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THE $\mathcal{NF}$-NUMBER OF TWO COMPLETE GRAPHS JOINED BY A COMMON VERTEX

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Abstract

Let Δ be a simplicial complex on the vertex set V. For $m = 1, 2, 3, \ldots$, the notion of m-th $\mathcal{NF}$-complex of Δ, $\delta^{(m)}_{\mathcal{NF}}(\Delta)$, was introduced by HIBI and MAHMOOD in [5], where $\delta^{(m)}_{\mathcal{NF}}(\Delta) = \delta_{\mathcal{NF}}(\delta^{(m-1)}_{\mathcal{NF}}(\Delta))$ with setting $\delta^{(1)}_{\mathcal{NF}}(\Delta) = \delta_{\mathcal{NF}}(\Delta)$ such that $\delta_{\mathcal{NF}}(\Delta)$ is the Stanley–Reisner complex of the facet ideal of Δ. The $\mathcal{NF}$-number of Δ is the least positive integer q for which $\delta^{(q)}_{\mathcal{NF}}(\Delta) \simeq \Delta$. In this paper, we investigated the $\mathcal{NF}$-number of two copies of complete graphs K_n joined by one common vertex $\{u\}$. At the end, we also provided an explicit example for the case of two copies of K_5 joined by common vertex $\{u\}$.

Key words: simplicial complex, graph, facet ideal, Stanley–Reisner ideal, minimal vertex cover

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1. Introduction. The Stanley–Reisner ideal of a simplicial complex [5] was introduced in 1974 by STANLEY [7] and REISNER [6] independently. On the other hand, VILLARREAL [8] introduced the notion of edge ideal in 1990, which was then generalized by FARIDI [1] in 2002 to the facet ideal of a simplicial complex.

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The readers are referred to [2] for more details on Stanley–Reisner theory. For a simplicial complex Δ, the Stanley–Reisner complex of the facet ideal of Δ is again a simplicial complex which was termed as the $\mathcal{NF}$-complex in [5]. The $\mathcal{NF}$-complex of Δ is denoted by $\delta_{\mathcal{NF}}(\Delta)$. Since this $\mathcal{NF}$-complex is again a simplicial complex on V, we can consider the following sequence of complexes arising from Δ:

$$\Delta,\ \delta_{\mathcal{NF}}(\Delta),\ \delta_{\mathcal{NF}}(\delta_{\mathcal{NF}}(\Delta)),\ \delta_{\mathcal{NF}}(\delta_{\mathcal{NF}}(\delta_{\mathcal{NF}}(\Delta))),\ \dots\ .$$

Now with the notation $\delta^{(k)}_{\mathcal{NF}}(\Delta) = \delta_{\mathcal{NF}}(\delta^{(k-1)}_{\mathcal{NF}}(\Delta))$ with setting $\delta^{(1)}_{\mathcal{NF}}(\Delta) = \delta_{\mathcal{NF}}(\Delta)$, as used in [5], the above sequence of complexes can be written as

$$\Delta,\ \delta^{(1)}_{\mathcal{NF}}(\Delta),\ \delta^{(2)}_{\mathcal{NF}}(\Delta),\ \delta^{(3)}_{\mathcal{NF}}(\Delta),\ \dots\ .$$

Since the vertex set V of all these $\mathcal{NF}$-complexes is finite, all the $\mathcal{NF}$-complexes in these sequences cannot be distinct as shown below.

Lemma 1.1 ([5], Lemma 1.4)**.** *Consider Δ to be the simplicial complex on m vertices. Then there exists a positive integer q with*

$$\delta^{(q)}_{\mathcal{NF}}(\Delta) = \Delta.$$

The smallest positive integer k for which $\delta^{(k)}_{\mathcal{NF}}(\Delta)$ is isomorphic to Δ is called $\mathcal{NF}$-number of Δ. In Table 1, the $\mathcal{NF}$-numbers of path graph and cyclic graph have been computed through Macaulay2 [3] upto 11 vertices. In this paper we have shown that the $\mathcal{NF}$-number of two copies of complete graph K_n joined by a common vertex is $2n+1$, Theorem 3.8. We proved our main Theorem 3.8 by investigating all the intermediate $\mathcal{NF}$-complexes from 1 to $2n$. We studied these intermediate complexes in some steps based on the behaviour of these complexes. These intermediate steps are included in Lemma 3.1, Proposition 3.4, Lemma 3.5, Proposition 3.6, and Proposition 3.7. At the end, we also show an explicit example for the case of two copies of K_5 joined by a common vertex $\{u\}$.

