

ON THE PROBABILITY DISTRIBUTION OF RELAXATION
LENGTH IN A PARAMETRIC ONE-DIMENSIONAL
TROPICAL SANDPILE MODEL

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Abstract

A special case of the one-dimensional tropical sandpile model is introduced, incorporating a parameter that affects the deformation of the hypersurface. We focus on study of relaxation behaviour towards finite stationary points. Particularly, for two stationary points randomly taken from $[0, 1]$ we derived probability distribution function of relaxation length. Numerical experiments for uniformly distributed stationary points are given and coincide with the theoretical inferences.

Key words: sandpile model, tropical geometry, self-organized criticality, relaxation, power law **2020 Mathematics Subject Classification:** 14T90, 60D05, 65C05

Introduction. BAK, TANG and WIESENFELD introduced in [1] a sandpile model that can reach a critical state with a power-law distribution of avalanches and $1/f$ noise in the mass-fluctuation power spectrum. The sandpile model is a cellular automaton that has especially simple rule of updating scheme and exhibits behaviour of self-organized criticality (SOC).

Under the study of complexity in nature, SOC was discovered in statistical physics and describes a common mechanism of how small perturbation of dynamical systems under a certain criticality state leads to power law for temporal, spatial or other types of fluctuations. In nature there are a lot of examples of

such systems, for instance earthquakes are part of the relaxation mechanism of the earth's crust, where Omori's law describes hyperbolic decaying of aftershocks frequency with time, while Gutenberg–Richter law states that the distribution of earthquake numbers above a certain magnitude level m follows power law [2].

The question of whether SOC is a fundamental property of dynamical systems is opened, for example in [3] authors report that brain signals reflect $1/f$ scaling for some frequency bands, but in the same time neuronal activity is not consistent with critical states. However, there remains interest in research output generated from the SOC hypothesis.

Tropical sandpile (TS) model with SOC behaviour was presented recently in [4] as an independent from applications geometric object. The description of this object starts from the definition of tropical series F acting on the compact convex subset $\Omega \subset \mathbb{R}^n$:

$$(1) \quad \begin{aligned} F(\mathbf{x}) &= \bigoplus_{\mathbf{v} \in \mathbb{Z}^n} (c_{\mathbf{v}} \odot \mathbf{x}^{\odot \mathbf{v}}), \quad \mathbf{x}^{\odot \mathbf{v}} = x_1^{\odot v_1} \odot x_2^{\odot v_2} \odot \cdots \odot x_n^{\odot v_n}, \\ F: \Omega &\rightarrow [0, \infty), \quad c_{\mathbf{v}} \in \mathbb{R}, \\ F(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega, \end{aligned}$$

where $c_{\mathbf{v}}$ are the so-called monomial coefficients. Tropical geometry is a relatively recent field in mathematics and here we follow tradition to use notations of symbols $\oplus$ and $\odot$ as tropical sum and multiplication, respectively. So function $F(\mathbf{x})$ can be written in the classical arithmetic form

$$(2) \quad F(\mathbf{x}) = \inf_{\mathbf{v} \in \mathbb{Z}^n} \left(c_{\mathbf{v}} + \sum_{i: v_i \in \mathbb{Z}^n} v_i x_i \right).$$

It is obvious that from the point of view of linear algebra tropical series is a minimum of an infinite number of linear functions. From (2) it follows that there is a hypersurface $C(F)$ which consists of all points $\mathbf{x} \in \Omega$, where the function $F(\mathbf{x})$ is non-smooth.

