

ON IMPROVED ANKENY AND RIVLIN TYPE POLYNOMIAL INEQUALITY

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Abstract

For a polynomial $p(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$ of degree n , we denote

$$M(p, R) := \max_{|z|=R \geq 0} |p(z)|$$

and $M(p, 1) := \|p\|$. A well-known result of Ankeny and Rivlin [Pacific J. Math., 5(2)(1955), 849–862] states that if $p(z) \neq 0$ in $|z| < 1$ and $R \geq 1$, then

$$M(p, R) \leq \left(\frac{R^n + 1}{2} \right) \|p\|.$$

This inequality has been sharpened by Dalal and Govil [Anal. Theory Appl., 36(2)(2020), 225–234], who proved that for $p(z) \neq 0$ in $|z| < 1$, $R \geq 1$ and any N , $1 \leq N \leq n$,

$$M(p, R) \leq \frac{R^n + 1}{2} \|p\| - \frac{n}{2} \|p\| \left(1 - \frac{2|a_n|}{\|p\|} \right) \int_1^R \frac{(r-1)r^{N-1}}{r + \frac{2|a_n|}{\|p\|}} dr.$$

In this paper, we sharpen the above inequality of Dalal and Govil.

Key words: polynomial, inequality, maximum modulus

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1. Introduction. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree n , and for $R \geq 0$, let

$$M(p, R) := \max_{|z|=R} |p(z)|.$$

We will denote $M(p, 1)$ by $\|p\|$. Then, by the maximum modulus principle, $M(p, R)$ is a strictly increasing function of R and is defined for $0 \leq R < \infty$. Also, it is a simple deduction from the maximum modulus principle (see [1, pp. 158, Problem 269]) that for $R \geq 1$,

$$(1.1) \quad M(p, R) \leq R^n \|p\|.$$

The result is best possible and equality holds if and only if $p(z) = \lambda z^n$, λ being a non-zero complex number.

Inequality (1.1) gives an estimate of $|p(z)|$ on $|z| = 1$ as an upper bound. It is a natural desire to seek improved bounds of (1.1) under appropriate assumptions. In many attempts, ANKENY and RIVLIN [2] were able to improve the bound of (1.1) by restricting all the zeros of the polynomial in $|z| \geq 1$. In fact, they proved the following result.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$(1.2) \quad M(p, R) \leq \left(\frac{R^n + 1}{2} \right) \|p\|, \quad R \geq 1.$$

The above inequality is best possible and equality holds for polynomials having all their zeros on the unit circle.

Several related papers and research monographs have been studied by numerous authors (see, for example, [3–8]).

A refinement of inequality (1.2) was given by DALAL and GOVIL [9], who proved the following.

Theorem 1.2. *If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$ and any N , $1 \leq N \leq n$,*

$$(1.3) \quad M(p, R) \leq \frac{R^n + 1}{2} \|p\| - \frac{n}{2} \|p\| \left(1 - \frac{2|a_n|}{\|p\|} \right) \int_1^R \frac{(r-1)r^{N-1}}{r + \frac{2|a_n|}{\|p\|}} dr.$$

The quest to strengthen the existing results has motivated us to build up this work. We also discuss the consequences of our results in the form of a corollary and remarks. Moreover, we also give a computational example comparing the bound of our result with those of the previously known results. But before we jump into the proof of the main result and its consequences, we need to discuss some lemmas.

2. Lemmas. The first lemma is due to GOVIL [10].

Lemma 2.1. *If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n , then for $R \geq 1$,*

$$M(p, R) \leq R^n \left(1 - \frac{(\|p\| - |a_n|)(R^2 - 1)}{R^2\|p\| + |a_n| + \frac{R|a_{n-1}|\|p\|}{\|p\| - |a_n|}} \right) \|p\|.$$

The above inequality is best possible and equality holds for $p(z) = \lambda z^n$, λ being a non-zero complex number.

Lemma 2.2. *If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n and $r \geq 1$, then*

$$(2.1) \quad f(x) = \left(1 - \frac{(x - n|a_n|)(r^2 - 1)}{r^2x + n|a_n| + \frac{x(n-1)|a_{n-1}|r}{x - n|a_n|}} \right) x$$

is an increasing function of x for $x \geq n|a_n|$.

Proof. Putting $a = n|a_n|$ and $b = (n-1)|a_{n-1}|r$ in (2.1), we have

$$f(x) = \left(1 - \frac{(x-a)^2(r^2-1)}{\{(r^2x+a)(x-a) + xbr\}^2} \right) x.$$

When $x = a$, the result is obvious.