Table 1

n	$\Delta = P_n$	$\mathcal{NF}$-Number	$\Delta = C_n$	$\mathcal{NF}$-Number
2	P_2	4	C_2	–
3	P_3	5	C_3	4
4	P_4	2	C_4	2
5	P_5	16	C_5	2
6	P_6	48	C_6	12
7	P_7	47	C_7	8
8	P_8	552	C_8	26
9	P_9	27 833	C_9	139
10	P_{10}	38 471	C_{10}	284
11	P_{11}	290 815	C_{11}	196

2. Preliminaries. This section contains some basic definitions from the literature to make this paper self-contained.

Definition 2.1. Let k be a field and let I be a squarefree monomial ideal of the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$. The Stanley–Reisner complex of I, denoted by $\Delta_{\mathcal{N}}(I)$, is a simplicial complex on n vertices, such that any subset of V is a face of $\Delta_{\mathcal{N}}(I)$ if and only if the corresponding monomial $x_{i_1}x_{i_2}\ldots x_{i_s} \notin I$. The ideal I is then called the Stanley–Reisner ideal of the complex $\Delta_{\mathcal{N}}(I)$.

There is a nice relation between the minimal vertex covers of a simplicial complex and the minimal primary decomposition of $\mathcal{I}_{\mathcal{F}}(\Delta)$ given in [1], stated below. In this paper the set of all minimal vertex covers of Δ would be denoted by $\Gamma(\Delta)$.

Proposition 2.2. *Let V be any vertex set of a simplicial complex Δ. Any subset of V is called a minimal vertex cover of Δ iff the corresponding ideal of the polynomial ring R is a minimal prime ideal of $\mathcal{I}_{\mathcal{F}}(\Delta)$.*

Proposition 5.3.10 of [8] suggests a way of computing the Stanley–Reisner complex of a squarefree monomial ideal I as below:

Proposition 2.3. *Let k be a field, and I be a squarefree monomial ideal of the polynomial ring R. A minimal prime ideal of any subset $\{v_{i_1}, v_{i_2}, \ldots, v_{i_q}\}$ of $V = \{v_1, v_2, \ldots, v_n\}$ belongs to $\mathfrak{F}(\Delta_{\mathcal{N}}(I))$ if and only if the corresponding ideal is generated by $\{x_1, x_2, \ldots, x_n\} \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_q}\}$.*

Definition 2.4. Let V be a vertex set of a simplicial complex Δ, then any facet ideal of Δ as $\mathcal{NF}$-complex is called the Stanley–Reisner complex. Let us denote the $\mathcal{NF}$-complex of Δ by $\delta_{\mathcal{NF}}(\Delta)$. Also, for any $k \in \mathbb{N}$, we describe the k-th $\mathcal{NF}$-complex of Δ as $\delta^{(k)}_{\mathcal{NF}}(\Delta) = \delta_{\mathcal{NF}}\left(\delta^{(k-1)}_{\mathcal{NF}}(\Delta)\right)$; set $\delta^{(1)}_{\mathcal{NF}}(\Delta) = \delta_{\mathcal{NF}}(\Delta)$. The smallest positive integer $q > 0$ such that $\delta^{(q)}_{\mathcal{NF}}(\Delta) \simeq \Delta$ is called the $\mathcal{NF}$-number of Δ.

Proposition 2.5. *Let V be a vertex set of a simplicial complex Δ. Then for any $k \in \mathbb{N}$, we have $\delta^{(k)}_{\mathcal{NF}}(\Delta) = \left\langle V - M \colon M \in \Gamma\left(\delta^{(k-1)}_{\mathcal{NF}}(\Delta)\right)\right\rangle$.*

Proof. By definition, for any $k \in \mathbb{N}$, $\delta^{(k)}_{\mathcal{NF}}(\Delta)$ is the Stanley–Reisner complex of the facet ideal of the complex $\delta^{(k-1)}_{\mathcal{NF}}(\Delta)$. Proposition 2.3 tells that the minimal primary components of the facet ideal of $\delta^{(k-1)}_{\mathcal{NF}}(\Delta)$ is given by the minimal vertex covers of the complex $\delta^{(k-1)}_{\mathcal{NF}}(\Delta)$. Now we can employ Proposition 2.3 to bring about the conclusion. $\square$