For a certain point $\mathbf{p} \in \Omega$ we will define operator G_p on the set $B(\Omega)$ of all possible tropical series $F(\mathbf{x})$ of the form (1), i.e. $G_p: B(\Omega) \rightarrow B(\Omega)$ in such a way that $G_p F(\mathbf{x})$ breaks F at point $\mathbf{p} \notin C(F)$ by increasing monomial coefficient $c_{\mathbf{v}}$. If $\mathbf{p} \in C(F)$, then $G_p F(\mathbf{x}) = F(\mathbf{x})$ so G_p is an idempotent operator:

$$(3) \quad G_p F(\mathbf{x}) = \min(U(\mathbf{x}) | U \in B(\Omega), U \geq F, \mathbf{p} \in C(U)).$$

For $\dim(\Omega) \geq 3$ it is hard to imagine what is happening with $F(\mathbf{x})$ after applying operator G_p , however, for $\dim(\Omega) = 2$ there are quite enough illustrations for understanding for instance in [5]. In one-dimensional case, i.e. $n = 1: \Omega \subset \mathbb{R}$, $F(\mathbf{x})$ is a tropical polynomial divided into parts, where each part is presented as a monomial (see Fig. 1). From Fig. 1 it is clearly seen that if $\mathbf{p}$ is next to the

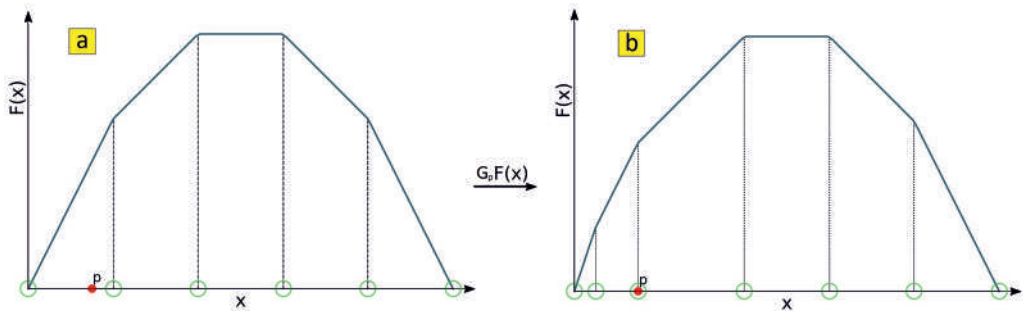


Fig. 1. Demonstration of applying operator G_p (expression (3)) to function $F(\mathbf{x})$ in one-dimensional case $\Omega \subset \mathbb{R}$. $C(F)$ is the sequence of green points

$\partial\Omega$, then $G_p F(\mathbf{x})$ increases the existing monomial coefficient c_v and adds a new monomial with the next integer v .

The described mathematical basis of function $F(\mathbf{x})$ and operator G_p allows to introduce TS model as a dynamical system, following [6]. Suppose we have a finite sequence of points $\{\mathbf{p}_i\}_{i=1}^M$ in some convex subset $\Omega \subset \mathbb{R}^n$. Then each state of this system is a function $F_m(\mathbf{x})$ – the result of applying operator G_p to $F_0(\mathbf{x})$ m times and corresponding hypersurface – $C(F_m)$, where $F_0(\mathbf{x}) \equiv 0$ is the initial state. For instance, $F_N(\mathbf{x}) = G_{\mathbf{p}_1} G_{\mathbf{p}_2} \dots G_{\mathbf{p}_N} F_0(\mathbf{x})$ is the result of sequential application (for each $\mathbf{p}_i$) of operator G_p to function $F_0(\mathbf{x})$. If $N > M$ it is assumed to use cyclic order of G_p application, i.e. $(G_{\mathbf{p}_1} G_{\mathbf{p}_2} \dots G_{\mathbf{p}_N})^d F_0(\mathbf{x})$, $d \geq 2$. But, generally, the order of operator G_p application is not necessarily sequential, i.e. any scheme can be chosen. When a whole sequence $\{\mathbf{p}_i\}_{i=1}^M$ becomes a part of hypersurface $C(F_L)$ for a certain integer $L > 0$, then it is considered the final state of the system and called *relaxation*, while $L(\{\mathbf{p}_i\}_{i=1}^M)$ is *relaxation length*. Here two reasonable questions arise which were touched upon in [5] as well:

1. What can we say about probability distribution of $L(\{\mathbf{p}_i\}_{i=1}^M)$ under uniformly distributed points $\{\mathbf{p}_i\}_{i=1}^M$ in Ω ?
2. Does $C(F_L)$ depend on the order of operator G_p application under relaxation procedure?

Below, these questions will be treated for a specific TS model which has not been studied yet.

