

MODULI DIFFERENCE OF INITIAL INVERSE  
COEFFICIENTS OF BAZILEVIČ FUNCTIONS

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Abstract

Let  $f$  be analytic in the unit disk  $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ , and  $\mathcal{B}_1(\alpha, i\beta)$  represent the class of Bazilevič functions  $f$  satisfying  $(zf'(z)/f(z))(f(z)/z)^{\alpha+i\beta} = p(z)$ , where  $\operatorname{Re}(p(z)) > 0$ ,  $\alpha \geq 0$ , and  $\beta \in \mathbb{R}$ . The function  $f$  has the series expansion  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  for  $z \in \mathbb{D}$ . Let  $F$  denote the inverse function of  $f$ , defined in some disk  $|w| \leq r_0(f)$ , with series expansion  $F(w) = w + \sum_{n=2}^{\infty} A_n w^n$ . In this article, we establish sharp bounds on  $||A_3| - |A_2||$  for the class  $\mathcal{B}_1(\alpha, i\beta)$ .

**Key words:** inverse coefficients, Bazilevič functions, univalent functions, coefficient difference

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**1. Introduction.** Let  $\mathcal{A}$  denote the class of all analytic functions in the unit disk  $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$  satisfying the normalization conditions  $f(0) = 0$  and  $f'(0) = 1$ . The subclass  $\mathcal{S} \subset \mathcal{A}$  consists of all univalent (one-to-one) functions in  $\mathbb{D}$ . Every function  $f \in \mathcal{S}$  has a series expansion of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A domain  $\Omega \subseteq \mathbb{C}$  is starlike with respect to  $z_0 \in \mathbb{C}$  if for every  $z \in \Omega$ , the line segment joining  $z$  and  $z_0$  lies entirely in  $\Omega$ . An analytic function  $f$  is starlike

if  $f(\mathbb{D})$  is starlike with respect to the origin. The class of all univalent starlike functions, denoted  $\mathcal{S}^*$ , is characterized analytically by

$$f \in \mathcal{S}^* \iff \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{D}.$$

In 1985, DE BRANGES [1] proved the celebrated Bieberbach conjecture, establishing that for  $f \in \mathcal{S}$ ,  $|a_n| \leq n$  for all  $n \geq 2$ , with equality achieved by the Koebe function  $k(z) = z/(1-z)^2$  and its rotations. This breakthrough naturally led to questions about coefficient differences. POMMERENKE [2] conjectured in 1973 that for  $f \in \mathcal{S}^*$ ,  $||a_{n+1}| - |a_n|| \leq 1$  for  $n \geq 2$ . This was proved by LEUNG [3] in 1978, with equality attained by  $f(z) = z/(1-\rho z)(1-\sigma z)$ , where  $|\rho| = |\sigma| = 1$ .

A function  $f$  is close-to-convex with argument  $\beta$ , where  $|\beta| \leq \pi/2$ , if there exists a starlike function  $g$  such that

$$\operatorname{Re} \left( e^{i\beta} \frac{zf'(z)}{g(z)} \right) > 0, \quad z \in \mathbb{D}.$$

The class of all such functions is denoted  $\mathcal{K}_\beta$ . Every close-to-convex function is univalent, though the converse is false. For  $\beta = 0$ , we obtain the standard class  $\mathcal{K} := \mathcal{K}_0$  of close-to-convex functions. KOEPF [4] showed in 1985 that for  $f \in \mathcal{K}$ ,  $||a_3| - |a_2|| \leq 1$ , but the problem remains open for  $n \geq 3$ .

In 1955, BAZILEVIČ [5] introduced a generalization of starlike and close-to-convex functions. The class  $\mathcal{B}(\alpha, i\beta)$ , where  $\alpha \geq 0$  and  $\beta \in \mathbb{R}$ , consists of functions  $f \in \mathcal{S}$  of the form

$$(1.2) \quad f(z) = \left( (\alpha + i\beta) \int_0^z g^\alpha(t) p(t) t^{i\beta-1} dt \right)^{1/(\alpha+i\beta)},$$

where  $g \in \mathcal{S}^*$  and  $\operatorname{Re}(p(z)) > 0$ . SHEIL-SMALL [6] studied the special case  $\alpha = 0$  in 1972, showing that functions in this subclass satisfy

$$(1.3) \quad \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{i\beta} \right) > 0, \quad z \in \mathbb{D}.$$

It is worth mentioning that for  $\beta = 0$ , the class  $\mathcal{B}(\alpha, 0)$  has been studied extensively (see, [7–11]).

