

## ON CONNECTEDNESS, GIRTH AND DIAMETER OF GAP POSET GRAPHS

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### Abstract

In this article, we introduce the graphs associated with gap posets of numerical semigroups. We prove that gap poset graphs are either bipartite graphs or graphs with girth 3. Also, we characterize gap poset graphs in terms of connectedness, girth and diameter.

**Key words:** numerical semigroup, partial order, complete graph, bipartite graph, diameter

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**1. Introduction** Graph theory plays a crucial role in understanding the concepts differently and solving different problems in mathematics. Nowadays graph theory has many applications in different fields such as in computer science, chemistry, biology, engineering, and other sciences as well (see [1–3]). In the past few decades, algebraic graph theory has been a great development area (see [4–6]). The study of graphs to different algebraic structures is a very interesting and beneficial field of research as this study plays a role of a bridge between algebra and graph theory by using properties of one field into the other and vice versa (see [7–13]).

An ordered pair  $(\mathcal{V}, \mathcal{E})$  of a set of nodes (also known as vertices)  $\mathcal{V}$  and a set of edges  $\mathcal{E}$  defines a graph  $\mathcal{G}$ . The end points of an edge are known as adjacent vertices. The cardinality of  $\mathcal{E}$  and  $\mathcal{V}$  is said to be the size and the order of a graph

$\mathcal{G}$ , respectively. An alternating pattern of vertices and edges,  $v_1e_1v_2e_2\ldots v_ne_nv_{n+1}$  for  $n \geq 0$ , is called a path in a graph. If the starting and ending vertices of a path are the same, then the path is said to be a closed path or a cycle. The number of edges in the shortest path joining two vertices  $v_x, v_y$  in a graph is the distance between them and is denoted by  $d(v_x, v_y)$ . The diameter of a graph, represented as  $\text{diam}(\mathcal{G})$ , is the length of the shortest path connecting the most distant vertices. The girth of a graph, written as  $\text{gr}(\mathcal{G})$ , is regarded as the length of its shortest cycle. If there is no closed path in  $\mathcal{G}$ , we define  $\text{gr}(\mathcal{G}) = \infty$ . If for any  $v_x, v_y \in \mathcal{G}$ , a path connecting  $v_x, v_y$  exists, then  $\mathcal{G}$  is called a connected graph. Otherwise  $\mathcal{G}$  is a disconnected graph. For a disconnected  $\mathcal{G}$ , we define  $\text{diam}(\mathcal{G}) = \infty$ . A graph is complete, if for any  $v_x, v_y \in \mathcal{G}$ , an edge connecting  $v_x, v_y$  exists. Symbolically, it is written as  $K_n$ . If the collection  $\mathcal{V}$  can be grouped into two disjoint sets  $\mathcal{A}$  and  $\mathcal{B}$  such that each edge in the graph has one endpoint in  $\mathcal{A}$  and the other endpoint in  $\mathcal{B}$ , then the graph  $\mathcal{G}$  is called a bipartite graph. An edgeless graph has no edges. For more deep concepts related to graph theory, see [14, 15].

A relation on a nonempty set is a partial ordering, if it is reflexive, anti-symmetric, and transitive. Due to the structural properties, lattices, chains, antichains, and order ideals of posets have important applications in many different areas of mathematics. The modern study of posets was significantly advanced by BIRKHOFF's pioneering work on lattice theory, which laid the foundation for the algebraic treatment of ordered structures [16]. Subsequently, STANLEY highlighted the role of posets in combinatorics, particularly in the study of order polytopes and enumerative combinatorics [17]. Additionally, posets have been extensively studied in the context of algebraic structures such as distributive lattices and Boolean algebras, leading to significant results in universal algebra and logic [18].

