Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 75, No 5, 2022

MATHEMATICS
Algebra

NOTES ON 1-ABSORBING PRIME IDEALS

El Mehdi Bouba, Mohammed Tamekkante*, Ünsal Tekir,#, Suat Koç*****

Received on November 29, 2021
Presented by V. Drensky, Member of BAS, on January 27, 2022

Abstract

Let R be a commutative ring with a nonzero identity. A proper ideal I of R is said to be a 1-absorbing prime ideal if $xyz \in I$ for some nonunits $x, y, z \in R$, then $xy \in I$ or $z \in I$. It is well known that *prime ideal* $\Rightarrow$ 1-*absorbing prime ideal* $\Rightarrow$ *primary ideal* $\Rightarrow$ semi-primary ideal, that is, the class of 1-absorbing prime ideals comes between the classes of prime ideals and primary ideals. Also, the above right arrows are not reversible. In this article, we characterize rings over which every 1-absorbing prime ideal is prime and every primary ideal is 1-absorbing prime. Also, by comparing 1-absorbing prime ideals and other some classical ideals such as 2-absorbing ideals and semi-primary ideals, we characterize Noetherian divided rings and von Neumann regular rings.

Key words: primary ideal, 1-absorbing prime ideal, 1-absorbing primary ideal

2020 Mathematics Subject Classification: 13A15, 13A99

1. Introduction. Throughout this paper, we consider only commutative rings with nonzero identity. Let R always denote such a ring. The concept of prime ideals and its generalizations have a distinguished place in Commutative Algebra since not only they are used in the classification of commutative rings but also they have some applications to other areas such as Algebraic Geometry, General Topology, Cryptology and Graph Theory. See, for example, [15] and [20]. A proper ideal P of R is said to be a *prime ideal* if $xy \in P$ for some $x, y \in R$

#Corresponding author.
DOI:10.7546/CRABS.2022.05.01

implies that $x \in P$ or $y \in P$ [19]. The set of all prime ideals and maximal ideals of R will be designated by $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$, respectively. A ring R with $|\mathrm{Max}(R)| = 1$ is said to be a *local ring*. Otherwise, we say that R is non-local. For every proper ideal I of R, the radical of I is defined as $\sqrt{I} := \{x \in R \colon x^n \in I$ for some positive integer $n\}$. Note that $\sqrt{I}$ is the intersection of every prime ideal of R that contains I. In particular, $\sqrt{0}$ denotes the set of all nilpotent elements of R. A ring R is called a *reduced ring* if $\sqrt{0} = (0)$. Recall that a proper ideal Q of R is said to be a *primary ideal* if whenever $xy \in Q$ for some $x, y \in R$, then $x \in Q$ or $y \in \sqrt{Q}$. Note that if Q is a primary ideal, then $p = \sqrt{Q}$ is a prime ideal. The converse need not be true in general. For instance, let $R = k[X, Y]$, where k is a field, and $Q = (X^2, XY)$. Then $\sqrt{Q} = (X)$ is a prime ideal. Since $XY \in Q$, $X \notin Q$ and $Y \notin \sqrt{Q}$, it follows that Q is not a primary ideal of R. A proper ideal Q of R with prime radical is said to be a *semi-primary* ideal of R [12]. Thus we have the following irreversible right arrows of ideals:

$$\text{prime ideal} \;\Rightarrow\; \text{primary ideal} \;\Rightarrow\; \text{semi-primary ideal.}$$

A commutative ring R is called a π-*regular ring* if for each $a \in R$ there exist $x \in R$ and $n \in \mathbb{N}$ such that $a^n = a^{2n}x$. Note that the reversibility of the above first right arrow characterizes π-regular rings. Also, Gilmer [12] studied the reversibility of the above second right arrow in commutative rings with nonzero identity.

