

DIFFERENTIAL INCLUSIONS AND QUASI-LYAPUNOV FUNCTIONS

Martin Ivanov^{1✉}, Mikhail Krastanov^{1,2}, Nadezhda Ribarska^{1,2}

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Abstract

A sufficient condition for existence of a solution of a differential inclusion with a uniformly bounded right-hand side that has nonempty closed (possibly nonconvex) values is obtained. An Olech-type result is obtained as a corollary. An example, which originates from the Fuller problem from optimal control theory, is given to demonstrate the applicability of the main result.

Key words: differential inclusions with nonconvex right-hand side, existence of solutions, quasi-Lyapunov function, invariance, Fuller problem

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1. Introduction. We study the existence of local solutions of differential inclusions of the form

$$\dot{x} \in F(x), \quad x(0) = x_0,$$

where $x_0 \in \mathbb{R}^n$ and $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multi-valued mapping with nonempty values, i.e. the existence of $T > 0$ and an absolutely continuous function $x: [0, T] \rightarrow \mathbb{R}^n$ such that $x(0) = x_0$ and $\dot{x}(t) \in F(x(t))$ for almost every $t \in [0, T]$.

The purpose of this paper is to announce a recent result in the case where the multi-valued mapping F is uniformly bounded and has nonempty closed (possibly nonconvex) values. Two applications of this result are briefly outlined: for proving an Olech-type result and for obtaining existence of a solution of a differential inclusion arising from a classical optimal control problem.

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We make use of the formalism proposed in [1], which in a way is a development of the patchy vector fields of ANCONA and BRESSAN [2]. This approach consists in constructing a “suitable” relatively open partition of the domain of the multi-valued mapping and approximating it on each element of the partition with an upper semi-continuous mapping which has convex compact values. The approximating mappings should be close to the original ones, and each element of the partition must be weakly forward invariant with respect to its corresponding approximating mapping. This construction works well in the case where the relatively open partition is locally finite. In such a case, a uniform limit of ε -solutions is an ε -solution of the differential inclusion. For the case where locally finite partitions of the above kind cannot be constructed, a more complicated structure (consisting of countably many relatively open partitions, each refining the previous one) is developed in [1] (cf. Definition 2.5). An example using this generalized structure is given in [1] as well as in the last section of this paper.

An existence result for differential inclusions with mixed-type conditions on the right-hand side is obtained. Similar results were proved by OLECH [3] and ŁOJASIEWICZ [4] (see also [5] and [6]). These results unify the upper semi-continuous and lower semi-continuous cases. For example, in Łojasiewicz’s result, lower semi-continuity on an open set and convexity and upper semi-continuity on its complement are assumed. Convexity is also assumed in [1] on the set of points, where the right-hand side lacks lower semi-continuity. Here, instead of convexity, we assume that the field “runs away” from this set. More precisely, we assume the existence of a non-negative function that vanishes only at the points from this set and increases along approximate solutions of the considered inclusion at least by a constant rate. This guarantees that uniform limits of such solutions stay outside the set of points, where the right-hand side lacks lower semi-continuity, and allows us to use the results from [1].

The function in question (called quasi-Lyapunov function) is assumed to be piecewise smooth. The reason is the fact that this research began as a search for conditions guaranteeing the existence of solutions of inclusions whose right-hand side belongs to some o-minimal structure (for example, subanalytic sets). In such a situation, piecewise smoothness is a natural assumption. It is possible to look for non-smooth variants.

We show that the main result is applicable to an inclusion that originates from the optimal feedback for the Fuller problem from optimal control theory. This problem is one of the first examples for chattering control – the optimal control switches infinitely many times on a finite time interval (cf. [7]). We construct a quasi-Lyapunov function in this example which is semi-algebraic.

Detailed proofs of the results presented in this paper can be found in [8].

2. Preliminaries. The multi-valued mappings under consideration are defined on a nonempty subset D of $\mathbb{R}^n$ and the solutions of the inclusions have to stay in D in order to be well defined. Moreover, one of the key ideas of the presented

approach is the weak forward invariance of the elements of the partitions with respect to the corresponding approximating mappings, so we recall the following notions related to invariance.

