

RELATION BETWEEN EXPONENTIAL MOMENT PROBLEM AND CLASSICAL MOMENT PROBLEM

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Abstract

The main result of the paper is an interesting relation between the solution of the truncated exponential moment problem and truncated classical moment problem, considered on the half-line or on a compact interval.

Key words: exponential polynomials, Hausdorff moment problem, Stieltjes moment problem

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1. Introduction. The main result of the present paper is a curious relation between the exponential moment problem and the classical moment problem [1]. The generalized moment problem which contains the exponential moment problem has been considered for the first time by MARKOFF in his original paper of 1886, in the form

$$\int_a^b u_j(x) d\mu(x) = c_j$$

where μ is a Borel measure on $[a, b]$, and $\{u_j\}_{j=1}^N$ is a set of functions which satisfy some appropriate conditions, see [2], also results and historical remarks in [3]. For the particular case of functions of the form

$$u_j(x) = e^{\lambda_j x},$$

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where λ_j are pairwise different real constants, this problem has been studied by F. Hausdorff (1921–1923) and S. Bernstein (1928), whose results may be found together with historical remarks in [1] and [3].

The interest to the exponential polynomials, i.e. linear combinations of exponential functions $\{e^{\lambda_j x}\}$ for real and complex numbers λ_j is motivated by the fact that the polyharmonic functions in special domains are split into sums which contain special instances of exponential functions, cf. [4]. Recently, they have found numerous applications in Approximation theory, Spline theory, CAGD, Image processing, Moment problems, Data Analysis, and others, as may be seen in [5–16].

The paper is organized as follows: in Section 2 we recall basic facts and background material about exponential polynomials. In Section 3 we present and prove our main results, Theorem 2 and Theorem 9.

2. Exponential polynomials. Exponential polynomials in a general setting have been given a special attention, e.g. in [17]. In the present paper we will consider exponential polynomials for some special parameters. For simplicity's sake, in the present paper we will assume that the vector of the “frequencies” $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_N)$ (in the engineering literature frequencies are called only the imaginary parts of the constants λ_j) consists of pairwise different real numbers λ_i and $N = 2n - 1$ is an **odd integer**. We assume that $\mathbb{S}$ is a linear functional on the linear span

$$U_N = \left\langle e^{\lambda_0 x}, \dots, e^{\lambda_N x} \right\rangle_{\text{lin}}$$

hence, the dimension of U_N is $2n$. Let us recall that the function (sometimes called fundamental) $\Phi(x) := \Phi_{(\lambda_0, \dots, \lambda_N)}(x)$, is defined as the unique element of the space U_N which satisfies

$$\Phi^{(j)}(0) = 0 \quad \text{for } j = 0, 1, \dots, N-1; \quad \Phi^{(N)}(0) = 1,$$

and $\Phi^{(j)}$ denotes the j -th derivative of the function Φ . Next, we consider a special basis of the space U_N defined by the set

$$B(x) := \{b_j(x)\}_{j=0}^N,$$

where we have put

$$(1) \quad b_j(x) := b_{N,j}(x) := j! \cdot \Phi_{(\lambda_0, \dots, \lambda_N)}^{(N-j)}(x) \quad \text{for } j = 0, \dots, N.$$

For the proof that the functions $\{b_j\}_{j=0}^N$ form a basis of the space U_N we refer to [10, 11]. The “moment” linear functional $\mathbb{S} : U_N \rightarrow \mathbb{R}$ is defined by the finite number of “exponential moments”:

$$(2) \quad \widehat{c}_j := \widehat{c}_{N,j} := \int j! \cdot \Phi_{(\lambda_0, \dots, \lambda_N)}^{(N-j)}(x) d\mu(x) = \int b_j(x) d\mu(x) \quad \text{for } j = 0, \dots, N.$$

3. Main results. First, we will state the notations and results which we use for proving our main Theorem.

For a number $x \in \mathbb{C}$ and a natural number $m \geq 1$, we use the standard notation for the Pochhammer symbol (as in [18]):

$$(x)_0 := 1, \quad (x)_m := x(x+1) \cdots (x+m-1) \quad \text{for } m \geq 1.$$

Note that $(1)_m = m!$. We shall need the following result of CHAMMAM [19] about determinants of Hankel matrices, see also formula (1.1) of CHU [18].

