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ON SOLVABILITY AND AVERAGE CODEGREE OF A FINITE GROUP

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Abstract

Let $\mathrm{Irr}_2(G)$ be the set of all complex irreducible characters χ such that their codegrees $\chi^c(1) = \dfrac{|G : \ker \chi|}{\chi(1)}$ are 1 or even, and let

$$\mathrm{acod}_2(G) = \frac{1}{|\mathrm{Irr}_2(G)|} \sum_{\chi \in \mathrm{Irr}_2(G)} \chi^c(1).$$

We show that, a group G with $\mathrm{acod}_2(G) < \dfrac{53}{4}$ is solvable.

Key words: simple group, irreducible character codegree, nonsolvable group

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1. Introduction. All groups under consideration are finite. Let G be a group and let $\mathrm{Irr}(G)$ denote the set of all complex irreducible characters of G. Isaacs, Loukaki and Moretó in [1] defined the average character degree of a finite group by $\mathrm{acd}(G) = \dfrac{1}{|\mathrm{Irr}(G)|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)$. The average degree contains fruitful structural information; see [1–7], for instance.

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Qian et al. in [8] defined the codegree of a complex irreducible character $\chi \in \mathrm{Irr}(G)$ of a group G as $\chi^c(1) = \dfrac{|G : \ker \chi|}{\chi(1)}$. Some scholars studied the group structure by using the codegree, for example, Ahanjideh in [9] showed the non-divisibility of groups by using character codegrees.

Lewis and Yan in [10] investigated the influence of codegree sum of a group on group structure and they showed that, if G is a nonsolvable group, then $\sum_{\chi \in \mathrm{Irr}(G)} \chi^c(1) \geq 68$ with equality if and only if $G \cong \mathrm{PSL}(2,5)$. Wang et al. defined the average codegree of a finite group as

$$\mathrm{acod}(G) = \frac{1}{|\mathrm{Irr}(G)|} \sum_{\chi \in \mathrm{Irr}(G)} \chi^c(1)$$

and showed the relation between the non-solvability and non-supersolvability of finite groups and the average codegrees; see [11, Theorems A and B].

As a generalization of $\mathrm{acod}(G)$, we consider only the character codegrees which are 1 or divisible by 2. So we let

$$\mathrm{Irr}_2(G) = \{\chi \in \mathrm{Irr}(G) | \chi^c(1) \text{ is 1 or even}\},$$

then define

$$\mathrm{acod}_2(G) = \frac{1}{|\mathrm{Irr}_2(G)|} \sum_{\chi \in \mathrm{Irr}_2(G)} \chi^c(1)$$

as the average codegree of a group G.

First we show the following result corresponding to that of [11] and [10].

Theorem 1.1. *Let G be a nonsolvable group. Then $\mathrm{acod}_2(G) \geq \dfrac{53}{4}$ with equality if and only if $G \cong A_5$.*

Observe that $\mathrm{acod}_2(A_5) = \dfrac{1}{4} \cdot \left[1 + \dfrac{60}{3} + \dfrac{60}{3} + \dfrac{60}{5}\right] = \dfrac{53}{4}$, so the lower bound is best possible.

Note that there exist solvable groups G with $\mathrm{acod}_2(G) > \dfrac{53}{4}$, for instance, let G be SmallGroup(60,3), then $\mathrm{acod}_2(G) = \dfrac{227}{17} \approx 13.35 > 13.25 = \dfrac{53}{4}$ by [12]. So in Theorem 1.1, the condition "a nonsolvable group" is necessary.

Duality to Theorem 1.1, we also have the following result which is correspondent to Theorem 1.2 of [13]: a finite group G with $\mathrm{acd}_2(G) < 5/2$ is solvable.

Theorem 1.2. *A finite group G with $\mathrm{acod}_2(G) < \dfrac{53}{4}$ is solvable.*

Lewis and Yan showed that there exist the series of nonsolvable groups $\{G\}$ and $\{H\}$ such that $\mathrm{fcod}(G) \to \infty$ when $|G| \to \infty$, $\mathrm{fcod}(H) \to \infty$ when $|H| \to \infty$; see Proposition 2.3(2) of [10]. Comparing to this result of [10], we also have the following result.

Proposition 1.3. *Let G be a non-abelian simple group. Then* $\mathrm{acod}_2(G)$ *is closing to* ∞ *when* $|G|$ *is closing to* ∞.

We conjecture that, Proposition 1.3 is true for nonsolvable groups.