One-dimensional parametric TS model. It is possible to algorithmically figure out how relaxation goes for dynamical TS system when $n = 1$ and $\Omega = [0, 1]$. According to definitions (2) and (3) for some point $p \in [0, 1]$, there is a distance $d(p) = \min(p - u_1, u_2 - p)$ where $u_2 > u_1$ are the nearest points of $C(F)$ to p . Note that for any F boundary $\partial\Omega$ is always included in the hypersurface, i.e. $\partial\Omega \in C(F)$. Operator G_p moves points u_1, u_2 on a distance $d(p)$ towards each other as shown in Fig. 1. For simplification p and u_1, u_2 we will call a *stationary*

(motionless) and *mobile* points, respectively. On the $\partial\Omega$, i.e. in both points 0 and 1 it is assumed to be an infinite number of mobile points. If we have a sequence of stationary points $\{p_i\}_{i=1}^M$, then for each intermediate state (step) $F_k = G_{p_{k-1}} F_{k-1}$ two nearest to p_k mobile points always move towards each other simultaneously, and for initial state F_0 two mobile points move from points 0 and 1. Theoretical and numerical inferences for this dynamical model are done in [5], particularly the author states that relaxation terminates always after finite steps, i.e. there always exists a finite sequence of G_p applications such that $G_{p_{k_1}} G_{p_{k_2}} \dots G_{p_{k_L}} F_0$ leads to relaxation. It is also noticed there that $L(\{p_i\}_{i=1}^M)$ obeys power law which means that the TS model exhibits SOC behaviour.

We now present a modified version of the described TS model by introducing a parameter which is responsible for the movement of mobile points with different velocities, i.e. if we have $v_1 \neq v_2$ for each pair of closest to p mobile points, then $d(p)$ becomes unequal for both of them, i.e. in this case we have

$$d_{u_1}(p) = v_1 t^*, \quad d_{u_2}(p) = v_2 t^*, \quad c = v_1/v_2 > 0, \quad t^* = \min((p - u_1)/v_1, (u_2 - p)/v_2).$$

Thus we provide a rule as to how do mobile points move relatively to stationary points, hence it is possible to resolve the problem about probability distribution of $L(\{p_i\}_{i=1}^M)$.

Distribution of relaxation length L . For a given random vector $\{p_i\}_{i=1}^M \sim f$, where $f(x_1, \dots, x_M)$ is the probability density function, the length of relaxation L is a random variable. The case when $M = 1$ is quite trivial since relaxation ends in one step. If $M = 2$, we have vector two stationary points P, Q randomly distributed on $[0, 1]$ and the relaxation procedure is represented as a sequence $(G_p G_q)^m F_0$, therefore it is more convenient to study L^* – the number of applications of operator G_p . We denote $P(L^* = N)$ as probability of relaxation length equal to N . The relationship between L^* and L is obvious, i.e. it is piecewise linear, so $L = 2L^*$ or $L = 2L^* - 1$ depending on which operator was the last to participate in relaxation G_p or G_q .

Proposition 1. *For identical and symmetric distributions of P, Q on $\Omega = [0, 1]$ the probability function of relaxation length depends on the parameter $c > 0$ and is defined as follows:*

a) for $N = 1$:

$$P(L^* = 1) = \int\limits_{\substack{0 < p < c/(c+1) \\ 1-p/c < q < 1}} f(p, q) dp dq + \int\limits_{\substack{c/(c+1) < p < 1 \\ 0 < q < (1-p)c}} f(p, q) dp dq,$$

b) for $N \geq 2$:

$$(4) \quad P(L^* = N) = I(c, N) + I(1/c, N), \quad I(c, N) = \iint_{D(c, N)} f(p, q) dp dq,$$

where $D(c, N)$ is a locus – concave areas which are determined through specific unification and intersection of inequalities:

$$(5a) \quad p > q: \left[\begin{cases} c(N-1)(p-q) > q \\ c(N-2)(p-q) < q \\ (N-2)(p-q) < c-p-cp \end{cases} ; \right. \\ \left. \begin{cases} c(N-1)(p-q) < q \\ (N-1)(p-q) > c-p-cp \\ (N-2)(p-q) < c-p-cp \end{cases} \right]$$

$$(5b) \quad q > p: \left[\begin{cases} c(N-1)(q-p) > p \\ c(N-2)(q-p) < p \\ (N-1)(q-p) < c-p-cq \end{cases} . \right. \\ \left. \begin{cases} c(N-2)(q-p) < p \\ (N-1)(q-p) > c-p-cq \\ (N-2)(q-p) < c-p-cq \end{cases} \right]$$

The author believes that the given below proof of this proposition has no analogues that can be found in the literature.