3. The $\mathcal{NF}$-Number of two complete graphs joined by a common vertex. Let $n \in \mathbb{N}$ with $n \geq 3$; for any positive integer m, let $[m] = \{1, 2, \ldots, m\}$ and $\overline{[m]} = [m] \cup \{0\}$. Furthermore suppose that $V_1 = \{x_1, x_2, \ldots, x_{n-1}\}$ and $V_2 = \{y_1, y_2, \ldots, y_{n-1}\}$. Define a simplicial complex Δ on $V = V_1 \cup V_2 \cup \{u\}$, as follows: $\Delta = \langle \{x_i, x_j\}, \{y_i, y_j\}, \{x_i, u\}, \{y_i, u\} \colon i \neq j \in [n-1] \rangle$. Geometrically, this Δ represents copies of two complete graphs on $V_1 \cup \{u\}$ and $V_2 \cup \{u\}$, respectively, joined by common vertex $\{u\}$. Let us denote Δ by $K_n \bullet K_n$. The graph of $K_5 \bullet K_5$ is shown in Fig. 1.

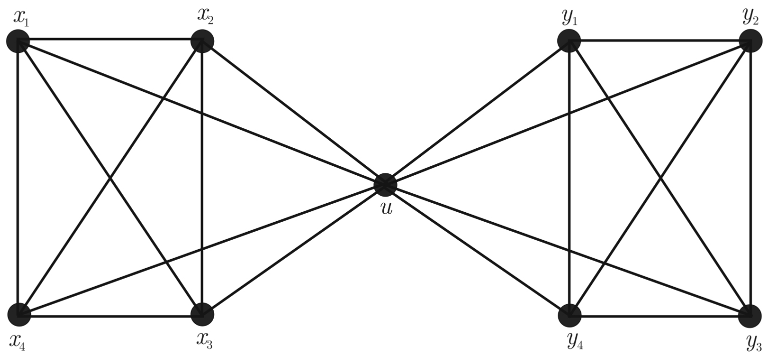

Fig. 1. $\Delta = K_5 \bullet K_5$

The following lemma computes the first three $\mathcal{NF}$-complexes of Δ.

Lemma 3.1. *For the simplicial complex $\Delta = K_n \bullet K_n$ on V, the following statements hold true:*

(i) $\delta^{(1)}_{\mathcal{NF}}(\Delta) = \langle \{u\}, \{x_i, y_j\} \colon i, j \in [n-1] \rangle$;

(ii) $\delta^{(2)}_{\mathcal{NF}}(\Delta) = \langle V - \{V_j \cup \{u\}\} \colon j \in \{1, 2\} \rangle$;

(iii) $\delta^{(3)}_{\mathcal{NF}}(\Delta) = \langle \{V \setminus \{x_i, y_j\}\} \colon i, j \in [n-1] \rangle$.

Proof. Let M be any minimal vertex cover of Δ. If $u \in M$, then we claim that M misses exactly one vertex from each V_i. If M misses two vertices from some V_i, then the edge formed by those two vertices will not be incident with any element of M. Therefore M will be of the form $\{u\} \cup (V_1 - \{x_i\}) \cup (V_2 - \{y_j\})$. If $u \notin M$, then M will contain both V_1 and V_2. Therefore, $M = V_1 \cup V_2$.

$$\Rightarrow \delta^{(1)}_{\mathcal{NF}}(\Delta) = \langle \{u\}, \{x_i, y_j\} \colon i, j \in [n-1] \rangle.$$

Now for the proof of (ii), take M to be any minimal vertex cover of $\delta^{(1)}_{\mathcal{NF}}(\Delta)$. Since $\{u\}$ is a facet of $\delta^{(1)}_{\mathcal{NF}}(\Delta)$, this means that $u \in M$. Note that the other facets of $\delta^{(1)}_{\mathcal{NF}}(\Delta)$ form a complete bipartite graph. This means that $M = V_1 \cup \{u\}$ or $M = V_2 \cup \{u\}$, as required.