Note that  $\mathcal{B}(0, 0) = \mathcal{S}^*$  and  $\mathcal{B}(1, 0) = \mathcal{K}$ . The case  $\alpha \geq 0$ ,  $\beta \in \mathbb{R}$  with  $g(z) = z$  was investigated by SINGH [12], who defined the subclass  $\mathcal{B}_1(\alpha, i\beta)$  by

$$(1.4) \quad \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\alpha+i\beta} \right) > 0, \quad z \in \mathbb{D}.$$

For  $f \in \mathcal{S}$ , the inverse function  $F = f^{-1}$  exists in some disk  $|w| \leq r_0(f)$  and has the expansion

$$(1.5) \quad F(w) = w + \sum_{n=2}^{\infty} A_n w^n.$$

From the identity  $f(F(w)) = w$ , comparing coefficients in (1.1) and (1.5) yields

$$(1.6) \quad A_2 = -a_2 \quad \text{and} \quad A_3 = 2a_2^2 - a_3.$$

LÖWNER [13] established in 1923 the sharp estimate  $|A_n| \leq (2n)!/(n!(n+1)!)$  for  $n = 2, 3, \dots$  in the classes  $\mathcal{S}$  and  $\mathcal{S}^*$ , with equality for the Koebe function. ALI [14] determined sharp bounds on inverse coefficients for strongly starlike functions. LIBERA and ZŁOTKIEWICZ [15] proved that inverse coefficient bounds for convex functions satisfy  $|A_i| \leq 1$  for  $i = 1, 2, \dots, 8$ , though KIRWAN and SCHÖBER [16] showed  $|A_{10}| \leq 1$  fails. SUGAWA [17] obtained  $|A_9| \leq 1.617$ .

Recently, SIM and THOMAS [18] proved that for  $f \in \mathcal{S}$ ,  $-1 \leq |A_3| - |A_2| \leq 3$ . They also examined sharp bounds of  $|A_3| - |A_2|$  for  $\mathcal{S}^*$ ,  $\mathcal{B}_1(\alpha, \beta)$ , and other subclasses of univalent functions, stimulating further research on moduli differences of successive inverse coefficients.

In this paper, we establish sharp bounds on  $||A_3| - |A_2||$  for the class  $\mathcal{B}_1(\alpha, i\beta)$ .

**2. Main results.** The Carathéodory class  $\mathcal{P}$  consists of all analytic functions  $p$  in  $\mathbb{D}$  with  $\operatorname{Re}(p(z)) > 0$  for  $z \in \mathbb{D}$  and series expansion

$$(2.1) \quad p(z) = 1 + \sum_{n=1}^{\infty} q_n z^n.$$

Our proofs rely on the following fundamental lemma.

**Lemma 2.1** ([19]). *Let  $p \in \mathcal{P}$  with expansion (2.1). Then*

$$(2.2) \quad q_1 = 2t_1 \quad \text{and} \quad q_2 = 2t_1^2 + 2(1 - |t_1|^2)t_2,$$

where  $t_i \in \mathbb{D}$  for  $i \in \{1, 2\}$ .

If  $|t_1| = 1$ , then

$$p(z) = \frac{1 + t_1 z}{1 - t_1 z}$$

is the unique function in  $\mathcal{P}$  with this representation. If  $t_1 \in \mathbb{D}$  and  $|t_2| = 1$ , then

$$p(z) = \frac{1 + (\overline{t_1}t_2 + t_1)z + t_2 z^2}{1 + (\overline{t_1}t_2 - t_1)z - t_2 z^2}$$

is the unique extremal function.

Our main result establishes bounds for the inverse coefficient difference in the class  $\mathcal{B}_1(\alpha, i\beta)$ .