For the case when $x > a$, we have the first derivative of $f(x)$ with respect to x and we simplify to get

$$f'(x) = \frac{A(a, b, r, x)}{\left(r^2x + a + \frac{xbr}{x-a} \right)^2} \geq 0$$

since

$$A(a, b, r, x) = \left(\frac{xbr}{x-a} \right)^2 + \frac{2xabr}{x-a} + r^2(a^2 + x^2) + 2ax + \frac{2x^2br}{x-a} \geq 0.$$

This ends the proof of Lemma 2.2. □

Lemma 2.3. *If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then*

$$(2.2) \quad \|p'\| \leq \frac{n}{1 + s_\mu} \|p\|,$$

where, here and throughout this paper,

$$s_\mu = k^{\mu+1} \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|} k^{\mu-1} + 1}{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|} k^{\mu+1} + 1} \right\}.$$

This inequality is sharp, and equality holds for the polynomial

$$p(z) = (z^m + k^m)^{\frac{n}{m}},$$

where n is a multiple of m .

This inequality is due to QAZI [11, Lemma 1].

Lemma 2.4. *The function $g(N)$ given by*

$$g(N) = \frac{(r^2 - 1)r^{N-1}}{r^2 + ar + b}$$

is a nonnegative and increasing function of N for $N \geq 1$, $r \geq 1$ and some real a and b .

Proof. The proof of the lemma simply follows by the first derivative test. $\square$

3. Main result. We prove the following generalized sharpened form of Theorem 1.2 due to Dalal and Govil [9] under some conditions which, in turn, sharpens Theorem 1.1 due to Ankeny and Rivlin [2].

Theorem 3.1. *If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n and $p(z) \neq 0$ for $|z| < k$, $k \geq 1$, then for $R \geq 1$ and any N , $1 \leq N \leq n$,*

$$(3.1) \quad M(p, R) \leq \frac{R^n + s_\mu}{1 + s_\mu} \|p\| - \frac{n}{1 + s_\mu} \|p\| \left(1 - \frac{|a_n|(1 + s_\mu)}{\|p\|} \right) \\ \times \int_1^R \frac{r^{N-1}(r^2 - 1)}{r^2 + \frac{(n-1)|a_{n-1}|}{\frac{n}{1+s_\mu}\|p\| - n|a_n|} r + \frac{|a_n|(1+s_\mu)}{\|p\|}} dr.$$

Inequality (3.1) is best possible and equality holds for $p(z) = z^n + k^n$.

Proof. If $n = 0$, then the polynomial $p(z)$ is constant, so $M(p, R) = \|p\|$, for all R , and the theorem holds trivially. Since $p(z)$ is of degree $n \geq 1$, hence for each θ , where $0 \leq \theta < 2\pi$, we have

$$(3.2) \quad \left| p(Re^{i\theta}) - p(e^{i\theta}) \right| \leq \int_1^R \left| p'(re^{i\theta}) \right| dr \\ \leq \int_1^R \left(1 - \frac{(\|p'\| - n|a_n|)(r^2 - 1)}{\|p'\| r^2 + n|a_n| + \frac{\|p'\| r(n-1)|a_{n-1}|}{(\|p'\| - n|a_n|)}} \right) \|p'\| r^{n-1} dr$$

by applying Lemma 2.1 to the polynomial $p'(z)$, which is of degree $n - 1$.

Now, by Lemma 2.2, the integrand in (3.2) is an increasing function of $\|p'\|$, therefore on applying Lemma 2.3 to inequality (3.2), we have for $0 \leq \theta < 2\pi$,

$$(3.3) \quad \left| p(Re^{i\theta}) - p(e^{i\theta}) \right| \leq \frac{n}{1 + s_\mu} \|p\| \int_1^R r^{n-1} dr - \frac{n}{1 + s_\mu} \|p\| \left(\frac{n}{1 + s_\mu} \|p\| - n|a_n| \right)$$

$$\begin{aligned} & \times \int_1^R \frac{r^{n-1}(r^2 - 1)}{\frac{n}{1+s_\mu} \|p\| r^2 + n|a_n| + \frac{\frac{n}{1+s_\mu} \|p\| (n-1) |a_{n-1}|}{\frac{n}{1+s_\mu} \|p\| - n|a_n|} r} dr \\ &= \frac{R^n - 1}{1 + s_\mu} \|p\| - \left(\frac{n}{1 + s_\mu} \|p\| - n|a_n| \right) \times \int_1^R \frac{r^{n-1}(r^2 - 1)}{r^2 + \frac{(n-1)|a_{n-1}|}{\frac{n}{1+s_\mu} \|p\| - n|a_n|} r + \frac{|a_n|(1+s_\mu)}{\|p\|}} dr. \end{aligned}$$

By Lemma 2.4 for every N where $1 \leq N \leq n$, we have

$$\int_1^R \frac{(r^2 - 1)r^{N-1}}{r^2 + ar + b} dr \leq \int_1^R \frac{(r^2 - 1)r^{n-1}}{r^2 + ar + b} dr.$$