One of the important generalizations of prime ideal is 2-absorbing ideal which was first studied by Badawi in [5]. A proper ideal I of R is said to be a *2-absorbing ideal* if whenever $xyz \in I$ for some $x, y, z \in R$, then $xy \in I$ or $xz \in I$ or $yz \in I$. Note that if I is a 2-absorbing ideal of R, then $\sqrt{I}$ is a prime ideal or $\sqrt{I} = P \cap Q$, where P, Q are distinct prime ideals of R that are minimal over I ([5], Theorem 2.4). In a very recent study, Yassine et al. [13] defined a new class of ideals between prime ideals and 2-absorbing ideals as follows: a proper ideal I of R is said to be a *1-absorbing prime ideal* if whenever $xyz \in I$ for some nonunits $x, y, z \in R$, then $xy \in I$ or $z \in I$. From [11] and [13], we also have the following irreversible right arrows of ideals:

$$\text{prime ideal} \;\Rightarrow\; \text{1-absorbing prime ideal} \;\Rightarrow\; \text{2-absorbing ideal}$$
$$\text{prime ideal} \;\Rightarrow\; \text{1-absorbing prime ideal} \;\Rightarrow\; \text{primary ideal} \;\Rightarrow\; \text{semi-primary.}$$

The first goal of our paper is to study reversibility of the above right arrows in commutative rings with nonzero identity. Among many results in this paper, we show that every nonzero proper ideal of R is a 1-absorbing prime ideal if and only if either $R \cong k_1 \times k_2$, where k_1, k_2 are fields, or $(R, \mathfrak{m})$ is local with $\mathfrak{m} = \sqrt{0} \neq (0)$ such that $\mathfrak{m}^2 \subseteq (x)$ for every nonzero $x \in R$ (see Theorem 2). We also prove that in a Noetherian ring R, every primary (semi-primary) ideal is a 1-absorbing prime if and only if R is a von Neumann regular ring (i.e. its factor ring R/I is a reduced ring for every ideal I of R) or $(R, \mathfrak{m})$ is local with $\mathfrak{m}^2 = (0)$ (see Proposition 5).

We characterize rings in which every 2-absorbing ideal is a 1-absorbing prime ideal (see Proposition 6 and Theorem 4). A commutative ring R is said to be a *UN-ring* if $\sqrt{0}$ is a maximal ideal of R [10] and R is said to be a *divided ring* if its every prime ideal is comparable with principal ideals [7]. For more details on divided rings and UN-rings, we refer [16] and [17] to the reader. In particular we show that if R is a Noetherian ring, then every 2-absorbing ideal is a 1-absorbing prime ideal if and only if R is a UN-ring or R is a domain with unique nonzero prime ideal if and only if R is a divided ring (see Theorem 5).

2. Notes on 1-absorbing prime ideals. Recall from [8] that a proper ideal P of R is said to be a 2-prime ideal if whenever $xy \in P$ for some $x, y \in R$, then $x^2 \in P$ or $y^2 \in P$. It is easy to see that every 1-absorbing prime ideal is also a 2-prime ideal. To see this, choose $x, y \in R$ such that $xy \in P$. Then note that $x^2y \in P$ gives $x^2 \in P$ or $y^2 \in P$ since P is a 1-absorbing prime ideal. But the converse is not true in general. For instance, let $R = k[X, Y]$ and $P = (X^2, XY, Y^2)$, where k is a field. Then by Proposition 2.4 (2) in [14] P is a 2-prime ideal of R. Since $X(1 + Y)X \in P$ and $X(1 + Y)$, $X \notin P$, we have that P is not a 1-absorbing prime ideal of R.

In this section, we compare 1-absorbing prime ideals and other classical ideals such as prime, primary, semi-primary, 2-absorbing and 2-prime ideals.