Definition 2.1. Let $D \subseteq \mathbb{R}^n$ and $\varphi: [0, T) \rightarrow \mathbb{R}^n$ be an absolutely continuous function. It is said that the set D is forward invariant with respect to φ if for each $t \in [0, T)$ with $\varphi(t) \in D$ there exists $\delta > 0$ such that $\varphi(\tau) \in D$ for all $\tau \in [t, t + \delta)$.

Definition 2.2. Let $S \subseteq D \subseteq \mathbb{R}^n$ and $F: D \rightrightarrows \mathbb{R}^n$ be a multi-valued mapping. It is said that S is weakly forward invariant with respect to F if for every $x_0 \in S$ there exist $\delta > 0$ and a trajectory φ of $\dot{x} \in F(x)$ defined on $[0, \delta)$ such that $\varphi(0) = x_0$ and $\varphi(t) \in S$ for all $t \in [0, \delta)$.

In the following definitions we make precise what is meant by a “suitable” partition and by a refinement of it.

Definition 2.3 ([9]). Let X be a topological space, α_0 be an ordinal number and

$$\mathcal{U} = \{U_\alpha : 1 \leq \alpha < \alpha_0\}$$

be a well-ordered family of subsets of X . It is said that $\mathcal{U}$ is a relatively open partition of X if:

- (i) U_α is contained in $X \setminus \left(\bigcup_{\beta < \alpha} U_\beta \right)$ and it is relatively open in it for every α ;
- (ii) $X = \bigcup_{1 \leq \alpha < \alpha_0} U_\alpha$.

Note that each element of a relatively open partition is an intersection of an open set and a closed set, i.e. it is locally closed.

Definition 2.4 ([1]). Let $\mathcal{U}^1$ and $\mathcal{U}^2$ be two relatively open partitions of X . It is said that $\mathcal{U}^2$ is a refinement of $\mathcal{U}^1$ ($\mathcal{U}^2 \prec \mathcal{U}^1$) if:

- (i) for each element $U^2 \in \mathcal{U}^2$ there exists an element $U^1 \in \mathcal{U}^1$ with $U^2 \subseteq U^1$;
- (ii) if $U^2 \ni U_{\beta_i}^2 \subseteq U_{\alpha_i}^1 \in \mathcal{U}^1$, $i = 1, 2$ with $\alpha_1 < \alpha_2$, then $\beta_1 < \beta_2$.

The following definition is a slight modification of the original Definition 3.1 from [1].

Definition 2.5 ([1]). Let D be a locally closed subset of $\mathbb{R}^n$ and $F: \overline{D} \rightrightarrows \mathbb{R}^n$ be a multi-valued mapping with nonempty values. Let $\varepsilon > 0$ be fixed and let $\mathcal{U}$ be a σ -relatively open partition of D (that is, $\mathcal{U} = \bigcup_{m=1}^{\infty} \mathcal{U}^m$, where each $\mathcal{U}^m$ is a relatively open partition of D) such that $\mathcal{U}^{m+1} \prec \mathcal{U}^m$ for all $m \in \mathbb{N}$. Let **Green** $\mathcal{U}$ be a disjoint subfamily of $\mathcal{U}$ and let $\mathcal{G} = \{G_U : U \in \mathbf{Green}\mathcal{U}\}$ be a family of multi-valued mappings.

It is said that $(\mathcal{U}, \mathcal{G})$ is an invariant partial ε -approximation of F if:

- (i) if $U \in (\mathbf{Green}\mathcal{U}) \cap \mathcal{U}^m$, then $U \in \mathcal{U}^k$ for all $k \geq m$;

- (ii) for each point $x \in \cup(\mathbf{Green}\mathcal{U})$ there exists a neighbourhood of x which intersects at most finitely many elements of $\mathbf{Green}\mathcal{U}$;
- (iii) the set $\cup(\mathbf{Green}\mathcal{U})$ is open in D ;
- (iv) $(\mathbf{Green}\mathcal{U}, \mathcal{G})$ is an invariant ε -approximation of F , i.e.
 - (a) $G_U: \overline{U} \rightrightarrows \mathbb{R}^n$ is an upper semi-continuous multi-valued mapping with nonempty convex compact values for each $U \in \mathbf{Green}\mathcal{U}$;
 - (b) for each $U \in \mathbf{Green}\mathcal{U}$ and for each $x \in \overline{U}$ it is true that

$$G_U(x) \cap T_U^B(x) \subseteq F(x) + \varepsilon \overline{B},$$

where $T_U^B(x)$ denotes the Bouligand tangent cone to U at x ;

- (c) for each $U \in \mathbf{Green}\mathcal{U}$ and for each $x \in U$ the intersection $G_U(x) \cap T_U^B(x)$ is nonempty.