The structure of this short paper is as follows. In Section 2, we mainly give the invariant $\mathrm{acod}_2(S)$ for $S \in \{\mathrm{PSL}(2,q), \mathrm{Sz}(q)\}$; in Section 3, we show the proofs of Theorems 1.1, 1.2 and Proposition 1.3. The other notations and notions are standard, please see [14] for instance.

2. Some needed results. In this section, we mainly give some basic results which will be used to prove the main theorem.

The following two results can control group structures.

Lemma 2.1. *Let* $G = \mathrm{PSL}(2,q)$ *with* $q = p^n$.

(1) *If* $p = 2$, *then* $\mathrm{acod}_2(G) = \dfrac{q^3 - q^2 + q + 1}{q}$;

(2) *If* $p > 2$, *then*

$$\mathrm{acod}_2(G) = \begin{cases} \dfrac{(q+1)(q^2+17q-4)+8}{2(q+13)}, & \text{if } q \equiv -1 \pmod 4; \\ \dfrac{(q-1)(q^2+15q+4)+8}{2(q+11)}, & \text{if } q \equiv 1 \pmod 4. \end{cases}$$

Let p be a prime divisor of an integer $n > 1$. Then denote the p-part of n by n_p, that is, for certain s, $p^s \big| n$ but $p^{s+1} \nmid n$. Write $n_p = p^s$.

Proof. We see that $\chi^c(1) = \dfrac{|G : \ker\chi|}{\chi(1)}$ for a character $\chi \in \mathrm{Irr}(G)$, and that $\ker\chi = \{1\}$ for a non-principal character $\chi \in \mathrm{Irr}(G)$. Note that $|G| = \dfrac{q(q^2-1)}{\gcd(2,q-1)}$, so $\chi \notin \mathrm{Irr}_2(G)$ if and only if $|G|_2 = \chi(1)_2$.

(1) If $p = 2$, then $|G|_2 = q$, and so by p. 403 of [15], we have that

$$\mathrm{acod}_2(G) = \frac{1 + \frac{q(q^2-1)}{q+1} \cdot (\frac{q}{2} - 1) + \frac{q(q^2-1)}{q-1} \cdot \frac{q}{2}}{1 + (\frac{q}{2} - 1) + \frac{q}{2}} = \frac{q^3 - q^2 + q + 1}{q}.$$

(2) If $p \geq 3$, then $\gcd(q-1, q+1) = 2$ and we see that, if $q \equiv -1 \pmod 4$, then $\dfrac{q-1}{2}$ is odd, and $(q+1)_2 = |G|_2$; if $q \equiv 1 \pmod 4$, then $\dfrac{q+1}{2}$ is odd, and $(q-1)_2 = |G|_2$. It follows from p. 402 of [15], that

$$\mathrm{acod}_2(G) = \begin{cases} \dfrac{1 + \frac{q(q^2-1)}{2} \cdot \left(\frac{1}{q} + \frac{2}{\frac{q-1}{2}} + \frac{\frac{q-3}{4}}{q-1} \right)}{1 + 1 + 2 + \frac{q-3}{4}}, & \text{if } q \equiv -1 \pmod 4; \\ \dfrac{1 + \frac{q(q^2-1)}{2} \cdot \left(\frac{1}{q} + \frac{2}{\frac{q+1}{2}} + \frac{\frac{q-5}{4}}{q+1} \right)}{1 + 1 + 2 + \frac{q-5}{4}}, & \text{if } q \equiv 1 \pmod 4, \end{cases}$$

$$= \begin{cases} \dfrac{(q+1)(q^2+17q-4)+8}{2(q+13)}, & \text{if } q \equiv -1 \pmod 4; \\ \dfrac{(q-1)(q^2+15q+4)+8}{2(q+11)}, & \text{if } q \equiv 1 \pmod 4. \end{cases}$$

This completes the proof. □

Lemma 2.2. *Let $G = \mathrm{Sz}(q)$ with $q = 2^{2m+1}$. Then*

$$\mathrm{acod}_2(G) = \frac{1 + (2\sqrt{2q} + q)q(q^2+1)}{q+2}.$$

Proof. We see that $|G| = q^2(q-1)(q^2+1)$ and for a non-principal irreducible character $\chi \in \mathrm{Irr}(G)$, $\ker\chi = \{1\}$. So by [16], we have

$$\begin{aligned} &\mathrm{acod}_2(G) \\ &= \frac{1 + q^2(q-1)(q^2+1)\left(\frac{2}{\frac{1}{2}(q-1)\sqrt{2q}} + \frac{\frac{1}{4}(q+\sqrt{2q})}{(q-1)(q-\sqrt{2q}+1)} + \frac{\frac{1}{2}(q-2)}{q^2+1} + \frac{\frac{1}{4}(q-\sqrt{2q})}{(q-1)(q+\sqrt{2q}+1)}\right)}{1 + 2 + \frac{1}{4}(q+\sqrt{2q}) + \frac{1}{2}(q-2) + \frac{1}{4}(q-\sqrt{2q})} \\ &= \frac{1 + (2\sqrt{2q}+q)q(q^2+1)}{q+2}. \end{aligned}$$