Lemma 1. *Let R be a ring. The following statements are satisfied:*

1. *If I is a 1-absorbing prime ideal that is not a prime ideal, then $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m}^2 \subseteq I \subsetneq \mathfrak{m}$.*

2. *Every 1-absorbing prime ideal is primary.*

Proof. The proof of both statements 1. and 2. follows directly by combining Theorem 2.4 from [13] and Lemma 2.1 from [11]. □

Proposition 1. *Let R be a ring. Then, there exists a 1-absorbing prime ideal of R which is not prime if and only if R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 \neq \mathfrak{m}$.*

Proof. (⇒) Suppose that there exists a 1-absorbing prime ideal I of R which is not prime. Then, by Lemma 1, R is a local ring (with maximal ideal $\mathfrak{m}$) such that $\mathfrak{m}^2 \subseteq I \subsetneq \mathfrak{m}$. Therefore, we have the desired result.

(⇐) R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 \neq \mathfrak{m}$. Then, by the definition of 1-absorbing prime ideal, $\mathfrak{m}^2$ is 1-absorbing prime ideal which is not prime. Thus, R contains a 1-absorbing prime ideal which is not a prime ideal. □

Proposition 2. *Let R be a ring and let I and J be two 1-absorbing prime ideals of R that are not prime. Then $I \cap J$ and $I + J$ are 1-absorbing prime ideals of R.*

Proof. First by Lemma 1, R is local (with maximal ideal $\mathfrak{m}$) and $\mathfrak{m}^2 \subseteq I \cap J$. Let $xyz \in I \cap J$ for some nonunit elements $x, y, z \in R$. Since $x, y \in \mathfrak{m}$, then $xy \in \mathfrak{m}^2$ and so we have $xy \in I \cap J$. Thus, $I \cap J$ is a 1-absorbing prime ideal of R. Similarly, we prove that $I + J$ is a 1-absorbing prime ideal of R. □

Theorem 1. *Let R be a ring. Then, (0) is a 1-absorbing prime ideal of R if and only if R is a domain or $(R, \mathfrak{m})$ is a local ring such that $\mathfrak{m}^2 = (0)$.*

Proof. $(\Rightarrow)$ Suppose that (0) is a 1-absorbing prime ideal of R and R is not a domain. Then, (0) is a 1-absorbing prime ideal of R that is not a prime. Hence, by Lemma 1, R is a local ring with a maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 \subseteq (0)$. Thus, $(R, \mathfrak{m})$ is a local ring such that $\mathfrak{m}^2 = (0)$.

$(\Leftarrow)$ Clear. □

In [13], the authors give a partial characterization of rings in which every nonzero proper ideal is 1-absorbing prime ([13], Theorem 1.6). Now, in the following theorem, we give a complete characterization of this class of rings.

Theorem 2. *Let R be a ring. The following are equivalent:*

1. *Every nonzero proper ideal of R is 1-absorbing prime.*

2. *$R \cong k_1 \times k_2$, where k_1 and k_2 are fields or R is a local ring with maximal ideal $\mathfrak{m} = \sqrt{0} \neq (0)$ such that $\mathfrak{m}^2 \subseteq (x)$ for every nonzero $x \in \mathfrak{m}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is non-local. Then R has at least two maximal ideals. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two distinct maximal ideals of R. Assume that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$. Then by using [13], Theorem 2.4 and our hypothesis, we conclude that $\mathfrak{m}_1 \cap \mathfrak{m}_2$ is prime ideal of R. Thus $\mathfrak{m}_1 = \mathfrak{m}_2$, a contradiction. So $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, and then $R \cong R/\mathfrak{m}_1 \times R/\mathfrak{m}_2$. Consequently, R is isomorphic to a product of two fields. Now, suppose that R is local with maximal ideal $\mathfrak{m}$. Then, by our hypothesis every nonzero proper ideal of R is 2-absorbing. So, by [5], Theorem 3.4 we get the desired result.