Proof. The case $N = 1$ is trivial, the locus of points (p, q) where relaxation process terminates after $G_p F_0$ or $(G_p G_q) F_0$ are two triangles, numerical experiment justified it and shows them as a blue area in Fig. 2b or 2c. For the case $N \geq 2$, we will imagine that p and q are close to each other and $q < p < c/(c+1)$. First of all note that a mobile point from any end of the interval $[0, 1]$ passes a distance equal to $p - q$ for a strictly one sequence $(G_p G_q) F_0$. Further, let us consequently figure out two different scenarios when relaxation terminates: $\underbrace{G_p \dots G_p}_{N-2} G_q G_p$ and

$\underbrace{G_p \dots G_p}_{N-2} G_q G_p G_q$. The curly bracket indicates count of G_p application. Keeping

in mind that while mobile point moves from p to q with velocity v_2 , the left mobile point moves in direction to point q with time $(p - q)/v_2$ (i.e. passes the distance $(p - q)c$). The first sequence corresponds to a situation where taken from the left border mobile point covered the distance $d_N = (N - 2)(p - q)c < q$, then for termination process in three steps $G_p G_q G_p$ its required $0 < (q - d_N)/c < p - q$ that gives us two inequalities of (5a). The third inequality is a restriction that is imposed on the right mobile taken from the right border, so that it should not reach point p on the $(N - 1)$ -th application of operator G_p : $1 - p - p/c - (N - 2)(p - q)/c > 0$. The same reasoning is correct for the second sequence, when intermediate state of relaxation process is ready to be terminated after four steps, i.e. $(G_p G_q)^2$. While the mobile point is moving from q to p the second mobile point taken from the right end of the interval $[0, 1]$ should pass distance $d_N = (N - 2)(p - q)/c < 1 - p - p/c$. Note that here we have already taken into account the fact that after the first step $G_p F_0$ this mobile point passes a distance equal to $1 - p/c$. Then the relaxation