For the proof of (iii), it is clear that any subset of the form $\{x_i, y_j\}$, where $i, j \in [n-1]$, is a minimal vertex cover of $\delta^{(2)}_{\mathcal{NF}}(\Delta)$, and all the minimal vertex covers of $\delta^{(2)}_{\mathcal{NF}}(\Delta)$ are of this form.

$$\Rightarrow \delta^{(3)}_{\mathcal{NF}}(\Delta) = \langle V - \{x_i, y_j\} \colon i, j \in [n-1] \rangle. \qquad \square$$

Before moving ahead, we would like to set two more notations to express special classes of subsets of V. We will use these frequently in the rest of this section. For $i, j \in \overline{[n-1]} = [n-1] \cup \{0\}$, let

$$\mathcal{M}_{(i,j,k)} = \{F \subset V \colon |F \cap V_1| = i, |F \cap V_2| = j, |F \cap \{u\}| = k\}$$

and

$$\mathcal{M}^c_{(i,j,k)} = \{V \setminus F \colon F \in \mathcal{M}_(i,j,k)\}.$$

In particular one has

$$\mathcal{M}^c_{(i,j,k)} = \mathcal{M}_{((n-1)-i,(n-1)-j,1-k)}.$$

The set $\mathcal{M}_{(i,j,k)}$ consists of all those subsets of V with cardinalities $i+j+k$ such that these sets contain i_j elements from V_j for each $j \in [n-1]$. Let $\mathcal{M}^c_{(i,j,k)} = \{V \setminus F \colon F \in \mathcal{M}_{(i,j,k)}\}$. Note that $\mathcal{M}^c_{(i,j,k)} = \mathcal{M}_{((n-1)-i,(n-1)-j,1-k)}$. Let us keep the following lemma handy for its repeated use in this section.

Lemma 3.2. *Let (i_1, i_2, k) and $(j_1, j_2, k) \in \overline{[n-1]} \times \overline{[n-1]} \times \{0, 1\}$ be two non-zero ordered 3-tuples, and let $F \in \mathcal{M}^c_{(i_1,i_2,k)}$. If there exists $t \in \{1, 2\}$ such that $j_t > i_t$ or $k = 1$ in G, then $G \cap F \neq \varphi$ for all $G \in \mathcal{M}_{(j_1,j_2,k)}$.*

Proof. Any set F in $\mathcal{M}^c_{(i_1,i_2,k)}$ has $(n-1)-i_1$ elements from V_1, $(n-1)-i_2$ elements from V_2 and $1-k$ elements from $\{u\}$. By definition; whereas, any set G of $\mathcal{M}_{(j_1,j_2,k)}$ has j_t elements of the same set V_t. Since $j_t > i_t$ or G has $k = 1$, where 1 represents common vertex $\{u\}$, so F and G have common vertex $\{u\}$ or by the pigeonhole principle, F and G have at least $j_t - i_t$ elements of V_t in common. □

Corollary 3.3. *Let (i_1, i_2, k) and $(j_1, j_2, k) \in \overline{[n-1]} \times \overline{[n-1]} \times \{0, 1\}$ be two non-zero ordered 3-tuples, of non-negative integers such that $i_1 + i_2 = m$ and $j_1 + j_2 = m + 1$ or H has common vertex $\{u\}$, then every subset of $\mathcal{M}_{(j_1,j_2,k)}$ intersect with all subsets of $\mathcal{M}^c_{(i_1,i_2,k)}$.*

Proof. Let $G \in \mathcal{M}_{(j_1,j_2,k)}$ as $j_1 + j_2 = m+1$. Now if $F \in \mathcal{M}^c_{(i_1,i_2,k)}$ and sum of i_1 and i_2 is m, it means that there would exist at least one t such that the t-th entry of $\mathcal{M}_{(j_1,j_2,k)}$ would be greater than its corresponding entry in $\mathcal{M}^c_{(i_1,i_2,k)}$ or G has common vertex $\{u\}$. Then the result follows from the above lemma. □