A submonoid of  $\mathbb{N}$  (including 0) that has a finite complement in  $\mathbb{N}$  is a numerical semigroup denoted by  $\mathcal{S}$ . The set of gaps  $\mathcal{G}$  of  $\mathcal{S}$ , is the set of natural numbers not in  $\mathcal{S}$ , i.e.,  $\mathcal{G} = \mathbb{N} \setminus \mathcal{S}$ . The multiplicity of  $\mathcal{S}$ , denoted by  $m(\mathcal{S})$ , is the least nonzero element of  $\mathcal{S}$ , while the Frobenius number  $F(\mathcal{S})$ , is the greatest element of  $\mathcal{G}$ . A numerical semigroup is said to be symmetric numerical semigroup, if  $F(\mathcal{S})$  is odd and for every  $x \in \mathbb{Z} \setminus \mathcal{S}$ , we have  $F(\mathcal{S}) - x \in \mathcal{S}$ . A pseudo-symmetric numerical semigroup, is a numerical semigroup, in which  $F(\mathcal{S})$  is even and for every  $x \in \mathbb{Z} \setminus \mathcal{S}$ , either we have  $F(\mathcal{S}) - x \in \mathcal{S}$  or  $x = F(\mathcal{S})/2$ . A numerical semigroup satisfying  $F(\mathcal{S}) < 2m(\mathcal{S})$  is called an elementary numerical semigroup. Every numerical semigroup is finitely generated, i.e., there exists  $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{S}$  such that  $\mathcal{S} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ . The minimal number of elements that generates  $\mathcal{S}$  is known as embedding dimension of  $\mathcal{S}$ . For detailed study related to numerical semigroups, see [19].

The order relation  $\prec$  on  $\mathcal{G}$  is defined as:  $x \prec y$  if and only if  $y - x \in \mathcal{S}$ . It is easy to see that  $(\mathcal{G}, \prec)$  is a poset. A simple graph associated with the gap poset  $\mathcal{G}$  of a numerical semigroup  $\mathcal{S}$ , denoted by  $J_{\mathcal{G}}(\mathcal{S})$  is a graph such that

$$V(J_{\mathcal{G}}(\mathcal{S})) = \{v_x : x \in \mathcal{G}\},$$

and

$$E(J_{\mathcal{G}}(\mathcal{S})) = \{v_x v_y \Leftrightarrow x \prec y \text{ or } y \prec x\}.$$

In this article, we discuss the different graph parameters of the gap poset graphs  $J_{\mathcal{G}}(\mathcal{S})$ .

**2. Connectedness, girth and diameter of gap poset graphs.** We start this section by characterizing two extreme classes of graphs, the class of complete graphs and the class of edgeless graphs.

**Proposition 1.** *Every complete graph  $K_n$  can be viewed as a gap poset graph  $J_{\mathcal{G}}(\mathcal{S})$  of a numerical semigroup of multiplicity 2.*

**Proof.** Since  $|K_n| = n$ , therefore we can define a numerical semigroup  $\mathcal{S} = \langle 2, 2n+1 \rangle$ . It is easy to see that the gap poset for  $\mathcal{S}$  is  $\mathcal{G} = \{1, 3, 5, \dots, 2n-1\}$ . Clearly  $|J_{\mathcal{G}}(\mathcal{S})| = |\mathcal{G}| = n$  and every two vertices of  $J_{\mathcal{G}}(\mathcal{S})$  are adjacent as the difference of two odd integers is an even integer.  $\square$

**Theorem 2.** *Let  $J_{\mathcal{G}}(\mathcal{S})$  be a graph associated with the gap poset  $\mathcal{G}$  of a numerical semigroup  $\mathcal{S}$ . Then the following statements are equivalent:*

- (i)  $J_{\mathcal{G}}(\mathcal{S})$  is an edgeless graph.
- (ii)  $\mathcal{S}$  is an interval numerical semigroup of maximal embedding dimension.