All results in [1] remain valid with this modified definition, as the existence of tangent velocities is required only at every point of the element, rather than at every point of its closure. Roughly speaking, the meaning of Definition 2.5 is that the domain can be divided into two parts: “good points” (the set $\cup(\mathbf{Green}\mathcal{U})$) and “bad points” (the complement of $\cup(\mathbf{Green}\mathcal{U})$ in D). The set of “good points” can be partitioned into a locally finite family of subsets each of which is weakly forward invariant with respect to a suitably chosen approximating mapping. Depending on the problem, additional efforts are necessary to deal with the “bad points”. In [1] a scheme is suggested for proofs of the existence of solutions of differential inclusions using this structure.

The following proposition is a slight modification of Proposition 2.3 from [1] that is stated for an invariant ε -approximation with a locally finite relatively open partition of the domain (cf. [1, Definition 2.2]). Since $\mathbf{Green}\mathcal{U}$ is locally finite and open in D by definition, every point of $\cup(\mathbf{Green}\mathcal{U})$ has a neighbourhood V (in D) that is contained in $\cup(\mathbf{Green}\mathcal{U})$ and intersects finitely many elements of $\mathbf{Green}\mathcal{U}$. Thus, there exists a positive integer m such that $V \cap \mathcal{U}^m$ is a relatively open partition of $V \cap D$ with finitely many elements. Then we are in the situation of the proposition cited from [1]. This idea is implemented in Theorem 3.4 from the same paper.

Proposition 2.6. *Let D be a locally closed subset of $\mathbb{R}^n$ and $F: \overline{D} \rightrightarrows \mathbb{R}^n$ be uniformly bounded. Let $\mathcal{U}$ be a σ -relatively open partition of D and $(\mathcal{U}, \mathcal{G})$ be an invariant partial ε -approximation of F . Then:*

- (i) *for each point $x_0 \in \cup(\mathbf{Green}\mathcal{U})$ there exist $T > 0$ and an ε -solution $\varphi: [0, T) \rightarrow \cup(\mathbf{Green}\mathcal{U})$ of the inclusion $\dot{x} \in F(x)$ starting from x_0 such that each element of $\mathbf{Green}\mathcal{U}$ is forward invariant with respect to φ ;*

- (ii) for each absolutely continuous function $\varphi: [0, T] \rightarrow \cup(\mathbf{Green}\mathcal{U})$ with respect to which the elements of $\mathbf{Green}\mathcal{U}$ are forward invariant, the function

$$t \mapsto \alpha(t), \text{ where } \varphi(t) \in U_{\alpha(t)} \text{ and } U_{\alpha(t)} \in \mathbf{Green}\mathcal{U},$$

is monotone increasing;

- (iii) if $\varphi_k: [0, T] \rightarrow \cup(\mathbf{Green}\mathcal{U})$, $k = 1, 2, \dots$, is a sequence of ε -solutions of $\dot{x} \in F(x)$ with respect to which the elements of $\mathbf{Green}\mathcal{U}$ are forward invariant and $\varphi: [0, T] \rightarrow \cup(\mathbf{Green}\mathcal{U})$ is its uniform limit, then φ is an ε -solution of $\dot{x} \in F(x)$.