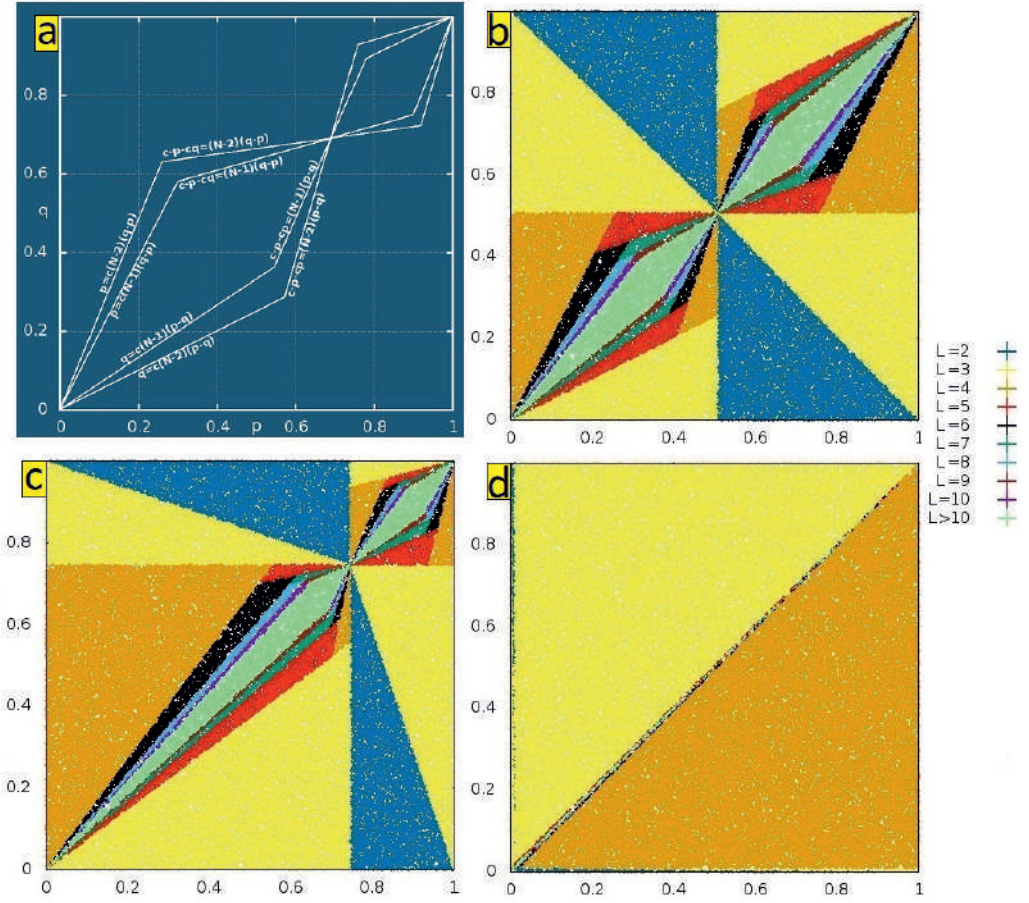


Fig. 2. a) Theoretical locus of $L^* = N$, $N \geq 2$. Numerically determined areas for different values of relaxation length L depending on parameter c : b) $c = 1$, c) $c = 3$, d) $c = 0.01$

process terminates in four steps if $0 < 1 - p - p/c - d_N < p - q$ – the fourth and fifth inequalities of expression (5a). The sixth inequality of (5a) is movement restriction of the left mobile point and can be obtained as well as the third inequality. Similar findings can be done for the case $q > p$ and we will have (5b).

Now it remains to notice the convenient symmetry with respect to the point $c/(c+1)$. The locus $D(c, N)$ was found under $p < c/(c+1)$. There is a symmetrically identical locus $D^*(c, N)$ under $p > c/(c+1)$ (see Fig. 2a), that has the same geometry as $D(c, N)$ and which can be determined by inequalities similarly to (5). But instead of deriving them, due to symmetry of probability functions of P and Q the following equality holds

$$\iint_{D^*(c, N)} f(p, q) dp dq = \iint_{D(1/c, N)} f(p, q) dp dq = I(1/c, N). \quad \square$$

Proposition 2. *The double integral $I(c, N)$ in expression (4) has an explicit form for calculations*

$$(6) \quad I(c, N) = \sum_{k=1}^2 \int_0^1 dx \int_0^{1-x} |J_{1,k}| f(h_{1,k}(x, y), s_{1,k}(x, y)) - |J_{2,k}| f(h_{2,k}(x, y), s_{2,k}(x, y)) dy,$$

$$h_{j,k}(x, y) = \frac{c}{c+1}x + p_{j,k}^*(c, N)y, \quad s_{j,k}(x, y) = \frac{c}{c+1}x + q_{j,k}^*(c, N)y,$$

$$J_{j,k} = \det \begin{pmatrix} c/(c+1) & p_{j,k}^* \\ c/(c+1) & q_{j,k}^* \end{pmatrix}, \quad j, k = 1, 2,$$

where $(p_{j,k}^*, q_{j,k}^*)$ are the corresponding points of the intersection of the lines which are the boundaries of the locus $D(c, N)$ (see Fig. 2a).

The proof of Proposition 2 is obvious, since it requires consequently to integrate the function $f(p, q)$ across concave areas of the locus $D(c, N)$ with the well-known technique of changing variables in a double integral. For instance, the square of $D(c, N)$ under $q < p < c/(c+1)$ can be calculated as the difference between the squares of two triangles determined by points $(0, 0)$, $(c/(c+1), c/(c+1))$, $(p_{j,k}^*, q_{j,k}^*)$, $k = 1, j = 1, 2$, where the first two points are common.