**Proof.** (i)  $\implies$  (ii). If  $\mathcal{S}$  has maximal embedding dimension but is not an interval numerical semigroup, then  $m(\mathcal{S}) + i \notin \mathcal{S}$  for some  $1 \leq i \leq m(\mathcal{S}) - 1$ . Then

$$m(\mathcal{S}) + i - i = m(\mathcal{S}) \in \mathcal{S}.$$

This implies that  $J_{\mathcal{G}}(\mathcal{S})$  is not an edgeless graph. Now if  $\mathcal{S}$  is an interval numerical semigroup with  $e(\mathcal{S}) < m(\mathcal{S})$ , then  $F(\mathcal{S}) > m(\mathcal{S})$  and, therefore,  $F(\mathcal{S}), F(\mathcal{S}) - m(\mathcal{S}) \in \mathcal{G}$ . This implies

$$F(\mathcal{S}) - (F(\mathcal{S}) - m(\mathcal{S})) = m(\mathcal{S}) \in \mathcal{S}.$$

This gives that there is an edge between the vertices  $v_{F(\mathcal{S})}$  and  $v_{F(\mathcal{S})-m(\mathcal{S})}$ , which gives contradiction against statement (i). Hence  $\mathcal{S}$  is an interval numerical semigroup of maximal embedding dimension.

(ii)  $\implies$  (i). If  $\mathcal{S} = \langle n, n+1, \dots, 2n-1 \rangle$ ,  $n \geq 2$ , then  $\mathcal{G} = \{1, 2, \dots, n-1\}$ . Clearly, every two elements of  $\mathcal{G}$  are incomparable, i.e., for every  $x, y \in \mathcal{G}$  and  $x > y$ , we have  $x - y \notin \mathcal{S}$ . This implies that  $J_{\mathcal{G}}(\mathcal{S})$  is an edgeless graph.  $\square$

The following results show that a gap poset graph is either a bipartite graph or it is a graph with girth 3.

**Lemma 3.** *Let  $\mathcal{S}$  be an elementary numerical semigroup. Then  $J_{\mathcal{G}}(\mathcal{S})$  is a bipartite graph.*

**Proof.** Since  $\mathcal{S}$  is an elementary numerical semigroup, therefore  $F(\mathcal{S}) < 2m(\mathcal{S})$ . This implies the gap set  $\mathcal{G}$  of  $\mathcal{S}$  is of the form  $\mathcal{G} = \mathcal{A} \cup \mathcal{B}$ , where  $\mathcal{A} = \{1, 2, \dots, m(\mathcal{S}) - 1\}$  and  $\mathcal{B} \subseteq \{m(\mathcal{S}) + 1, m(\mathcal{S}) + 2, \dots, 2m(\mathcal{S}) - 1\}$ . This gives

$V(J_{\mathcal{G}}(\mathcal{S})) = V_1 \cup V_2$ , where  $V_1 = \{v_x : x \in \mathcal{A}\}$  and  $V_2 = \{v_y : y \in \mathcal{B}\}$ . Clearly,  $V_1$  and  $V_2$  are disjoint as  $\mathcal{A} \cap \mathcal{B} = \emptyset$ . Also, there is no edge between any two vertices in  $V_1$ , as every two elements of  $\mathcal{A}$  are incomparable. Moreover, for  $1 \leq i < j \leq m(\mathcal{S}) - 1$

$$(m + j) - (m + i) = j - i \notin \mathcal{S}.$$

This implies that no two elements of  $\mathcal{B}$  are comparable, therefore, all vertices of  $V_2$  are independent. Consequently,  $J_{\mathcal{G}}(\mathcal{S})$  is a bipartite graph.  $\square$

**Example 1.** Let  $\mathcal{S} = \langle 5, 6, 7 \rangle$ . Then the gap poset of  $\mathcal{S}$  is  $\mathcal{G} = \{1, 2, 3, 4, 8, 9\}$ , clearly  $F(\mathcal{S}) = 9 < 2m(\mathcal{S}) = 10$ . The corresponding graph  $J_{\mathcal{G}}(\mathcal{S})$  is given in Fig. 1, which is a bipartite graph with the two independent sets  $\{v_1, v_2, v_3, v_4\}$  and  $\{v_8, v_9\}$ .