Definition 2.7 ([1]). Let D be a locally closed subset of $\mathbb{R}^n$ and $F: \overline{D} \rightrightarrows \mathbb{R}^n$ be a multi-valued mapping with nonempty values. Let $\mathcal{U}^i$ be a σ -relatively open partition of D and $(\mathcal{U}^i, \mathcal{G}^i)$ be an invariant partial ε_i -approximation of F , $i = 1, 2$. It is said that $(\mathcal{U}^2, \mathcal{G}^2)$ is a refinement of $(\mathcal{U}^1, \mathcal{G}^1)$ if:

- (i) $0 < \varepsilon_2 < \varepsilon_1$;
- (ii) for each $U^2 \in \mathbf{Green}\mathcal{U}^2$ there exists $U^1 \in \mathbf{Green}\mathcal{U}^1$ with $U^2 \subseteq U^1$;
- (iii) if $\mathbf{Green}\mathcal{U}^2 \ni U_i^2 \subseteq U_i^1 \in \mathbf{Green}\mathcal{U}^1$, $i = 1, 2$, with $U_2^1 \prec U_1^1$, then $U_2^2 \prec U_1^2$;
- (iv) if $\mathbf{Green}\mathcal{U}^2 \ni U^2 \subseteq U^1 \in \mathbf{Green}\mathcal{U}^1$, then $G_{U^2}(x) \subseteq G_{U^1}(x)$ for each $x \in \overline{U^2}$.

3. Main result. The following definition is motivated by the concept of stability and the related notion of a Lyapunov function. The function we define strictly increases at least by a constant rate along the trajectories, so they “move away” from the set of “bad points”. We require continuity on the whole domain and smoothness on each element of the partition of the set of “good points”.

Definition 3.1. Let D be a locally closed subset of $\mathbb{R}^n$ and $F: \overline{D} \rightrightarrows \mathbb{R}^n$ be a multi-valued mapping with nonempty values. Let $\mathcal{U}$ be a σ -relatively open partition of D and $(\mathcal{U}, \mathcal{G})$ be an invariant partial ε -approximation of F . It is said that $w: D \rightarrow \mathbb{R}$ is a quasi-Lyapunov function for $(\mathcal{U}, \mathcal{G})$ if:

- (i) $w(x) \geq 0$ for all $x \in D$ and $w(x) = 0$ if and only if $x \notin \cup(\mathbf{Green}\mathcal{U})$;
- (ii) w is a continuous function;
- (iii) the restriction $w|_U$ of w to U is a continuously differentiable function for all $U \in \mathbf{Green}\mathcal{U}$ (i.e. it is a restriction of a continuously differentiable function which is defined on an open set containing U);

(iv) for each $U \in \mathbf{Green}\mathcal{U}$ and for each $x \in U$ the following inequality holds:

$$\max_{y \in G_U(x) \cap T_U^B(x)} \langle \mathbf{grad} w(x), y \rangle \geq 1.$$

The maximum from condition (iv) of the above definition is required to be uniformly separated from zero by a positive number. Without loss of generality, we assume that this number is equal to 1. This guarantees that ε -solutions starting from “good points” always “stay away” from the “bad points” as well as uniform limits of such ε -solutions. In this way, using Proposition 2.6(iii), we can prove the existence of ε -solutions starting from the boundary of the set of “bad points”.

Proposition 3.2. *Let D be an intersection of an open subset S and a closed subset K of $\mathbb{R}^n$ and $F: \overline{D} \rightrightarrows \mathbb{R}^n$ be a uniformly bounded multi-valued mapping with nonempty values. Let $\mathcal{U}$ be a σ -relatively open partition of D and $(\mathcal{U}, \mathcal{G})$ be an invariant partial ε -approximation of F . Let $w: D \rightarrow \mathbb{R}$ be a quasi-Lyapunov function for $(\mathcal{U}, \mathcal{G})$.*

Then for each point $x_0 \in \overline{\cup(\mathbf{Green}\mathcal{U})} \cap D$ there exists an ε -solution $\varphi(\cdot)$ of $\dot{x} \in F(x)$ starting from x_0 and defined on the interval $[0, T)$ ($T = +\infty$ or $\varphi(t)$ tends to a point in $\mathbb{R}^n \setminus S$ whenever $t \rightarrow T$) such that $w(\varphi(t)) \geq w(x_0) + t$ for all $t \in [0, T)$. Moreover, if the starting point x_0 belongs to $\cup(\mathbf{Green}\mathcal{U})$, then each element of $\mathbf{Green}\mathcal{U}$ is forward invariant with respect to the corresponding ε -solution $\varphi(\cdot)$ starting from x_0 .