Theorem 1. *Let α, β be complex numbers and m a natural number. Then*

$$\det_{0 \leq i, j \leq m} \left[\frac{(\alpha)_{i+j}}{(1+\alpha+\beta)_{i+j}} \right] = \prod_{k=0}^m \frac{k! (\alpha)_k (1+\beta)_k}{(\alpha+\beta+k)_k (1+\alpha+\beta)_{2k}}.$$

We now present the main result of the paper:

Theorem 2. *Let N be odd and let the functions b_j be defined as in (1) for $j = 0, \dots, N$. For the function $\Phi_N = \Phi_{(\lambda_0, \dots, \lambda_N)}$ assume that*

$$\Phi_N^{(j)}(x) \geq 0 \text{ for all } x > 0 \text{ and all } j \geq 0.$$

Then, the following two statements hold:

1. *For every $x \in \mathbb{R}_+$, for every k with $2k \leq N$, and for every vector $p = (p_0, \dots, p_k) \in \mathbb{R}^{k+1}$, the inequality*

$$(3) \quad Q_1(p) := \sum_{i,j=0}^k p_i p_j b_{i+j}(x) \geq 0$$

holds, as well the inequality

$$(4) \quad Q_2(p) := \sum_{i,j=0}^k p_i p_j b_{i+j+1}(x) \geq 0.$$

2. *For every $x \in [0, 1]$, for every k with $2k+1 \leq N$, and for every vector $p = (p_0, \dots, p_k) \in \mathbb{R}^{k+1}$, the inequality*

$$(5) \quad \sum_{i,j=0}^k p_i p_j (b_{i+j}(x) - b_{i+j+1}(x)) \geq 0$$

holds.

Proof. The function Φ_N is a linear combination of exponential functions, hence it can be extended in a Taylor series convergent for all $x \in \mathbb{R}$, say

$$\Phi_N(x) = \sum_{s=N}^{\infty} a_s x^s \text{ with } a_N = \frac{1}{N!}.$$

By assumption we have $a_s = \Phi^{(s)}(0)/s! \geq 0$ for all $s = 0, \dots$. Note that for every

m with $0 \leq m \leq N$ holds

$$b_m(x) = m! \Phi_N^{(N-m)}(x) = m! \frac{d^{N-m}}{dx^{N-m}} \Phi_N(x) = m! \sum_{s=N}^{\infty} a_s \frac{d^{N-m}}{dx^{N-m}} x^s.$$

1. In order to prove inequality (3) for $x \in \mathbb{R}_+$, we have to show that for any $p \in \mathbb{R}^{k+1}$ the following inequality holds

$$Q_1(p) := \sum_{i,j=0}^k p_i p_j b_{i+j}(x) = \sum_{i,j=0}^k p_i p_j (i+j)! \sum_{s=N}^{\infty} a_s \frac{d^{N-(i+j)}}{dx^{N-(i+j)}} x^s \geq 0.$$

Clearly we have

$$Q_1(p) = \sum_{s=N}^{\infty} a_s \cdot \sum_{i,j=0}^k p_i p_j (i+j)! \frac{d^{N-(i+j)}}{dx^{N-(i+j)}} x^s.$$

For $s \geq N$ we have $\frac{d^N}{dx^N} x^s = \frac{s!}{(s-N)!} x^{s-N}$, hence

$$(i+j)! \frac{d^{N-(i+j)}}{dx^{N-(i+j)}} x^s = \frac{(i+j)! s! x^{s-(N-(i+j))}}{(s-(N-(i+j)))!} = \frac{(i+j)! s! x^{s+i+j-N}}{(s+i+j-N)!}.$$

It follows that

$$Q_1(p) = \sum_{s=N}^{\infty} a_s \cdot x^{s-N} s! \sum_{i,j=0}^k p_i p_j \frac{(i+j)!}{(s+i+j-N)!} x^{i+j}.$$

Obviously $x^{s-N} > 0$ for $x > 0$. Since $a_s \geq 0$, it suffices to prove that for every $x > 0$

$$\sum_{i,j=0}^k p_i p_j \frac{(i+j)!}{(s+i+j-N)!} x^{i+j}$$

is non-negative for $s = N$ and positive for $s > N$ and $p = (p_0, \dots, p_k) \neq 0$. Note that for $s = N$ we obtain

$$\sum_{i,j=0}^k p_i p_j x^{i+j} = \left(p_0 + p_1 x + \dots + p_k x^k \right)^2 \geq 0.$$

For $s > N$, it is sufficient to prove that the **Hankel matrix** defined by

$$\tilde{H}_{N,k,s}(x) = \begin{pmatrix} \frac{1}{(s-N)!} & \frac{x}{(s-N+1)!} & \dots & k! \frac{x^k}{(s-N+k)!} \\ \frac{x}{(s-N+1)!} & \dots & k! \frac{x^k}{(s-N+k)!} & (k+1)! \frac{x^{k+1}}{(s-N+k+1)!} \\ \dots & \dots & \dots & \dots \\ k! \frac{x^k}{(s-N+k)!} & \dots & \dots & (2k)! \frac{x^{2k}}{(s-N+2k)!} \end{pmatrix}$$

is strictly positive-definite for $s > N$. For proving the strict positive-definiteness, we apply the usual **determinant criterion** (see [20]), i.e. we have to verify that for every fixed $s > N$