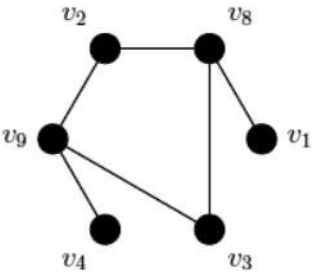


Fig. 1

**Corollary 4.** Let  $\mathcal{S}$  be an elementary numerical semigroup. Then  $J_{\mathcal{G}}(\mathcal{S})$  contains no cycle of odd length.

**Theorem 5.** The following statements are equivalent:

- (i)  $J_{\mathcal{G}}(\mathcal{S})$  contains a cycle of odd length.
- (ii) The girth of  $J_{\mathcal{G}}(\mathcal{S})$  is 3.
- (iii)  $\mathcal{S}$  is not an elementary numerical semigroup.

**Proof.** (i)  $\implies$  (ii). Consider  $J_{\mathcal{G}}(\mathcal{S})$  contains a cycle of odd length  $n$ , say

$$v_{x_1} \rightarrow v_{x_2} \rightarrow v_{x_3} \rightarrow \cdots \rightarrow v_{x_n} \rightarrow v_{x_1}, \quad n \geq 5.$$

Since there is an edge between  $v_{x_1}$  and  $v_{x_2}$ , therefore either  $x_1 - x_2 \in \mathcal{S}$  or  $x_2 - x_1 \in \mathcal{S}$ . Assume that  $x_2 - x_1 \in \mathcal{S}$ . Also an edge between  $v_{x_2}$  and  $v_{x_3}$  implies  $x_2 - x_3 \in \mathcal{S}$  or  $x_3 - x_2 \in \mathcal{S}$ . If  $x_3 - x_2 \in \mathcal{S}$ , then transitivity of posets gives  $x_3 - x_1 \in \mathcal{S}$  and, therefore,  $J_{\mathcal{G}}(\mathcal{S})$  contains a cycle  $v_{x_1} \rightarrow v_{x_2} \rightarrow v_{x_3} \rightarrow v_{x_1}$  of length 3, otherwise  $x_2 - x_3 \in \mathcal{S}$ . Continuing in this way, we get

$$x_4 - x_3, x_4 - x_5, x_6 - x_5, \dots, x_{n-1} - x_n \in \mathcal{S}.$$

As there is an edge between  $v_{x_1}$  and  $v_{x_n}$ , therefore either  $x_1 - x_n \in \mathcal{S}$  or  $x_n - x_1 \in \mathcal{S}$ . Note that if  $x_1 - x_n \in \mathcal{S}$ , then  $J_{\mathcal{G}}(\mathcal{S})$  contains a cycle  $v_{x_1} \rightarrow v_{x_2} \rightarrow v_{x_n} \rightarrow v_{x_1}$  of length 3 and if  $x_n - x_1 \in \mathcal{S}$ , then  $J_{\mathcal{G}}(\mathcal{S})$  contains a cycle  $v_{x_{n-1}} \rightarrow v_{x_n} \rightarrow v_{x_1} \rightarrow v_{x_{n-1}}$  of length 3. This implies the girth of  $J_{\mathcal{G}}(\mathcal{S})$  must be 3.

(ii)  $\implies$  (iii). Suppose  $F(\mathcal{S}) < 2m(\mathcal{S})$ . Then, from Lemma 3, we have that  $J_{\mathcal{G}}(\mathcal{S})$  is a bipartite graph and, therefore, it does not contain a cycle of odd length. This is a contradiction, as the girth of  $J_{\mathcal{G}}(\mathcal{S})$  is 3. So,  $F(\mathcal{S}) > 2m(\mathcal{S})$ , i.e.,  $\mathcal{S}$  is not an elementary numerical semigroup.