Assuming closedness of the values of the right-hand side and existence of successive refinements of the invariant partial ε -approximation, we can pass to the limit as $\varepsilon \rightarrow 0$ and obtain a solution of the considered differential inclusion.

The quasi-Lyapunov functions that are assumed to exist guarantee that approximate solutions starting from “good points” stay in the corresponding sets of “good points”. Then one could easily obtain a solution of the considered inclusion as a uniform limit of approximate solutions using similar arguments as in the proof of Theorem 2.6 from [1].

Theorem 3.3. *Let D be a locally closed subset of $\mathbb{R}^n$ and $F: \overline{D} \rightrightarrows \mathbb{R}^n$ be a uniformly bounded multi-valued mapping with nonempty closed values. Let $\mathcal{U}^k$ be a σ -relatively open partition of D , $(\mathcal{U}^k, \mathcal{G}^k)$ be an invariant partial $\frac{1}{k}$ -approximation of F , and $(\mathcal{U}^{k+1}, \mathcal{G}^{k+1})$ be a refinement of $(\mathcal{U}^k, \mathcal{G}^k)$ for all $k = 1, 2, \dots$. We assume that for each approximation $(\mathcal{U}^k, \mathcal{G}^k)$ there exists a quasi-Lyapunov function $w_k: D \rightarrow \mathbb{R}$ such that $w_{k_1} \geq w_{k_2}$ on D for every $k_1 \leq k_2$.*

Then for each point $x_0 \in \bigcap_{k=1}^{\infty} \left(\overline{\cup(\mathbf{Green}\mathcal{U}^k)} \right) \cap D$ there exist $T > 0$ and an absolutely continuous function $x: [0, T] \rightarrow D$ such that

$$\begin{cases} \dot{x}(t) \in F(x(t)) \text{ a.e. on } [0, T] \\ x(0) = x_0 \end{cases}.$$

The structure of an invariant partial ε -approximation of a multi-valued mapping has been used in [1] to obtain the existence of solutions of inclusions of the

form $\dot{x} \in F(x) + G(x)$, where F is lower semi-continuous on a G_δ subset of its domain D and upper semi-continuous and convex-valued on the complement, and G is upper semi-continuous and convex-valued on its domain D . One could easily adapt the proof of Theorem 4.1 from [1] (with G equal to the zero vector of $\mathbb{R}^n$) replacing the convexity assumption on the values of F at the “bad points” with the existence of a suitable Lyapunov-like function. Thus, using Theorem 3.3 from the present paper, we obtain the following corollary.

Corollary 3.4. *Let D be a locally closed subset of $\mathbb{R}^n$ and F be a uniformly bounded mapping with nonempty closed values defined on $\overline{D}$. We assume that there exist a sequence of positive reals ε_k , $k = 1, 2, \dots$, which tends to zero as k tends to infinity, relatively open and dense (in D) sets D_k such that F is ε_k -lower semi-continuous on D_k (we assume that $D_1 = D$), and there exists a sequence of continuous functions $w_k: D \rightarrow \mathbb{R}$ such that:*

- (i) $w_{k_1} \geq w_{k_2}$ on D for every $k_1 \leq k_2$;
- (ii) for each positive integer k we have $w_k(x) \geq 0$ for all $x \in D$ and $w_k(x) = 0$ if and only if $x \notin D_k$;
- (iii) for each positive integer k the restriction $w_k|_{D_k}$ of w_k to D_k is a continuously differentiable function;
- (iv) for each positive integer k and for each $x \in D_k$ the set $F_k(x)$ is non-empty, where $F_0(x) := F(x)$ and for each $k \geq 1$

$$F_k(x) := \{y \in F_{k-1}(x): \langle \mathbf{grad} w_k(x), y \rangle \geq 1\}.$$

We assume that for each positive integer k and for each $x \in D_k$ the inclusion $F(x) \subseteq T_D^B(x) \cap F_k(x) + \varepsilon_k \overline{B}$ holds.