The lemma is complete. □

Comparing to Lemma 2.2 of [11], we show the following result.

Lemma 2.3. *Let $S_1 \times S_2$ be the direct product of two groups S_1 and S_2, where S_1 and S_2 are non-abelian simple groups. Let $\chi_i \in \mathrm{Irr}(S_1)$ and $\eta_j \in \mathrm{Irr}(S_2)$. Then*

(1) $(\chi_i \times \eta_j)^c(1) = \chi_i^c(1)\eta_j^c(1)$;

(2) $\mathrm{acod}_2(S_1 \times S_2) > \mathrm{acod}_2(S_1)\mathrm{acod}_2(S_2)$.

Let $\mathrm{Irr}_{2'}(G) = \mathrm{Irr}(G) - \mathrm{Irr}_2(G)$ for a group G. Let p be a prime divisor of a positive integer n, we denote the p-part of n by n_p, that is, $n_p = p^s \mid n$ but $p^{s+1} \nmid n$.

Proof. (1) As S_1 and S_2 are non-abelian simple groups, 2 divides both $|S_1|$ and $|S_2|$, so the sets $\mathrm{Irr}_2(S_1) \setminus \{1_{S_1}\}$ and $\mathrm{Irr}_2(S_2) \setminus \{1_{S_2}\}$ are non-empty. Set $\chi_i \in \mathrm{Irr}(S_1)$ and $\eta_j \in \mathrm{Irr}(S_2)$. Then from Theorem 4.21 of [17], $\chi_i \times \eta_j \in \mathrm{Irr}(S_1 \times S_2)$. Observe that, for the principals 1_{S_1} and 1_{S_2}, $\ker(1_{S_1} \times 1_{S_2}) = S_1 \times S_2$ and for the non-principal characters $\chi_i \in \mathrm{Irr}(S_1)$ and $\eta_j \in \mathrm{Irr}(S_2)$, $\ker(\chi_i \times 1_{S_2}) = S_2$, $\ker(1_{S_1} \times \eta_j) = S_1$ and $\ker(\chi_i \times \eta_j) = \{1\}$. Now

$$\begin{aligned} (\chi_i \times \eta_j)^c(1) &= \frac{|S_1||S_2|}{|\ker(\chi_i \times \eta_j)|\chi_i(1)\eta_j(1)} \\ &= \begin{cases} 1_{S_1}^c(1)1_{S_2}^c(1), & \chi_i = 1_{S_1}, \eta_j = 1_{S_2}, \\ \chi_i^c(1)1_{S_2}^c(1), & \chi_i \neq 1_{S_1}, \eta_j = 1_{S_2} \\ 1_{S_1}^c(1)\eta_j^c(1), & \chi_i = 1_{S_1}, \eta_j \neq 1_{S_2} \\ \chi_i^c(1)\eta_j^c(1), & \chi_i \neq 1_{S_1}, \eta_j \neq 1_{S_2}. \end{cases} \end{aligned}$$

It follows that, $(\chi_i \times \eta_j)^c(1) = \chi_i^c(1)\eta_j^c(1)$.

(2) Note that, $\chi_i \times \eta_j \in \mathrm{Irr}_{2'}(S)$ if and only if one of the following holds:

(2a) $\chi_i(1)_2 = |S_1|_2$ and $\eta_j(1)_2 = |S_2|_2$;

(2b) $\chi_i(1)_2 = |S_1|_2$ and $|\ker(\chi_i \times 1_{S_2})|_2 = |S_2|_2$;

(2c) $\eta_j(1)_2 = |S_2|_2$ and $|\ker(1_{S_1} \times \eta_j)|_2 = |S_1|_2$.