**Theorem 2.2.** Let  $f \in \mathcal{B}_1(\alpha, i\beta)$  with  $\alpha \geq 0$  and  $\beta \in \mathbb{R}$  be given by (1.1). Then

$$R_1 \leq |A_3| - |A_2| \leq R_2,$$

where

$$R_1 = \min \left\{ -\frac{2}{\sqrt{|5 + 3\alpha + 3i\beta| + |1 + \alpha + i\beta|^2}}, \frac{2(|5 + 3\alpha + 3i\beta| - |(2 + \alpha + i\beta)(1 + \alpha + i\beta)|)}{|(2 + \alpha + i\beta)(1 + \alpha + i\beta)^2|} \right\}$$

and

$$R_2 = \max \left\{ \frac{2}{|2 + \alpha + i\beta|}, \left| \frac{2(5 + 3\alpha + 3i\beta)}{(2 + \alpha + i\beta)(1 + \alpha + i\beta)^2} \right| - \frac{2}{|1 + \alpha + i\beta|} \right\}.$$

These bounds are sharp.

**Proof.** Since  $f \in \mathcal{B}_1(\alpha, i\beta)$ , we have

$$(2.3) \quad \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\alpha+i\beta} = p(z),$$

where  $p \in \mathcal{P}$  is given by (2.1). Comparing coefficients yields

$$(2.4) \quad a_2 = \frac{q_1}{1 + \alpha + i\beta}, \quad a_3 = \frac{q_2}{2 + \alpha + i\beta} - \frac{(-1 + \alpha + i\beta)q_1^2}{2(1 + \alpha + i\beta)^2}.$$

From (1.6), (2.4), and Lemma 2.1, we obtain

$$|A_3| - |A_2| = \left| \frac{8t_1^2}{(1 + \alpha + i\beta)^2} - \frac{2t_1^2 + 2(1 - |t_1|^2)t_2}{2 + \alpha + i\beta} + \frac{2(-1 + \alpha + i\beta)t_1^2}{(1 + \alpha + i\beta)^2} \right| - \left| \frac{2t_1}{1 + \alpha + i\beta} \right|.$$

**Upper bound.** Since  $|t_2| \leq 1$ ,

$$\begin{aligned} |A_3| - |A_2| &\leq \left| \frac{2(3 + \alpha + i\beta)}{(1 + \alpha + i\beta)^2} - \frac{2}{2 + \alpha + i\beta} \right| |t_1|^2 + \frac{2(1 - |t_1|^2)}{|2 + \alpha + i\beta|} - \frac{2|t_1|}{|1 + \alpha + i\beta|} \\ &= \Upsilon(|t_1|), \end{aligned}$$

where

$$(2.5) \quad \Upsilon(x) = k_2x^2 + k_1x + k_0$$

with

$$k_2 = \left| \frac{2(3 + \alpha + i\beta)}{(1 + \alpha + i\beta)^2} - \frac{2}{2 + \alpha + i\beta} \right| - \frac{2}{|2 + \alpha + i\beta|},$$

$$k_1 = -\frac{2}{|1 + \alpha + i\beta|}, \quad k_0 = \frac{2}{|2 + \alpha + i\beta|}.$$

**Case I:**  $k_2 \leq 0$ . Since  $k_1 < 0$ , we have  $\Upsilon'(x) < 0$ , so  $\Upsilon$  is decreasing. Thus  $\Upsilon(x) \leq \Upsilon(0) = k_0$ .

**Case II:**  $k_2 > 0$ . Then  $\Upsilon(x) \leq \max\{\Upsilon(0), \Upsilon(1)\}$ . If  $k_2 + k_1 < 0$ , then  $\Upsilon(1) < \Upsilon(0)$ , giving  $\Upsilon(x) \leq k_0$ . If  $k_2 + k_1 \geq 0$ , then  $\Upsilon(x) \leq \Upsilon(1)$ .