(iii)  $\implies$  (i). If  $F(\mathcal{S}) > 2m(\mathcal{S})$ , then  $F(\mathcal{S}), F(\mathcal{S}) - m(\mathcal{S}), F(\mathcal{S}) - 2m(\mathcal{S}) \in \mathcal{G}$ . Since  $F(\mathcal{S}) - (F(\mathcal{S}) - m(\mathcal{S})) = m(\mathcal{S})$ ,  $F(\mathcal{S}) - (F(\mathcal{S}) - 2m(\mathcal{S})) = 2m(\mathcal{S})$ ,  $(F(\mathcal{S}) - m(\mathcal{S})) - (F(\mathcal{S}) - 2m(\mathcal{S})) = m(\mathcal{S}) \in \mathcal{S}$ , therefore  $J_{\mathcal{G}}(\mathcal{S})$  contains a cycle  $v_{F(\mathcal{S})} \rightarrow v_{F(\mathcal{S})-m(\mathcal{S})} \rightarrow v_{F(\mathcal{S})-2m(\mathcal{S})} \rightarrow v_{F(\mathcal{S})}$  of length 3. This completes the proof.  $\square$

**Proposition 6.** *Let  $\mathcal{S}$  be a numerical semigroup and  $e$  be a positive even integer. Then  $\text{gr}(J_{\mathcal{G}}(\mathcal{S})) \in \{3, e, \infty\}$ .*

**Proof.** The proof of this result follows from Lemma 3 and Theorem 5.  $\square$

We continue this section by collecting some results on the connectedness, girth and diameter of the gap poset graphs  $J_{\mathcal{G}}(\mathcal{S})$ .

**Proposition 7.** *Let  $\mathcal{S}$  be a symmetric numerical semigroup. Then the following statements hold:*

- (i) *The graph  $J_{\mathcal{G}}(\mathcal{S})$  is a connected graph with at least one vertex of degree  $|\mathcal{G}| - 1$ .*
- (ii)  $\text{diam}(J_{\mathcal{G}}(\mathcal{S})) \leq 2$ .
- (iii)  $\text{gr}(J_{\mathcal{G}}(\mathcal{S})) \in \{3, \infty\}$ .

**Proof.** (i). Since  $\mathcal{S}$  is symmetric, therefore for every  $x \in \mathcal{G}$  and  $x \neq F(\mathcal{S})$ , we have  $F(\mathcal{S}) - x \in \mathcal{S}$ . This implies that each vertex  $v_x \in V(J_{\mathcal{G}}(\mathcal{S}))$  has an edge with the vertex  $v_{F(\mathcal{S})}$ . So,  $J_{\mathcal{G}}(\mathcal{S})$  is a connected graph with  $\deg(v_{F(\mathcal{S})}) = |\mathcal{G}| - 1$ .

(ii). From (i), it follows that  $d(v_{F(\mathcal{S})}, v_x) = 1$  for all  $x \in \mathcal{G}$  and  $x \neq F(\mathcal{S})$ . Now for any  $v_x, v_y \in V(J_{\mathcal{G}}(\mathcal{S}))$ , we have

$$d(v_x, v_y) \leq d(v_x, v_{F(\mathcal{S})}) + d(v_{F(\mathcal{S})}, v_y) = 1 + 1 = 2.$$

This implies that  $\text{diam}(J_{\mathcal{G}}(\mathcal{S})) \leq 2$ .

(iii). Clearly, if there is an edge between vertices  $v_x, v_y \in V(J_{\mathcal{G}}(\mathcal{S}))$ , where  $v_x \neq v_{F(\mathcal{S})} \neq v_y$ , then  $\text{gr}(J_{\mathcal{G}}(\mathcal{S})) = 3$ , otherwise  $J_{\mathcal{G}}(\mathcal{S})$  is isomorphic to a star graph and therefore has an infinite girth.  $\square$

**Example 2.** Let  $\mathcal{S} = \langle 4, 5, 6 \rangle$  be a symmetric numerical semigroup. The gap poset of  $\mathcal{S}$  is  $\mathcal{G} = \{1, 2, 3, 7\}$ . The corresponding graph  $J_{\mathcal{G}}(\mathcal{S})$  is given in Fig. 2. It is a connected graph with  $\text{diam}(J_{\mathcal{G}}(\mathcal{S})) = 2$  and  $\text{gr}(J_{\mathcal{G}}(\mathcal{S})) = \infty$ .