Proposition 3.4. *For the simplicial complex $\Delta = K_n \bullet K_n$ on V for any $k \in \mathbb{N}$ such that $4 \leq k \leq n+2$,*

$$\delta^{(k)}_{\mathcal{NF}}(\Delta) = \left\langle \mathcal{M}^c_{(k-2,0,0)} \cup \mathcal{M}^c_{(0,k-2,0)} \bigcup_{\substack{i_1+i_2=k-3\\ i_1,i_2\in[n-1]}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=k-4\\ i_1,i_2\in\overline{[n-1]}}} \mathcal{M}^c_{(i_1,i_2,1)} \right\rangle.$$

Proof. Let us prove it by induction. For $k = 4$, as $k - 3 = 1$, and because both $i_1, i_2 \geq 1$, the equation $i_1 + i_2 = 1$ has no solution. It means that the collection

$$\bigcup_{\substack{i_1+i_2=k-3\\ i_1,i_2\in[n-1]}} \mathcal{M}^c_{(i_1,i_2,0)}$$

is empty, so the first step requires us to show that

$$\delta^{(4)}_{\mathcal{NF}}(\Delta) = \left\langle \mathcal{M}^c_{(2,0,0)} \cup \mathcal{M}^c_{(0,2,0)} \cup \mathcal{M}^c_{(0,0,1)} \right\rangle.$$

By definition, $\delta^{(4)}_{\mathcal{NF}}(\Delta) = \delta_{\mathcal{NF}}(\delta^{(3)}_{\mathcal{NF}}(\Delta))$. From part (iii) of Lemma 3.1, we know that each facet of $\delta^{(3)}_{\mathcal{NF}}(\Delta)$ misses exactly one vertex from both V_1 and V_2. This means that any subset of V containing at least two elements either from V_1 or V_2 or has only common vertex u would intersect with every facet of $\delta^{(3)}_{\mathcal{NF}}(\Delta)$.

$$\Rightarrow \delta^{(4)}_{\mathcal{NF}}(\Delta) = \left\langle \mathcal{M}^c_{(2,0,0)} \cup \mathcal{M}^c_{(0,2,0)} \cup \mathcal{M}^c_{(0,0,1)} \right\rangle .$$

For the inductive step, let our formula be valid for some k between 4 and $n+2$. By Corollary 3.3, we can find

$$\delta^{(k+1)}_{\mathcal{NF}}(\Delta) = \left\langle \mathcal{M}^c_{(k-1,0,0)} \cup \mathcal{M}^c_{(0,k-1,0)} \bigcup_{\substack{i_1+i_2=k-2 \\ i_1,i_2\in[n-1]}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=k-3 \\ i_1,i_2\in\overline{[n-1]}}} \mathcal{M}^c_{(i_1,i_2,1)} \right\rangle . \quad \square$$

Lemma 3.5. *Let Δ be a simplicial complex $\Delta = K_n \bullet K_n$ on V. Then for $k = n+3$, we have*

$$\delta^{(n+3)}_{\mathcal{NF}}(\Delta) = \left\langle \mathcal{M}^c_{(n-1,0,0)} \cup \mathcal{M}^c_{(0,n-1,0)} \bigcup_{\substack{i_1+i_2=n \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=n-1 \\ i_1,i_2\in[n-1]\setminus\{n-1\}}} \mathcal{M}^c_{(i_1,i_2,1)} \right\rangle .$$

Proof. For $k = n+2$ the expression $\mathcal{M}^c_{(k-2,0,0)} \cup \mathcal{M}^c_{(0,k-2,0)}$ has no solution. Therefore, by Proposition 3.4, we have

$$\delta^{(n+2)}_{\mathcal{NF}}(\Delta) = \left\langle \bigcup_{\substack{i_1+i_2=n-1 \\ i_1,i_2\in[n-1]}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=n-2 \\ i_1,i_2\in\overline{[n-1]}}} \mathcal{M}^c_{(i_1,i_2,1)} \right\rangle .$$

Note that if $i_1 + i_2 = n-1 : i_1, i_2 \in [n-1]$ and $i_1 + i_2 = n-2 : i_1, i_2 \in \overline{[n-1]}$, then by Corollary 3.3, $\Gamma\left(\delta^{(n+2)}_{\mathcal{NF}}(\Delta)\right)$ is minimal vertex cover of $\delta^{(n+2)}_{\mathcal{NF}}(\Delta)$. Hence follows the result. $\square$