We know that, if $(\chi_i, \eta_j) \in \mathrm{Irr}_{2'}(S_1) \times \mathrm{Irr}_{2'}(S_2)$, we have $\chi_i \times \eta_j \in \mathrm{Irr}_{2'}(S_1 \times S_2)$. Thus

$$|\mathrm{Irr}_{2'}(S)| = |\mathrm{Irr}_{2'}(S_1 \times S_2)| > |\mathrm{Irr}_{2'}(S_1) \times \mathrm{Irr}_{2'}(S_2)| = |\mathrm{Irr}_{2'}(S_1)||\mathrm{Irr}_{2'}(S_2)|.$$

It follows that, $|\mathrm{Irr}_2(S)| = |\mathrm{Irr}_2(S_1 \times S_2)| < |\mathrm{Irr}_2(S_1)||\mathrm{Irr}_2(S_2)|$ and so,

$$\begin{aligned}
\mathrm{acod}_2(S) &= \frac{1}{|\mathrm{Irr}_2(S)|}\sum_{i,j}(\chi_i \times \eta_j)^c(1) \\
&= \frac{1}{|\mathrm{Irr}_2(S)|}\sum_{i,j}\chi_i^c(1)\eta_j^c(1) \\
&> \frac{1}{|\mathrm{Irr}_2(S_1)||\mathrm{Irr}_2(S_2)|}\sum_{i,j}\chi_i^c(1)\eta_j^c(1) \\
&= \mathrm{acod}_2(S_1)\mathrm{acod}_2(S_2).
\end{aligned}$$

In particular, $\mathrm{acod}_2(S_1 \times S_1) > \mathrm{acod}_2(S_1)^2$. □

Example 2.4. It is easy to get from [12] that $\mathrm{acod}_2(A_5 \times A_5) = \frac{4369}{22} \approx 198.59$ and $\mathrm{acod}_2(A_5)^2 = \left(\frac{53}{4}\right)^2 \approx 175.56$, and so, $\mathrm{acod}_2(A_5 \times A_5) > \mathrm{acod}_2(A_5)^2$.

Remark 2.5. Let p be a prime, and let $\mathrm{Irr}_p(G) = \{\chi \in \mathrm{Irr}(G) | \chi^c(1) = 1 \text{ or } p \mid \chi^c(1)\}$. Write $\mathrm{acod}_p(G) = \frac{1}{|\mathrm{Irr}_p(G)|}\sum_{\chi_i \in \mathrm{Irr}_p(G)}\chi_i^c(1)$. We similarly as the computation of Lemma 2.3, have $\mathrm{acod}_p(S_1 \times S_2) > \mathrm{acod}_p(S_1)\mathrm{acod}_p(S_2)$.

Let $V(\chi) = \{\chi(x) : x \in G\}$ be the set of the values assumed by χ.

Lemma 2.6. *Let $V(\chi) = \{\chi(1), 0, a_1, \cdots, a_n\}$ where the a_i's are algebraic conjugate. Assume G is a nonsolvable group, then G has n irreducible characters which are faithful, in particular, at least one of such characters belongs to* $\mathrm{Irr}_2(G)$.

Proof. Let G be a group of Lie type or the symmetric group or a sporadic group, we can find a faithful character by [18], [19], and [14], respectively. So by the Theorem of [20], we get the desired result. □

3. Proofs of Theorem 1.1 and Proposition 1.3. Let G be a finite group with $\mathrm{acod}_2(G) < \frac{53}{4}$. If G is abelian, then we are done, so we assume that G is non-abelian. First we deal with the case when N is non-abelian.

Proposition 3.1. *Let G be a finite group and let N be a non-abelian minimal normal subgroup of G, then* $\mathrm{acod}_2(G) \geq \frac{53}{4}$.

Proof. Assume, to the contrary, that the result is false and that G is a minimal contraexample. Thus $\mathrm{acod}_2(G) < \dfrac{53}{4}$. Since N is nonabelian, G is nonsolvable. It follows from Lemma 2.6 that there is a character $\chi \in \mathrm{Irr}_2(G)$ such that $\chi^c(1) < \left[\dfrac{53}{4}\right] = 13 < 14$. Notice that $\chi^c(1) > 1 + \chi(1)$, so by Lemma 2.6, $|G| < 182$. Thus we have from Lemma 2.3 and [14, p. 239], that N is isomorphic to $\mathrm{PSL}(2,q)$ where q is equal to 5 or 7, and so, G is possibly isomorphic to

$$\mathrm{PSL}(2,5), \mathrm{PSL}(2,5).2, \mathrm{PSL}(2,5) \times C_3, \mathrm{PSL}(2,5) \times C_2 \ \text{ or } \ \mathrm{PSL}(2,7).$$

If $G \cong \mathrm{PSL}_2(q)$ with $q = 5, 7$, then by Proposition 2.1, we have

$$\mathrm{acod}_2(\mathrm{PSL}(2,q)) = \begin{cases} \dfrac{53}{4} & q = 5, \\ 33 & q = 7, \end{cases}$$

a contradiction.