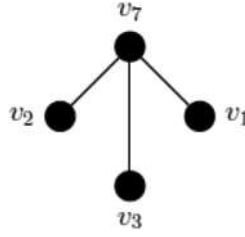


Fig. 2

**Example 3.** Let  $\mathcal{S} = \langle 3, 5 \rangle$  be a symmetric numerical semigroup. The gap poset of  $\mathcal{S}$  is  $\mathcal{G} = \{1, 2, 4, 7\}$ . The corresponding graph  $J_{\mathcal{G}}(\mathcal{S})$  is given in Fig. 3. It is a connected graph with  $\text{diam}(J_{\mathcal{G}}(\mathcal{S})) = 2$  and  $\text{gr}(J_{\mathcal{G}}(\mathcal{S})) = 3$ .

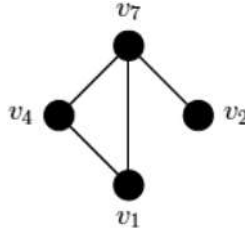


Fig. 3

**Theorem 8.** Let  $\mathcal{S}$  be a pseudo symmetric numerical semigroup. Then  $J_{\mathcal{G}}(\mathcal{S})$  is a connected graph if and only if  $\mathcal{S}$  is not an elementary numerical semigroup.

**Proof.** ( $\Rightarrow$ ) Suppose on the contrary, that  $\mathcal{S}$  is an elementary numerical semigroup, i.e.,  $F(\mathcal{S}) < 2m(\mathcal{S})$ . Since  $\mathcal{S}$  is pseudo symmetric, therefore either  $F(\mathcal{S}) = 2m(\mathcal{S}) - 2$  or  $F(\mathcal{S}) = 2m(\mathcal{S}) - 4$ .

Case 1. If  $F(\mathcal{S}) = 2m(\mathcal{S}) - 2$ , then  $g(\mathcal{S}) = m(\mathcal{S})$  and, therefore,  $\mathcal{G} = \{1, 2, 3, \dots, m(\mathcal{S}) - 1, F(\mathcal{S})\}$ . Note that  $\frac{F(\mathcal{S})}{2} \in \{1, 2, 3, \dots, m - 1\}$  and  $F(\mathcal{S}) - \frac{F(\mathcal{S})}{2} = \frac{F(\mathcal{S})}{2} \notin \mathcal{S}$ . This implies that there is no edge between the vertex  $v_{F(\mathcal{S})}$  and the vertices  $v_x$ , where  $v_x \in V(J_{\mathcal{G}}(\mathcal{S}))$  and  $v_x \neq v_{F(\mathcal{S})}$  and, therefore,  $J_{\mathcal{G}}(\mathcal{S})$  is disconnected.

Case 2. If  $F(\mathcal{S}) = 2m(\mathcal{S}) - 4$ , then  $g(\mathcal{S}) = m(\mathcal{S}) - 1$ . This gives that  $\mathcal{S}$  is an interval numerical semigroup of maximal embedding dimension, so from Theorem 2, it follows that  $J_{\mathcal{G}}(\mathcal{S})$  is an edgeless graph.

Consequently, both cases give a contradiction.

( $\Leftarrow$ ) Assume that  $F(\mathcal{S}) > 2m(\mathcal{S})$ . Since  $\mathcal{S}$  is pseudo symmetric, therefore for every  $x \in \mathcal{G}$ , either  $F(\mathcal{S}) - x \in \mathcal{S}$  or  $x = \frac{F(\mathcal{S})}{2}$ . Also,  $F(\mathcal{S}) - \frac{F(\mathcal{S})}{2} = \frac{F(\mathcal{S})}{2} \notin \mathcal{S}$ . This gives that every vertex  $v_x \in V(J_{\mathcal{G}}(\mathcal{S}))$  is connected with  $v_{F(\mathcal{S})}$  except  $v_{\frac{F(\mathcal{S})}{2}}$ . Now, in order to show that  $J_{\mathcal{G}}(\mathcal{S})$  is connected, we only need to show that  $v_{\frac{F(\mathcal{S})}{2}}$  has an edge with  $v_x$  for some  $x \in \mathcal{G}$  and  $x \neq \frac{F(\mathcal{S})}{2}$ . Since we have  $2m(\mathcal{S}) < F(\mathcal{S})$ ,