Proposition 3.6. *Consider the simplicial complex $\Delta = K_n \bullet K_n$ on V. Then for any $k \in \mathbb{N}$ such that $n+4 \leq k \leq 2n$, we have*

$$\delta^{(k)}_{\mathcal{NF}}(\Delta) = \left\langle \mathcal{M}^c_{(n-1,k-n-3,0)} \cup \mathcal{M}^c_{(k-n-3,n-1,0)} \cup \mathcal{M}^c_{(n-1,k-n-4,1)} \cup \mathcal{M}^c_{(k-n-4,n-1,1)} \bigcup_{\substack{i_1+i_2=k-3 \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=k-4 \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,1)} \right\rangle .$$

Proof. For $k = n+4$, $k-3 = n+1$ and $k-4 = n$. Let us first show that

$$\delta_{\mathcal{NF}}^{(n+4)}(\Delta) = \Big\langle \mathcal{M}^c_{(n-1,1,0)} \cup \mathcal{M}^c_{(1,n-1,0)} \cup \mathcal{M}^c_{(n-1,0,1)} \cup \mathcal{M}^c_{(0,n-1,1)} \bigcup_{\substack{i_1+i_2=n+1 \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=n \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,1)} \Big\rangle.$$

We need to prove that $\Gamma\left(\delta_{\mathcal{NF}}^{(n+3)}(\Delta)\right)$ is the minimal vertex cover for $\delta_{\mathcal{NF}}^{(n+3)}(\Delta)$. Since from Lemma 3.5,

$$\delta_{\mathcal{NF}}^{(n+3)}(\Delta) = \Big\langle \mathcal{M}^c_{(n-1,0,0)} \cup \mathcal{M}^c_{(0,n-1,0)} \bigcup_{\substack{i_1+i_2=n \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=n-1 \\ i_1,i_2\in[n-1]\setminus\{n-1\}}} \mathcal{M}^c_{(i_1,i_2,1)} \Big\rangle.$$

The desired result now immediately follows from Lemma 3.2 and Corollary 3.3.

Let it be true for some $n+4 < k < 2n$. By Corollary 3.3 and Lemma 3.2, we can easily see that,

$$\delta_{\mathcal{NF}}^{(k+1)}(\Delta) = \Big\langle \mathcal{M}^c_{(n-1,k-n-2,0)} \cup \mathcal{M}^c_{(k-n-2,n-1,0)} \cup \mathcal{M}^c_{(n-1,k-n-3,1)} \cup \mathcal{M}^c_{(k-n-3,n-1,1)} \bigcup_{\substack{i_1+i_2=k-2 \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,0)} \bigcup_{\substack{i_1+i_2=k-3 \\ i_1,i_2\in[n-1]\setminus\{1,n-1\}}} \mathcal{M}^c_{(i_1,i_2,1)} \Big\rangle. \quad \square$$

Proposition 3.7. *Let Δ ba a simplicial complex $\Delta = K_n \bullet K_n$ on V. Then for $k = 2n+1$, we have*

$$\delta_{\mathcal{NF}}^{(2n+1)}(\Delta) = \left\langle \mathcal{M}^c_{(n-1,n-2,0)} \cup \mathcal{M}^c_{(n-2,n-1,0)} \cup \mathcal{M}^c_{(n-1,n-3,1)} \cup \mathcal{M}^c_{(n-3,n-1,1)} \right\rangle.$$

Proof. For $k = 2n$ the equation $i_1 + i_2 = k-3 : i_1, i_2 \in [n-1] \setminus \{1, n-1\}$ has no solution. By Proposition 3.6, we have

$$\delta_{\mathcal{NF}}^{(2n)}(\Delta) = \Big\langle \mathcal{M}^c_{(n-1,n-3,0)} \cup \mathcal{M}^c_{(n-3,n-1,0)} \cup \mathcal{M}^c_{(n-1,n-4,1)} \cup \mathcal{M}^c_{(n-4,n-1,1)} \cup \mathcal{M}^c_{(n-2,n-2,1)} \Big\rangle.$$

Then by Lemma 3.2 the collections $\mathcal{M}_{(n-1,n-2,0)}$, $\mathcal{M}_{(n-2,n-1,0)}$, $\mathcal{M}_{(n-1,n-3,1)}$ and $\mathcal{M}_{(n-3,n-1,1)}$ would intersect with all the facets of $\delta_{\mathcal{NF}}^{(2n)}(\Delta)$. The result now follows. $\square$

Theorem 3.8. *For the simplicial complex* $\Delta = K_n \bullet K_n$ *on* V*, the* $\mathcal{NF}$*-number of* Δ *is* $2n+1$.