$$\det \tilde{H}_{N,k,s}(x) > 0 \quad \text{for all } k \geq 1.$$

Further we put

$$S := s - N \geq 1.$$

By extracting the powers of x first by columns and then by rows, we obtain the formula

$$\begin{aligned} \det \tilde{H}_{N,k,s}(x) &= \\ &= x^{k(k+1)} \det \begin{pmatrix} \frac{1}{S!} & \frac{1}{(S+1)!} & \cdots & k! \frac{1}{(S+k)!} \\ \frac{1}{(S+1)!} & \cdots & k! \frac{1}{(S+k)!} & (k+1)! \frac{1}{(S+k+1)!} \\ \cdots & \cdots & \cdots & \cdots \\ k! \frac{1}{(S+k)!} & \cdots & \cdots & (2k)! \frac{1}{(S+2k)!} \end{pmatrix}. \end{aligned}$$

Note that $x^{k(k+1)} > 0$ for every real number x since $k(k+1)$ is an even number.

To finish the proof, in Theorem 1, we put

$$\alpha = 1, \quad \beta = S - 1, \quad \text{so } 1 + \beta + \alpha = S + 1,$$

and in the obtained determinant, we divide each row by $S!$, which results in the above determinant. It is obviously positive since the right hand side in Theorem 1 is positive.

2. Further, we prove inequality (4) for every $x \in \mathbb{R}_+$, so we consider the quadratic form

$$Q_2(x) := \sum_{i,j=0}^k p_i p_j b_{i+j+1}(x).$$

Similar to the previous case, we see that

$$Q_2(p) = \sum_{s=N}^{\infty} a_s \cdot \sum_{i,j=0}^k p_i p_j (i+j+1)! \frac{d^{N-(i+j+1)}}{dx^{N-(i+j+1)}} x^s.$$

Then

$$(i+j+1)! \frac{d^{N-(i+j+1)}}{dx^{N-(i+j+1)}} x^s = \frac{(i+j+1)! s!}{(s+i+j+1-N)!} x^{s+i+j+1-N}.$$

Hence it suffices to show that

$$x \sum_{i,j=0}^k p_i p_j \frac{(i+j+1)!}{(s+i+j+1-N)!} x^{i+j}$$

is non-negative for $s = N$ and $x \geq 0$, and strictly positive for $x > 0$ and $p \neq 0$. In the case $s = N$ one can argue as before. For $s > N$ we have to show the positive definiteness of the corresponding Hankel matrix

$$\begin{aligned} & \tilde{H}'_{N,k,s}(x) \\ &= \begin{pmatrix} \frac{1}{(S+1)!} & \frac{2}{(S+2)!} & \cdots & (k+1)! \frac{1}{(S+k+1)!} \\ \cdots & & (k+1)! \frac{1}{(S+k+1)!} & (k+2)! \frac{1}{(S+k+2)!} \\ \cdots & & & \\ (k+1)! \frac{1}{(S+k+1)!} & & & (2k+1)! \frac{1}{(S+2k+1)!} \end{pmatrix}, \end{aligned}$$

where $S := s - N$. Now in Theorem 1 we put

$$\alpha = 2 \text{ and } \beta = S - 1, \text{ so } 1 + \alpha + \beta = S + 2,$$

and the proof follows as in item 1 of the present proof, but we have to divide each row by $(S+1)!$.

3. We consider the case of the compact interval $[0, 1]$. For every $x \in (0, 1)$, we want to prove inequality (5), hence we consider the quadratic form

$$Q_3(x) := \sum_{i,j=0}^k p_i p_j (b_{i+j}(x) - b_{i+j+1}(x)) = Q_1(x) - Q_2(x).$$

It follows that

$$Q_3(x) = \sum_{s=N}^{\infty} a_s \cdot \sum_{i,j=0}^k p_i p_j \left[\frac{(i+j)! s! x^{s+i+j-N}}{(s+i+j-N)!} - \frac{(i+j+1)! s! x^{s+i+j+1-N}}{(s+i+j+1-N)!} \right].$$

So

$$\begin{aligned} Q_3(x) &= \sum_{s=N}^{\infty} a_s \cdot \sum_{i,j=0}^k p_i p_j \frac{(i+j)! s! x^{s+i+j-N}}{(s+i+j+1-N)!} [(s+i+j+1-N) - (i+j+1)x] \\ &= \sum_{s=N}^{\infty} a_s \cdot \sum_{i,j=0}^k p_i p_j \frac{(i+j)! s! x^{s+i+j-N}}{(s+i+j+1-N)!} [s - N + (i+j+1)(1-x)]. \end{aligned}$$

The form $Q_3(x)$ is obviously positive if the following two inequalities hold:

$$\sum_{i,j=0}^k p_i p_j \frac{(i+j)! s! x^{s+i+j-N}}{(s+i+j+1-N)!} (s - N) \geq 0$$

and

$$\sum_{i,j=0}^k p_i p_j \frac{(i+j)! s! x^{s+i+j-N}}{(s+i+j+1-N)!} (i+j+1)(1-x) \geq 0.$$

The first inequality follows from Theorem 1 for $\alpha = 1$ and $\beta = s - N$, and the second inequality follows from Theorem 1 with $\alpha = 2$ and $\beta = s - N - 1$.