Now we will give several corollaries for the case when $P, Q \sim R[0, 1]$ are independent random variables with function $f(p, q) = f(p)f(q) = 1$.

Corollary 1. *The final form of $I(c, N)$ can be obtained using expression (6)*

$$(7) \quad I(c, N) = \frac{c^2 w_c / (c+1)}{2((N-1)w_c - c^2)(Nw_c - c^2)} + \frac{c^2 w_c / (c+1)}{2((N-1)w_c - c^2 - 1)(Nw_c - c^2 - 1)},$$

$$w_c = c^2 + c + 1.$$

According to Proposition 1 it is not complicated to check whether $P(L^* = 1) + P(L^* \geq 2)$ is equal to 1, since $\sum_{N=2}^{\infty} I(c, N)$ are infinity series similar to $\sum_{N=2}^{\infty} \frac{1}{N(N-1)}$ – telescoping series.

According to expressions (7) and (4) for a large N the probability $P(L^* = N)$ is asymptotically equal to $N^{-2}c/w_c$.

Corollary 2. *For $c = 1$, when nearest to the stationary point two mobile points move with equal velocity we get a special case of simple one-dimensional tropical sandpile model, which was examined in [5] and where the author derived a formula for $P(L^* = N)$, $N \geq 2$ that coincides with expressions (4) and (7), i.e. for $c = 1$:*

$$P(L^* = N) = I(c, N) + I(1/c, N) = 2I(1, N)$$

$$= \frac{3(9N^2 - 18N + 7)}{(3N-1)(3N-2)(3N-4)(3N-5)}.$$

Corollary 3. *There is a limiting case under $c \rightarrow 0$ for function $P(L^* = N)$, $N \geq 1$. First of all let us look at the numerical experiment under $v_2 \gg v_1$ (Fig. 2d), where it can be seen that the area $[0, 1]^2$ is divided strictly on two sides of points (p, q) such that $L = 3$ and $L = 4$ ($L^* = 2$). It means*

$$\lim_{c \rightarrow 0} P(L^* = N) = \begin{cases} \lim_{c \rightarrow 0} \frac{c}{(c+1)^2} + \lim_{c \rightarrow \infty} \frac{c}{(c+1)^2}, & N = 1 \\ \lim_{c \rightarrow 0} I(c, N) + \lim_{c \rightarrow \infty} I(1/c, N), & N \geq 2 \end{cases} = \begin{cases} 0, & N \neq 2 \\ 1, & N = 2 \end{cases}.$$

Corollary 4. *There is a “prediction function” $H: [0, 1]^2 \rightarrow \mathbb{N}$ that can directly determine relaxation length for any pair $(p, q) \in [0, 1]^2$ and helps to avoid computation of relaxation procedure algorithmically. It is not difficult to derive such a formula from expressions (5), for instance, if $0 < p < c/(c+1)$ and $1 - p/c > q$, then*

$$H(p, q) = \min \left(\left\lfloor \frac{\min(p, q)}{|p - q|c} \right\rfloor, \left\lfloor \frac{c - p - c \max(p, q)}{|p - q|} \right\rfloor \right) + 2,$$

where $\lfloor a \rfloor$ is the integer part of the number a . Actually, it can be done for other areas of the square $[0, 1]^2$.

Expression (7) establishes the power law for the relaxation process. It should be noticed that for a count of stationary points greater than 2 the problem of theoretical studying of relaxation length distribution function $P(L = N)$ becomes significantly more difficult. But below we will give some numerical analyses of relaxation length for $M > 2$.

Numerical results. The theoretical findings described earlier were confirmed by numerical experiments, several different numerical experiments were carried out. To confirm the validity of expressions (5) we conducted dropping numerous times two uniformly distributed points p and q , totally we got (p_i, q_i) , $i = \overline{1, 10^5}$ pairs of stationary points. For each pair and for various values of parameter $c = 1, 3, 0.01$ we calculate relaxation procedure and its length L , then points (p_i, q_i) corresponding to a certain length L we paint in the same colour, see Fig. 2b, c, d. We see that while L increases, the locus $D(c, N)$ compresses and becomes invisible already for $L = 10$, that is why it was decided to simply draw the area of (p, q) where $L > 10$. Figure 2 clearly demonstrates the connection between L and L^* , for instance considering the case when the relaxation terminates in five and six steps ($L^* = 3$) the locus $D(c, N)$ consists of two red and two black triangles where the set of (p, q) corresponds to $L = 5$ and $L = 6$, respectively.