(2) $\Rightarrow$ (1) If $R \cong k_1 \times k_2$, where k_1 and k_2 are fields, it clear that every nonzero proper ideal of R is prime and so is 1-absorbing prime. Now, suppose that R is a local ring with maximal ideal $\mathfrak{m} = \sqrt{0} \neq (0)$ such that $\mathfrak{m}^2 \subseteq (x)$ for every nonzero $x \in \mathfrak{m}$. Let I be a nonzero proper ideal of R. Then, there exists $x \in I$ such that $x \in \mathfrak{m} \setminus \{0\}$. Thus, $\mathfrak{m}^2 \subseteq (x) \subseteq I$. Now, let $abc \in I$ for some nonunit elements $a, b, c \in R$. Then, $ab \in I$ since $ab \in \mathfrak{m}^2 \subseteq I$, and so I is 1-absorbing prime of R. Thus, we conclude that every nonzero proper ideal of R is 1-absorbing prime. □

Recall that a ring R is said to be a von Neumann regular ring if for each $a \in R$ there exists $x \in R$ such that $a = a^2x$ [18]. Note that by [3] R is a von Neumann regular ring if and only if $I = \sqrt{I}$ for every ideal I of R, or equivalently R/I is a reduced ring.

Proposition 3. *Let R be a von Neumann regular ring and I be an ideal of R. Then the following statements are equivalent:*

1. *I is a 1-absorbing prime ideal of R.*

2. *I is a (semi)-primary ideal of R.*

3. *I is a prime ideal of R.*

Proof. (1) $\Rightarrow$ (2) Follows from Lemma 1.

(2) $\Rightarrow$ (3) Since R is a von Neumann regular ring, then $I = \sqrt{I}$ (by Theorem 3.1, p. 5 [3]), and so I is a prime ideal of R.

(3) $\Rightarrow$ (1) Clear. $\square$

Proposition 4. *Let $(R, \mathfrak{m})$ be a local Noetherian ring. Then every 1-absorbing prime ideal of R is prime if and only if R is a field.*

Proof. ($\Rightarrow$) Since R is a local ring with a maximal ideal $\mathfrak{m}$, then it is easy to see that $\mathfrak{m}^2$ is a 1-absorbing prime ideal of R. Hence, by hypothesis $\mathfrak{m}^2$ is a prime ideal of R, and so $\mathfrak{m}^2 = \mathfrak{m}$. So, by applying Nakayama's Lemma we conclude that $\mathfrak{m} = (0)$. Thus, R is a field.

($\Leftarrow$) Clear. $\square$

Remark 1. Consider the ring $R := \mathbb{Z}/4\mathbb{Z}$. Then, R is a local Noetherian ring over which the zero ideal $P = (\overline{0})$ is a 1-absorbing prime ideal which is not prime.

Proposition 5. *Let R be a Noetherian ring. Then, the following statements are equivalent:*

1. *Every semi-primary ideal is 1-absorbing prime.*
2. *Every primary ideal is 1-absorbing prime.*
3. *R is a von Neumann regular ring or $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m}^2 = 0$.*

Proof. (1) $\Rightarrow$ (2) Is trivial.

(2) $\Rightarrow$ (3) Suppose that R is a non-local ring. Then, by [13], Theorem 2.4 every primary ideal of R is prime. Thus, by [1], Theorem 3.1 R is a Noetherian von Neumann regular ring. Now, suppose that R is a local ring with maximal ideal $\mathfrak{m}$. Since $\mathfrak{m}^3$ is a primary ideal of R, then it is 1-absorbing prime. Hence $\mathfrak{m}^3 = \mathfrak{m}^2$, and so by Nakayama's Lemma we conclude that $\mathfrak{m}^2 = (0)$. Thus, R is a local ring with maximal ideal $\mathfrak{m} = \sqrt{0}$ such that $\mathfrak{m}^2 = 0$.