If we use the above inequality in (3.3), we get for every N , where $1 \leq N \leq n$,

$$\begin{aligned} (3.4) \quad \left| p(Re^{i\theta}) - p(e^{i\theta}) \right| &\leq \frac{R^n - 1}{1 + s_\mu} \|p\| - \left(\frac{n}{1 + s_\mu} \|p\| - n|a_n| \right) \\ &\quad \times \int_1^R \frac{r^{N-1}(r^2 - 1)}{r^2 + \frac{(n-1)|a_{n-1}|}{\frac{n}{1+s_\mu} \|p\| - n|a_n|} r + \frac{|a_n|(1+s_\mu)}{\|p\|}} dr, \end{aligned}$$

which clearly gives for every N , where $1 \leq N \leq n$,

$$\begin{aligned} (3.5) \quad \left| p(Re^{i\theta}) \right| &\leq \frac{R^n + s_\mu}{1 + s_\mu} \|p\| - \left(\frac{n}{1 + s_\mu} \|p\| - n|a_n| \right) \\ &\quad \times \int_1^R \frac{r^{N-1}(r^2 - 1)}{r^2 + \frac{(n-1)|a_{n-1}|}{\frac{n}{1+s_\mu} \|p\| - n|a_n|} r + \frac{|a_n|(1+s_\mu)}{\|p\|}} dr, \end{aligned}$$

which is equivalent to (3.1) and hence concludes the proof of Theorem 3.1. $\square$

Remark 3.2. Integral

$$I(N) = \int_1^R \frac{(r^2 - 1)r^{N-1}}{r^2 + sr + t} dr$$

can be expressed as a recurrence relation for some real s and t as shown below.

We define the function

$$f(N) = \int_1^R \frac{r^N}{r^2 + sr + t} dr,$$

which is defined for $N \geq 0$.

Then, we have for $N \geq 1$

$$f(N + 1) + sf(N) + f(N - 1) = \int_1^R r^{N-1} dr = \frac{R^N - 1}{N} = g(N).$$

Therefore, for $N \geq 1$

$$f(N+1) = g(N) - sf(N) - tf(N-1),$$

from which, it is easy to see that

$$(3.6) \quad I(N) = f(N+1) - f(N-1) \text{ for } N \geq 1.$$

Putting $N = n$ in (3.1), we obtain the following result.

Corollary 3.3. *If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n and $p(z) \neq 0$ for $|z| < k$, $k \geq 1$, then for $R \geq 1$,*

$$(3.7) \quad M(p, R) \leq \frac{R^n + s_\mu}{1 + s_\mu} \|p\| - \frac{n}{1 + s_\mu} \|p\| \left(1 - \frac{|a_n|(1 + s_\mu)}{\|p\|} \right) \\ \times \int_1^R \frac{r^{n-1}(r^2 - 1)}{r^2 + \frac{(n-1)|a_{n-1}|}{\frac{n}{1+s_\mu}\|p\| - n|a_n|}r + \frac{|a_n|(1+s_\mu)}{\|p\|}} dr.$$

Inequality (3.7) is best possible and equality holds for $p(z) = z^n + k^n$.

Remark 3.4. Due to Lemma 2.4, we can claim that Corollary 3.3 is the most improved version of Theorem 3.1.

Remark 3.5. As is well known, for any polynomial $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$, we have $|a_n| \leq \|p\|$. For $k = 1$, inequality (3.1) clearly sharpens inequality (1.1). Also, for $\mu = 1$, $k = 1$, Theorem 3.1 refines Theorem 1.2 as it is easy to verify that if

$$(3.8) \quad \|p\|^2 \geq 2 \left(1 - \frac{1}{n} \right) |a_{n-1}| \|p\| + 4|a_n|^2,$$

then for $1 \leq r \leq R$,

$$\frac{(r-1)(r^N-1)}{r + \frac{2|a_n|}{\|p\|}} \leq \frac{r^{N-1}(r^2-1)}{r^2 + \frac{(n-1)|a_{n-1}|}{\frac{n}{2}\|p\| - n|a_n|}r + \frac{|a_n|^2}{\|p\|}}.$$

4. Computation. For computation purposes, this section uses Wolfram Alpha for integrals. Now, we consider the polynomial $p(z) = (z+3)^2$, which also satisfies condition (3.8). Then $p(z) \neq 0$ for $|z| < 1$ and $\|p\| = 16$. If we take $R = 3$, then $M(p, 3) = 36$, and as it is easy to see that by using Theorem 1.1, we have

$$M(p, 3) \leq 80,$$

while by using Theorem 1.2 for $N = n$, we get

$$M(p, 3) \leq 80 - 14 \left\{ \frac{7}{4} + \frac{9}{32} \log \frac{5}{3} \right\} \approx 54.63.$$

Meanwhile, if we use Corollary 3.3 and (3.6) of Remark 3.2 for $s = 3/17$ and $t = 1/17$, we get

$$M(p, 3) \leq 80 - 14(3.62899 - 0.9743) \approx 42.83,$$

which is smaller than the bounds obtained by using Theorems 1.1 and 1.2. Thus, in general, Theorem 3.1 gives an improvement over Theorem 1.1 due to Ankeny and Rivlin [2] and Theorem 1.2 due to Dalal and Govil [9]. In particular, the bound obtained by Corollary 3.3 improves over those of Theorems 1.1 and 1.2 by about 46.46% and 21.6%, respectively. In fact, it is easy to see that by making appropriate choices of the polynomial $p(z)$, and the parameters N and R , this improvement can be made bigger and bigger.

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