Proof. Note that, by Proposition 3.7, we have

$$\delta_{\mathcal{NF}}^{(2n+1)}(\Delta) = \langle \{\mathcal{M}_{(0,1,1)}\} \cup \{\mathcal{M}^c_{(1,0,1)}\} \cup \{\mathcal{M}^c_{(0,2,0)}\} \cup \{\mathcal{M}^c_{(2,0,0)}\}\rangle = \Delta.$$

Now from Lemma 3.1, Proposition 3.4, Lemma 3.5, Proposition 3.6, and Proposition 3.7, we have that

$$\dim \delta_{\mathcal{NF}}^{(1)}(\Delta) = 1, \quad \dim \delta_{\mathcal{NF}}^{(2)}(\Delta) = n-2, \quad \dim \delta_{\mathcal{NF}}^{(3)}(\Delta) = 2n-4,$$

and

$$\dim \delta_{\mathcal{NF}}^{(k)}(\Delta) = \begin{cases} 2n-k+1 & \text{if } 4 \le k \le n+2, \\ n-1 & \text{if } k = n+3, \\ 2n-k+3 & \text{if } n+4 \le k \le 2n, \\ 1 & \text{if } k = 2n+1. \end{cases}$$

Note that $\dim \delta_{\mathcal{NF}}^{(1)}(\Delta) = 1$ has isolated vertex $\{u\}$, and $\dim \delta_{\mathcal{NF}}^{(2n+1)}(\Delta) = 1$ has no isolated vertex. So this shows that $\delta_{\mathcal{NF}}^{(2n+1)}(\Delta)$ is not isomorphic to $\delta_{\mathcal{NF}}^{(k)}(\Delta)$ for any $k \in [2n]$. Thus the $\mathcal{NF}$-number of Δ is $2n+1$. $\square$

Last we would like to compute $\mathcal{NF}$-number of $K_5 \bullet K_5$ to support our above theory.