(3) $\Rightarrow$ (1) The result follows directly from Proposition 3 and [11], Proposition 4.5. $\square$

Theorem 3. *Let R be a ring. Then, the following statements are equivalent:*

1. *Every semi-primary ideal is 1-absorbing prime.*
2. *R is a von Neumann regular ring or $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m}^2 = 0$.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is a non-local ring. Then, by [13], Theorem 2.4 every semi-primary ideal of R is prime, and so every primary ideal of R is prime. Hence, R is a von Neumann regular ring (by [1], Theorem 3.1). Now, suppose that R is a local ring with maximal ideal $\mathfrak{m}$. Then by using [1], Theorem 2.3 R has at most two prime ideals which are $\sqrt{0}$ and $\mathfrak{m}$. So, let $x \in \mathfrak{m}$. Then, $\sqrt{(x)} = \mathfrak{m}$ or $\sqrt{(x)} = \sqrt{0}$, and so (x) is a semi-primary ideal of R. Hence, by using our hypothesis the ideal (x) is a 1-absorbing prime. Thus, every principal proper ideal is 1-absorbing prime. Hence, by [11], Proposition 4.5, R is a local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{m}^2 = (0)$.

(2) $\Rightarrow$ (1) The result follows directly from Proposition 3 and [11], Proposition 4.5. $\square$

Corollary 1. *Let $(R, \mathfrak{m})$ be a local ring. Then every semi-primary ideal of R is 1-absorbing prime if and only if $\mathfrak{m}^2 = (0)$.*

Proof. ($\Rightarrow$) Since $(R, \mathfrak{m})$ is a local ring, then $R \cong R_{\mathfrak{m}}$. Then, by Theorem 3 and [3], Theorem 3.1, p. 5, we conclude that R is a field or $\mathfrak{m}^2 = 0$. Thus, $\mathfrak{m}^2 = (0)$.

($\Leftarrow$) This follows from Theorem 3. $\square$

Corollary 2. *Let R be a reduced ring. Then every semi-primary ideal of R is 1-absorbing prime if and only if R is a von Neumann regular ring.*

Recall from [4] that a proper ideal I of R is said to be a 2-*absorbing primary ideal* if whenever $abc \in I$ for some $a, b, c \in R$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Proposition 6. *Let R be a ring. Then, the following statements are equivalent:*

1. *Every 2-absorbing primary ideal is 1-absorbing prime.*

2. *$(R, \mathfrak{m})$ is a local ring and every semi-primary ideal is 1-absorbing prime.*

3. *$(R, \mathfrak{m})$ is a local ring with $\mathfrak{m}^2 = 0$.*

Proof. (1) $\Rightarrow$ (2) Let P and Q be two prime ideals of R. Since $P \cap Q$ is 2-absorbing (primary) ideal, we get that $\sqrt{P \cap Q} = P \cap Q$ is prime (by [13], Theorem 2.3), and so $P \subseteq Q$ or $Q \subseteq P$. Thus, prime ideals of R are comparable, as desired. Accordingly, R is local. On the other hand, every semi-primary ideal of R is 1-absorbing prime, since every semi primary ideal is 2-absorbing primary (see [4], Theorem 2.8).

(2) $\Rightarrow$ (3) Follows from Corollary 1.

(3) $\Rightarrow$ (1) This follows from [11], Proposition 4.5. $\square$

Theorem 4. *Let R be a ring. Then, every 2-absorbing ideal of R is 1-absorbing prime if and only if the two following conditions hold:*

1. *Prime ideals of R are comparable; in particular, R is local with maximal ideal $\mathfrak{m}$; and*

2. *If P is a minimal prime over a 2-absorbing ideal I, then $P = I$ or $P = \mathfrak{m}$.*

Proof. ($\Rightarrow$) 1. Let P and Q be two prime ideals of R. Since $P \cap Q$ is 2-absorbing ideal, we get that $\sqrt{P \cap Q} = P \cap Q$ is prime (by [13], Theorem 2.3), and so $P \subseteq Q$ or $Q \subseteq P$. Thus, prime ideals of R are comparable, as desired. Accordingly, R is local with maximal ideal $\mathfrak{m}$.

2. Let P be a minimal prime over a 2-absorbing ideal I. If I is prime, then $P = I$. If I is not prime ideal of R, then by Lemma 1 we have $\mathfrak{m}^2 \subseteq I \subseteq P$. Consequently, $P = \mathfrak{m}$.