Then for each point $x_0 \in D$ the differential inclusion $\dot{x} \in F(x)$ has a solution starting from x_0 .

4. An example. The example in this section originates from the following problem from optimal control theory known as the Fuller problem (cf., e.g., [7, Chapter 2]):

Minimize

$$\int_0^T |x(t)|^q dt \quad (q > 1)$$

subject to

$$\begin{aligned} \dot{x}(t) &= y(t), & \dot{y}(t) &= u(t), & u(t) &\in [-1, 1], \\ x(0) &= 0, & y(0) &= 0, & x(T) &= x_0, & y(T) &= y_0. \end{aligned}$$

This problem has been analyzed in Lemma 2.2 and Lemma 2.3 from [7]. For every final state (x_0, y_0) there exists $T_0 > 0$ such that for all $T > T_0$ the

problem has a unique optimal solution with a control function with a countable number of switching times. The switching points of the corresponding optimal trajectory determine a convergent sequence which tends to the origin. The curve $x = C(q)y^2 \operatorname{sgn} y$ is the switching set. The optimal control equals 1 on the left-hand side of the switching curve and equals -1 on the right-hand side. For every $q > 1$ the parameter $C(q)$ satisfies the inequality $0 < C(q) < 1/2$.

Taking into account the solution of this problem, we consider a differential inclusion with the following upper semi-continuous right-hand side:

$$F(x, y) = \begin{cases} \{(y, 1)\}, & \text{if } x < \frac{1}{4}y^2 \operatorname{sgn} y \\ \{(y, -1), (y, 1)\}, & \text{if } x = \frac{1}{4}y^2, y > 0 \\ \{(0, 1), (0, -1)\}, & \text{if } x = 0, y = 0 \\ \{(y, -1), (y, 1)\}, & \text{if } x = -\frac{1}{4}y^2, y < 0 \\ \{(y, -1)\}, & \text{if } x > \frac{1}{4}y^2 \operatorname{sgn} y \end{cases}.$$

To simplify the calculations, we have chosen the value $1/4$ for the parameter $C(q)$. The same kind of reasoning can be made for every value of $C(q)$ in the interval $(0, 1/2)$.

Classical existence results (cf., e.g., [5]) do not apply for the differential inclusion

$$(\dot{x}, \dot{y}) \in F(x, y), x(0) = x_0, y(0) = y_0,$$

when the starting point is $(x_0, y_0) = (0, 0)$ because the right-hand side is not convex-valued. Nevertheless, the existence of a solution of this inclusion starting from the origin follows from [7, Lemma 2.3].

Another way to prove the existence of such a solution is to use the results from Section 3. For this purpose, an invariant partial ε -approximation of F can be constructed. The origin is the only “bad point” here. Next, a quasi-Lyapunov function for this approximation can be constructed (see Fig. 1). Finally, applying

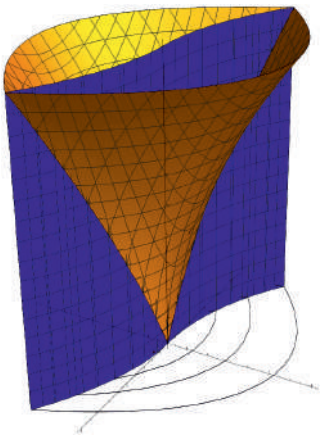


Fig. 1. The graph (in yellow) of the constructed quasi-Lyapunov function (the blue surface is the switching surface). The function is not Lipschitz at the origin. The norm of the gradient tends to infinity as we approach the origin. This guarantees the estimate in Definition 3.1(iv)

Theorem 3.3, we obtain the existence of a solution of the differential inclusion $(\dot{x}, \dot{y}) \in F(x, y)$ starting from the point $(0, 0)$.

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¹*Faculty of Mathematics and Informatics, Sofia University “St. Kliment Ohridski”,
5 James Bourchier Blvd, 1164 Sofia, Bulgaria
e-mail: mdobrinov@fmi.uni-sofia.bg, krastanov@fmi.uni-sofia.bg,
ribarska@fmi.uni-sofia.bg*

²*Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
Akad. G. Bonchev St, Bl. 8, 1113 Sofia, Bulgaria*