leq^{ess} K$. Hence $K \cap D = 0$. Thus $f|_D$ is a monomorphism. By C_2 condition, $f(D) \leq^{\oplus} M$. Then there is $g \in S$ such that $gf = \iota|_D$. Hence $(f - fgf)(K \oplus D) = (f - fgf)(L) = 0$, so $K \oplus D \leq \ker(f - fgf)$. Since $SocK \oplus D \leq^{ess} M$, $K \oplus D \leq^{ess} M$. It follows that $f - fgf \in \Delta$. Thus S/Δ is a regular ring. Then that $J(S) \subseteq \Delta$. Let $b \in \Delta$. Since $\ker b \cap \ker(1 - b) = 0$ and $\ker b \leq^{ess} M$, $\ker(1 - b) = 0$. Hence $(1 - b)M \leq^{\oplus} M$ by C_2 condition. However, $(1 - b)M \leq^{ess} M$, so $M = (1 - b)M$. Thence $\Delta \subseteq J(S)$. □

Theorem 3.11. *Assume R holds ascending chain condition (a.c.c.) on the right annihilators, $r(m)$, for $m \in M$ and M is weak π-extending with C_3. If S is Abelian (resp., M is strongly bounded), then there is an indecomposable decomposition for M.*

Proof. Suppose $\{L_\psi\}_{\psi\in\Psi}$ is an independent family of submodules of M and let $L = \bigoplus_{\psi\in\Psi} L_\psi$ be a local summand of M. Since S is Abelian, $L \trianglelefteq_p M$. Thus $SocL \trianglelefteq_p M$ by Lemma 2.1(i). (Since M is strongly bounded, $SocL \trianglelefteq M$ by ([12], Lemma 5.109). Thus $SocL \trianglelefteq_p M$). Hence $SocL \cap T = 0$ and $SocL \oplus T \leq^{ess} M$, for some $T \leq^{\oplus} M$ by Lemma 2.2. Take $L' = \bigoplus_{\psi\in\Phi} L_\psi$ for any finite subset Φ of Ψ. Then L' is a direct summand of M, so $L' \oplus T \leq^{\oplus} M$ by C_3 condition. Consequently, $L \oplus T$ is a local summand, so $L \oplus T$ is a complement in M by ([11], Lemma 4.5). However, $L \oplus T \leq^{ess} M$ which yields that $M = L \oplus T$. By ([9], Theorem 2.7), M has an indecomposable decomposition. □

Proposition 3.12. *Suppose R has a.c.c. on the right annihilators, $r(m)$, for $m \in M$ and M is weak π-extending with C_3. If S is Abelian (resp., M is strongly bounded), then the finite exchange property implies full exchange property.*

Proof. It is integrated by Theorem 3.11 and ([15], Corollary 6). □

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