To finish the application of Theorem 1, in both cases we divide each row of the particular determinant, obtained from it, by $(s - N + 1)!$. Finally, we recall that $0 \leq x \leq 1$. $\square$

Remark 3. Obviously, the results in Theorem 2 may be generalized to arbitrary compact interval $[a, b]$ which is a subset of $[0, \infty)$.

Proposition 4. *The inequality $\Lambda \geq 0$, i.e. $\lambda_j \geq 0$ for all j , implies*

$$\Phi_N^{(j)}(x) \geq 0 \text{ for all } x > 0 \text{ and all } j \geq N.$$

Indeed, this is proved by the Taylor expansion of the function Φ_N , see for an explicit expression [13], or [11], see also [10].

Let us recall the classical truncated Hausdorff moment problem, see formula (2.50) in [1], p. 74 (see also KREĬN-NUDEL'MAN [3]).

Theorem 5. *Let $\{s_k\}_{k=0}^{2n-1}$ be a finite sequence of real numbers for which the forms*

$$(6) \quad \sum_{i,k=0}^{n-1} s_{i+k+1} p_i p_k \geq 0, \quad \sum_{i,k=0}^{n-1} (s_{i+k} - s_{i+k+1}) p_i p_k \geq 0.$$

Then there exists a non-negative measure $d\sigma(x)$ such that

$$\int_0^1 x^k d\sigma(x) = s_k \quad \text{for } k = 0, 1, \dots, 2n-1.$$

Now we use the classical Hausdorff moment problem, to identify moment sequences arising in the setting of the exponential polynomials.

Theorem 6. *With the assumptions of Theorem 2, let the non-negative measure $d\mu$ be given on the interval $[0, 1]$. We put*

$$\widehat{c}_j := \int_0^1 b_j(x) d\mu(x) \quad \text{for } j = 0, 1, \dots, 2n-1,$$

where $b_j(x)$ are given by (1). Then there exists a non-negative measure $d\nu$ defined on the interval $[0, 1]$ which solves the classical Moment problem

$$\int_0^1 x^j d\nu(x) = \widehat{c}_j \quad \text{for } j = 0, 1, \dots, 2n-1.$$

Proof. In order to apply Theorem 5, we have to show that (6) holds. Indeed, by Theorem 2 we know that

$$\sum_{i,k=0}^{n-1} \widehat{c}_{i+k+1} p_i p_k = \int_0^1 \left(\sum_{i,k=0}^{n-1} p_i p_k b_{i+k+1}(x) \right) d\mu(x) \geq 0.$$

The second inequality in (6) is proved in a similar way. $\square$

Remark 7. For simplicity we have proved Theorem 2 only for odd $N = 2n - 1$, but it can be shown by the same methods that an analogue holds for even $N = 2n$. Moreover Theorem 6 may be treated in a similar way for $N = 2n$ using formula (2.51) instead of (2.50) in [1, p. 74].

Let us recall the classical full Stieltjes moment problem (on the half-line), see [1, p. 76, in section 6.5], (see also Kreĭn–Nudel'man [3]).

Theorem 8. *Let $\{s_k\}_{k=0}^\infty$ be a sequence of real numbers for which the forms*

$$\sum_{i,k=0}^m s_{i+k} p_i p_k \geq 0, \quad \sum_{i,k=0}^m s_{i+k+1} p_i p_k \geq 0$$

for every natural number m . Then there exists a non-negative measure $d\sigma(x)$ such that

$$\int_0^\infty x^k d\sigma(x) = s_k \quad \text{for } k = 0, 1, 2, \dots$$

Now we use Theorem 8 (called sometimes Stieltjes theorem), to identify moment sequences which appear in the setting of the exponential moment problem.

Theorem 9. *With the assumptions of Theorem 2, let the non-negative measure $d\mu$ be given on the interval $[0, \infty)$. We put*

$$\widehat{c}_j := \int_0^\infty b_j(x) d\mu(x) \quad \text{for } j = 0, 1, \dots,$$

where $b_j(x)$ are given by (1). Then there exists a non-negative measure $d\nu$ defined on the interval $[0, \infty)$ which solves the classical Moment problem

$$\int_0^\infty t^j d\nu(t) = \widehat{c}_j \quad \text{for } j = 0, 1, \dots$$

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