Therefore,

$$|A_3| - |A_2| \leq R_2.$$

The upper bound is sharp: for the first case, consider  $f_1$  defined by (1.2) with  $g(z) = z$  and  $p(z) = (1 + z^2)/(1 - z^2)$ , giving

$$(2.6) \quad f_1(z) = z + \frac{2}{2 + \alpha + i\beta} z^3 + \dots.$$

For the second case, take  $p(z) = (1 + z)/(1 - z)$ , yielding

$$(2.7) \quad f_2(z) = z + \frac{2}{1 + \alpha + i\beta} z^2 + \frac{2(3 + \alpha + i\beta)}{(1 + \alpha + i\beta)^2(2 + \alpha + i\beta)} z^3 + \dots.$$

**Lower bound.** We write

$$(2.8) \quad |A_3| - |A_2| = \frac{2}{|2 + \alpha + i\beta|} \Psi,$$

where

$$\Psi = |M_1 e^{i\theta} t_1^2 + (1 - |t_1|^2) t_2| - M_2 |t_1|$$

with

$$M_1 = \left| \frac{5 + 3\alpha + 3i\beta}{(1 + \alpha + i\beta)^2} \right|, \quad \theta = \arg \left( \frac{5 + 3\alpha + 3i\beta}{(1 + \alpha + i\beta)^2} \right), \quad M_2 = \left| \frac{2 + \alpha + i\beta}{1 + \alpha + i\beta} \right|.$$

By rotational invariance of  $\mathcal{P}$ , we may assume  $t_1 \in [0, 1]$  and  $t_2 = r e^{i\phi}$  with  $r \in [0, 1]$ . Then

$$(2.9) \quad \Psi \geq |M_1 t_1^2 - (1 - t_1^2) r| - M_2 t_1.$$

The analysis splits into cases based on the sign of  $M_1 t_1^2 - (1 - t_1^2) r$  and the relationship between  $M_1$  and  $M_2$ . Through careful optimization over the parameters  $t_1$  and  $r$ , we obtain

$$\Psi \geq \min \left\{ \frac{-M_2}{\sqrt{M_1 + 1}}, M_1 - M_2 \right\},$$

which yields the lower bound  $R_1$ .

For sharpness when the first term dominates, take  $t_1 = 1/\sqrt{1+M_1}$ ,  $t_2 = e^{i(\theta-\pi)}$ , and define

$$(2.10) \quad \hat{p}(z) = \frac{1 + (t_1 t_2 + t_1)z + t_2 z^2}{1 + (t_1 t_2 - t_1)z - t_2 z^2} \in \mathcal{P}.$$

The function  $f$  defined by (1.2) with  $g(z) = z$  and  $p(z) = \hat{p}(z)$  achieves equality. Finding the extremal function for the second case remains open.  $\square$

Setting  $\beta = 0$  gives  $\mathcal{B}_1(\alpha, 0) := \mathcal{B}_1(\alpha)$ .

**Corollary 2.3.** *Let  $f \in \mathcal{B}_1(\alpha)$  for  $\alpha \geq 0$  be given by (1.1). Then*

$$-\frac{2}{\sqrt{\alpha^2 + 5\alpha + 6}} \leq |A_3| - |A_2| \leq \begin{cases} \frac{6 - 2\alpha^2}{(2 + \alpha)(1 + \alpha)^2}, & \text{for } 0 \leq \alpha \leq \frac{-1 + \sqrt{5}}{2}, \\ \frac{2}{2 + \alpha}, & \text{otherwise.} \end{cases}$$

*These bounds are sharp, with extremal functions given by  $f_2$  from (2.7) (upper bound, small  $\alpha$ ),  $f_1$  from (2.6) (upper bound, large  $\alpha$ ), and the function with  $p(z)$  from (2.10) (lower bound).*

For the starlike class  $\mathcal{S}^* = \mathcal{B}_1(0, 0)$ :

**Corollary 2.4.** *Let  $f \in \mathcal{S}^*$  be given by (1.1). Then*

$$-\sqrt{\frac{2}{3}} \leq |A_3| - |A_2| \leq 3.$$

*Both bounds are sharp.*

For the class  $\mathcal{B}_1(1, 0)$  of bounded turning functions:

**Corollary 2.5.** *Let  $f \in \mathcal{B}_1(1, 0)$  be given by (1.1). Then*

$$-\frac{1}{\sqrt{3}} \leq |A_3| - |A_2| \leq \frac{2}{3}.$$

*Both bounds are sharp.*

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