Table 2

$\delta_{\mathcal{NF}}^{(1)}(\Delta)$	$\langle\{x_1,y_1\},\{x_2,y_1\},\{x_3,y_1\},\{x_4,y_1\},\{x_1,y_2\},\{x_2,y_2\},\{x_3,y_2\},\{x_4,y_2\},$ $\{x_1,y_3\},\{x_2,y_3\},\{x_3,y_3\},\{x_4,y_3\},\{x_1,y_4\},\{x_2,y_4\},\{x_3,y_4\},\{x_4,y_4\},\{u\}\rangle.$
$\delta_{\mathcal{NF}}^{(2)}(\Delta)$	$\langle\{x_1,x_2,x_3,x_4\},\{y_1,y_2,y_3,y_4\}\rangle.$
$\delta_{\mathcal{NF}}^{(3)}(\Delta)$	$\langle\{x_2,x_3,x_4,y_2,y_3,y_4,u\},\{x_1,x_3,x_4,y_2,y_3,y_4,u\},\{x_1,x_2,x_4,y_2,y_3,y_4,u\},$ $\{x_1,x_2,x_3,y_2,y_3,y_4,u\},\{x_2,x_3,x_4,y_1,y_3,y_4,u\},\{x_1,x_3,x_4,y_1,y_3,y_4,u\},$ $\{x_1,x_2,x_4,y_1,y_3,y_4,u\},\{x_1,x_2,x_3,y_1,y_3,y_4,u\},\{x_2,x_3,x_4,y_1,y_2,y_4,u\},$ $\{x_1,x_3,x_4,y_1,y_2,y_4,u\},\{x_1,x_2,x_4,y_1,y_2,y_4,u\},\{x_1,x_2,x_3,y_1,y_2,y_4,u\},$ $\{x_2,x_3,x_4,y_1,y_2,y_3,u\},\{x_1,x_3,x_4,y_1,y_2,y_3,u\},\{x_1,x_2,x_4,y_1,y_2,y_3,u\},$ $\{x_1,x_2,x_3,y_1,y_2,y_3,u\}\rangle.$
$\delta_{\mathcal{NF}}^{(4)}(\Delta)$	$\langle\mathcal{M}^c_{(2,0,0)}\cup\mathcal{M}^c_{(0,2,0)}\cup\mathcal{M}^c_{(0,0,1)}\rangle.$
$\delta_{\mathcal{NF}}^{(5)}(\Delta)$	$\langle\mathcal{M}^c_{(3,0,0)}\cup\mathcal{M}^c_{(0,3,0)}\cup\mathcal{M}^c_{(1,1,0)}\cup\mathcal{M}^c_{(1,0,1)}\cup\mathcal{M}^c_{(0,1,1)}\rangle.$
$\delta_{\mathcal{NF}}^{(6)}(\Delta)$	$\langle\mathcal{M}^c_{(4,0,0)}\cup\mathcal{M}^c_{(0,4,0)}\cup\mathcal{M}^c_{(2,1,0)}\cup\mathcal{M}^c_{(1,2,0)}\cup\mathcal{M}^c_{(2,0,1)}\cup\mathcal{M}^c_{(0,2,1)}\cup\mathcal{M}^c_{(1,1,1)}\rangle.$
$\delta_{\mathcal{NF}}^{(7)}(\Delta)$	$\langle\mathcal{M}^c_{(3,1,0)}\cup\mathcal{M}^c_{(1,3,0)}\cup\mathcal{M}^c_{(2,2,0)}\cup\mathcal{M}^c_{(3,0,1)}\cup\mathcal{M}^c_{(0,3,1)}\cup\mathcal{M}^c_{(2,1,1)}\cup\mathcal{M}^c_{(1,2,1)}\rangle.$
$\delta_{\mathcal{NF}}^{(8)}(\Delta)$	$\langle\mathcal{M}^c_{(4,0,0)}\cup\mathcal{M}^c_{(0,4,0)}\cup\mathcal{M}^c_{(3,2,0)}\cup\mathcal{M}^c_{(2,3,0)}\cup\mathcal{M}^c_{(3,1,1)}\cup\mathcal{M}^c_{(1,3,1)}\cup\mathcal{M}^c_{(2,2,1)}\rangle.$
$\delta_{\mathcal{NF}}^{(9)}(\Delta)$	$\langle\mathcal{M}^c_{(4,1,0)}\cup\mathcal{M}^c_{(1,4,0)}\cup\mathcal{M}^c_{(3,3,0)}\cup\mathcal{M}^c_{(4,0,1)}\cup\mathcal{M}^c_{(0,4,1)}\cup\mathcal{M}^c_{(3,2,1)}\cup\mathcal{M}^c_{(2,3,1)}\rangle.$
$\delta_{\mathcal{NF}}^{(10)}(\Delta)$	$\langle\mathcal{M}^c_{(4,2,0)}\cup\mathcal{M}^c_{(2,4,0)}\cup\mathcal{M}^c_{(4,1,1)}\cup\mathcal{M}^c_{(1,4,1)}\cup\mathcal{M}^c_{(3,3,1)}\rangle.$
$\delta_{\mathcal{NF}}^{(11)}(\Delta)$	$\langle\mathcal{M}^c_{(4,3,0)}\cup\mathcal{M}^c_{(3,4,0)}\cup\mathcal{M}^c_{(4,2,1)}\cup\mathcal{M}^c_{(2,4,1)}\rangle = \Delta.$

Example 3.9. Let $V = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, u\}$; let $\Delta = K_5 \bullet K_5$, where the sets $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$ are vertices of each copy, and $\{u\}$ is a common vertex. This means that

$$\begin{aligned}\Delta = \langle &\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{y_1, y_2\}, \{y_1, y_3\},\\ &\{y_1, y_4\}, \{y_2, y_3\}, \{y_2, y_4\}, \{y_3, y_4\}, \{x_1, u\}, \{x_2, u\}, \{x_3, u\}, \{x_4, u\}, \{y_1, u\},\\ &\{y_2, u\}, \{y_3, u\}, \{y_4, u\}\rangle.\end{aligned}$$

Let us see that the $\mathcal{NF}$-number of $\Delta = K_5 \bullet K_5$ is $2 \cdot 5 + 1 = 11$. We illustrate all calculations in Table 2.

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