therefore  $\frac{F(\mathcal{S})}{2} - m(\mathcal{S}) > 0$ . Then either  $\frac{F(\mathcal{S})}{2} - m(\mathcal{S}) \in \mathcal{G}$  or  $\frac{F(\mathcal{S})}{2} - m(\mathcal{S}) \in \mathcal{S}$ . If  $\frac{F(\mathcal{S})}{2} - m(\mathcal{S}) \in \mathcal{S}$ , then  $\frac{F(\mathcal{S})}{2} \in \mathcal{S}$ , which is not possible. Hence  $\frac{F(\mathcal{S})}{2} - m(\mathcal{S}) \in \mathcal{G}$ . Now for  $\frac{F(\mathcal{S})}{2}, \frac{F(\mathcal{S})}{2} - m(\mathcal{S}) \in \mathcal{G}$  we have

$$\frac{F(\mathcal{S})}{2} - \left( \frac{F(\mathcal{S})}{2} - m(\mathcal{S}) \right) = m(\mathcal{S}) \in \mathcal{S}.$$

This implies that there is an edge between  $v_{\frac{F(\mathcal{S})}{2}}$  and  $v_{\frac{F(\mathcal{S})}{2} - m(\mathcal{S})}$  and, therefore,  $(J_{\mathcal{G}}(\mathcal{S}))$  is a connected graph.  $\square$

**Proposition 9.** *Let  $J_{\mathcal{G}}(\mathcal{S})$  be a connected graph associated with the gaps poset of pseudo symmetric numerical semigroup  $\mathcal{S}$ . Then  $\text{gr}(J_{\mathcal{G}}(\mathcal{S})) = 3 = \text{diam}(J_{\mathcal{G}}(\mathcal{S}))$ .*

**Proof.** Since  $J_{\mathcal{G}}(\mathcal{S})$  is a connected graph, therefore  $\mathcal{S}$  is not an elementary numerical semigroup. Then Theorem 2 and Theorem 5 give  $\text{gr}(J_{\mathcal{G}}(\mathcal{S})) = 3 = \text{diam}(J_{\mathcal{G}}(\mathcal{S}))$ .  $\square$

**Example 4.** Let  $\mathcal{S} = \langle 3, 7, 11 \rangle$  be a pseudo symmetric numerical semigroup. The gap poset of  $\mathcal{S}$  is  $\mathcal{G} = \{1, 2, 4, 5, 8\}$ . The corresponding graph  $J_{\mathcal{G}}(\mathcal{S})$  is given in Fig. 4. It is a connected graph with  $\text{diam}(J_{\mathcal{G}}(\mathcal{S})) = 3 = \text{gr}(J_{\mathcal{G}}(\mathcal{S}))$ .

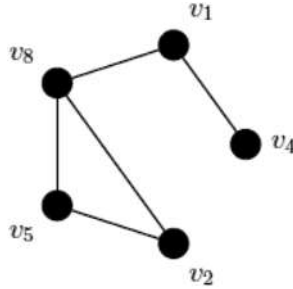


Fig. 4

**3. Conclusion** In this article, graphs associated with the gap poset have been discussed. We have examined some properties such as connectedness, completeness, girth and diameter of these graphs. Also, we showed that gap poset graphs are either bipartite graphs or graphs with girth 3.

In the future, we will enhance the characterization of these graphs by exploring some other graph-theoretic invariants, such as metric dimension, domination number, safe number, etc. Moreover, comparing gap poset graphs with other graph constructions may lead to further interesting discoveries on the interplay between graph theory and numerical semigroup theory.

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