If $G \cong \mathrm{PSL}_2(5).2$, we obtain $\mathrm{acod}_2(\mathrm{PSL}(2,5).2) = \dfrac{131}{7} \approx 18.7$, a contradiction.

If $G \cong \mathrm{PSL}(2,5) \times C_3$ or $G \cong \mathrm{PSL}(2,5) \times C_2$, then from [12] we obtain

$$\mathrm{acod}_2(\mathrm{PSL}(2,5) \times C_3) = \frac{365}{10} = 36.5 > \frac{53}{4} = 13.25,$$
$$\mathrm{acod}_2(\mathrm{PSL}(2,5) \times C_2) = \frac{189}{9} = 21 > 13.25,$$

a contradiction.

This completes the proof. □

Proposition 3.2. *Let G be a nonsolvable with an abelian minimal normal subgroup $N > 1$, then* $\mathrm{acod}_2(G) \geq \dfrac{53}{4}$.

Proof. Assume, to the contrary, that $\mathrm{acod}_2(G) < \dfrac{53}{4}$ and that G is a contraexample with minimal order $|G|$.

By hypothesis, G is nonsolvable, and so, by Lemma 2.6, G has a faithful irreducible character belonging to $\mathrm{Irr}_2(G)$. It follows that $|G| < 182$. Thus G has a section isomorphic to $\mathrm{PSL}_2(q)$ with $q = 5$ or 7; see p. 239 of [14]. We derive from the order of G that G is isomorphic to $\mathrm{SL}(2,5)$ (observe that the order $|\mathrm{SL}(2,7)|$ of $\mathrm{SL}(2,7)$ is $336 > 182$); see p. xvi of [14]. But in this case, $\mathrm{acod}_2(\mathrm{SL}(2,5)) = \dfrac{235}{8} > \dfrac{106}{8} = \dfrac{53}{4}$, a contradiction. □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Let N be a minimal normal subgroup of G. Then by Propositions 3.1 and 3.2, we have that $\mathrm{acod}_2(G) \geq \dfrac{53}{4}$. Then assume that

G is neither an almost simple group nor a semisimple group. Thus from the nonsolvability of G, $G \cong N$, and so we must show that $G \cong A_5$ if and only if $\mathrm{acod}_2(G) \leq \frac{53}{4}$. If $G \cong A_5$, then by p. 3 of [14], $\mathrm{acod}_2(G) = \frac{53}{4}$. Hence we need to show that if $\mathrm{acod}_2(G) \leq \frac{53}{4}$, then G is isomorphic to A_5. Note from Lemma 2.6 that $|G| < 182$, so G is isomorphic to $\mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,7)$. By p. 2–3 of [14], we similarly have that $G \cong \mathrm{PSL}(2,5)$. □

Proof of Theorem 1.2. Assume that the result is not true, then G is nonsolvable. By Theorem 1.1, we have that $\mathrm{acod}_2(G) \geq \frac{53}{4}$, a contradiction to the assumption. □

Now we will give the proof of Proposition 1.3.

Proof of Proposition 1.3. We see that when $|G| \to \infty$, $|\mathrm{Irr}_2(G)|$ is closing to ∞, and so for $\chi_i \in \mathrm{Irr}_2(G)$, we obtain

$$\mathrm{acod}_2(G) = \frac{\chi_1^c(1) + \chi_2^c(1) + \cdots + \chi^c_{|\mathrm{Irr}_2(G)|}(1)}{|\mathrm{Irr}_2(G)|} \to \max_{i \in [1, |\mathrm{Irr}_2(G)|]} \{\chi_i^c(1)\} \text{ if } |G| \to \infty.$$

As for a non-principal character $\eta \in \mathrm{Irr}_2(G)$, we get $\ker \eta = 1$. Choose a character from $\mathrm{Irr}_2(G)$, say χ, such that $\chi(1) > 1$ is minimal, then

$$\chi^c(1) = \frac{|G : \ker \chi|}{\chi(1)} = \frac{|G|}{\chi(1)}$$

is maximal, and when $|G| \to \infty$, $\chi^c(1) \to \infty$. Thus $\mathrm{acd}_2(G) \to \infty$ if $|G| \to \infty$. □

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