This experience helps to check whether formula (7) as a particular case of expression (4) is correct, i.e. we have done Monte-Carlo simulation for different numbers of points (p_i, q_i) , $i = \overline{1, K}$ to obtain empirical estimates of $P(L^* = N)$. Squares of areas where $P(L^* = 2), \dots, P(L^* > 5)$ were estimated as empirical frequencies. Due to the law of large numbers, while increasing the number of points K these estimates became close to theoretical probabilities (7) in the case when $c = 3$.

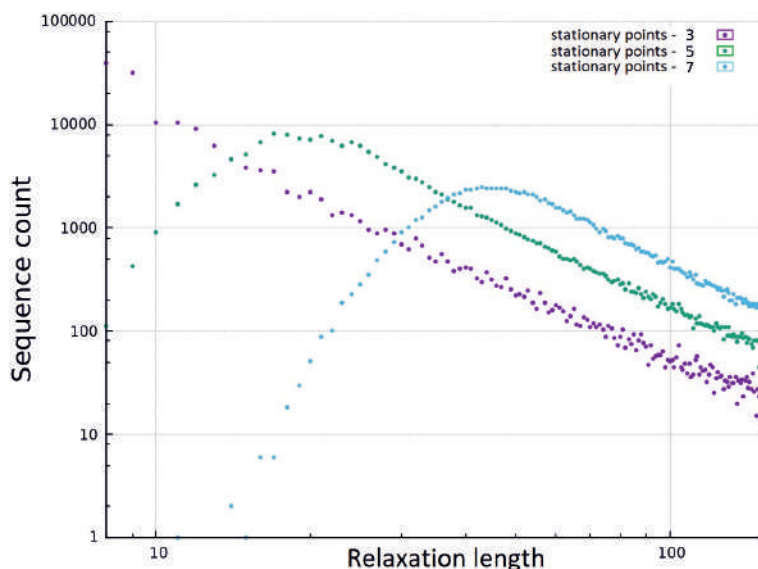


Fig. 3. Estimated frequency of relaxation length calculated for different counts of stationary points. Parameter $c = 3$; violet – 3 points, green – 5 points, blue – 7 points

In order to establish distribution law of relaxation length L for $M > 2$, a similar experiment was done for sequences $\{p_i\}_{i=1}^M$ of stationary points randomly distributed on $[0, 1]^M$. The estimated frequencies of L for different sequence length $M = 3, 5, 7$ are shown in Fig. 3. Setting up logarithmic scale on both axes it is concluded that for large values of L the power law holds, i.e. there exist $\mu(M) < 0$ and $r_M > 0$ $P(L = N) = r_M N^{\mu(M)}$ for large values of N . So we can say that our parametric dynamic system exhibits SOC behaviour [4].

Another question that we raised earlier concerns the final form of the hypersurface $C(F_L)$ which for our parametric sandpile model is the location of mobile points at the end of relaxation process. For $M = 6$, $c = 3$ we have considered two different sets $\{p_i\}_{i=1}^M$. Varying the order of operator G_p the application leads to different sequences of relaxation $G_{p_{k_1}} G_{p_{k_2}} \dots G_{p_{k_L}} F_0$ as shown in Fig. 4. We discovered that the final location of mobile points does not depend on the choice of relaxation algorithm, i.e. on the order of G_p application.

Remark. SOC property of dynamical systems always leads to power law dependency or distributions, but the converse statement is not always true. For TS model we strictly followed the definition of SOC given in [7], i.e. slowly driven by external force G_p system, that exhibits *avalanches* – the hypersurface $C(F)$ changes under G_p application and has power law interactions.

Conclusion. The presented results are a certain continuation of the ideas published earlier in [5]. Modification of well-known one-dimensional tropical sandpile model by introducing specific parameter is proposed. Two issues were raised

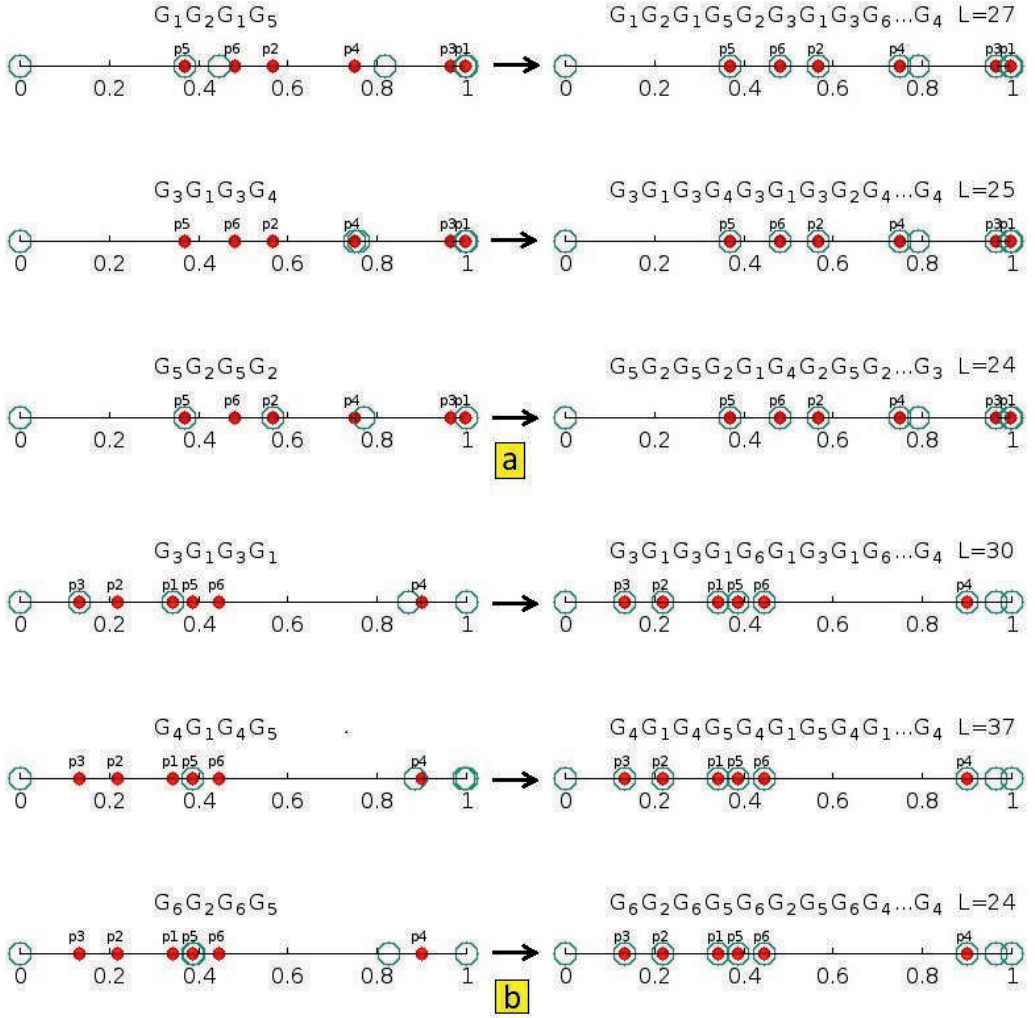


Fig. 4. Example of relaxation procedure demonstrated for two different sets of six stationary points and for case $c = 3$ – “a” and “b” and concluded into two parts: 1) the left part is a result of applying different operators four times from the start; 2) the right part is the finite result of relaxation

and resolved here. The function of probability distribution of relaxation length for two randomly distributed stationary points is given. The case of uniformly distributed stationary points is analyzed in detail. Both theoretical and numerical studies establish power law for relaxation length which indicates SOC behaviour of the introduced dynamical system.

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