(⇒) Let I be a 2-absorbing ideal of R. Since the prime ideals of R are comparable, we have $P^2 \subseteq I \subseteq P$, where P is a minimal prime over a 2-absorbing ideal I (by [5], Theorem 2.4). If $P = I$, we are done. If $P = \mathfrak{m}$, then it easy to prove that I is 1-absorbing prime since $\mathfrak{m}^2 \subseteq I$ and all nonunit elements of R are contained in $\mathfrak{m}$. □

Theorem 5. *Let R be a Noetherian ring. Then, the following statements are equivalent:*

1. *Every 2-absorbing ideal of R is 1-absorbing prime.*

2. *R is UN-ring or R is a domain with unique nonzero prime ideal.*

3. *R is a divided ring.*

Proof. (1) ⇒ (2) By Theorem 4, R is local with maximal ideal $\mathfrak{m}$. Suppose that R is not a UN-ring and let P be a non-maximal prime ideal of R. By using [9], Lemma 2.2, $P\mathfrak{m}$ is a 2-absorbing ideal of R, and so is 1-absorbing prime ideal. If $P\mathfrak{m}$ is not prime ideal, then by Lemma 1 we have $\mathfrak{m}^2 \subseteq P\mathfrak{m}$, and so $P = \mathfrak{m}$, a contradiction. Hence, $P\mathfrak{m}$ is prime ideal of R, and so $P\mathfrak{m} = P$. Since R is Noetherian, by Nakayama's Lemma, we get $P = (0)$. Thus, R is a domain and $\mathfrak{m}$ is the unique nonzero prime ideal of R.

(2) ⇒ (3) First, note that in both cases R is local with maximal ideal $\mathfrak{m}$. If R is a UN-ring, then R has exactly one prime ideal which is $\mathfrak{m}$, and so R is a divided ring. Now suppose R is a domain with unique nonzero prime ideal. Then, R has exactly two prime ideals which are (0) and $\mathfrak{m}$. Clearly, (0) and $\mathfrak{m}$ are comparable with any other principal ideal of R. Consequently, R is a divided ring.

(3) ⇒ (1) Suppose that R is a divided ring (R is local with maximal ideal $\mathfrak{m}$) and let I be a 2-absorbing ideal of R. Then, by [5], Theorem 2.4, $\sqrt{I} = P$ is a prime ideal of R such that $P^2 \subseteq I$. By using [13], Theorem 2.15, P^2 is a 1-absorbing prime ideal of R. If P^2 is not a prime ideal of R, then $\mathfrak{m}^2 = P^2 \subseteq I$, and so I is a 1-absorbing prime of R. Otherwise, P^2 is a prime ideal of R, and so $P^2 = P$. Since R is Noetherian, P is finitely genetared. Then, by [19], Theorem 1.8.22, P is a principal and generated by an idempotent element. Thus, $P = (0)$, and then $I = P = (0)$ is a (1-absorbing) prime ideal R. □

A ring R is said to be a *2-P ring* if its every 2-prime ideal is prime [9]. We know that every 1-absorbing prime ideal is a 2-prime ideal and the converse is not true. Now, we investigate rings over which every 2-prime ideal is a 1-absorbing prime ideal.

Theorem 6. *Let R be a Noetherian ring. If every 2-prime ideal of R is 1-absorbing prime ideal, then one of the following holds:*

1. *R is 2-P ring.*

2. *R is UN-ring.*

3. *R is a domain with unique nonzero prime ideal.*

Proof. If R is non-local ring, then by using [13], Theorem 2.3 and our hypothesis every 2-prime ideal of R is prime ideal. Thus, R is 2-P ring. Now suppose that R is a local (with maximal ideal $\mathfrak{m}$) but not a UN-ring. Let P be a non-maximal prime ideal of R. Then, $P\mathfrak{m}$ is a 1-absorbing prime ideal of R, since $P\mathfrak{m}$ is a 2-prime ideal (by [14], Lemma 3.2). If $P\mathfrak{m}$ is not prime ideal, then by Lemma 1 we have $\mathfrak{m}^2 \subseteq P\mathfrak{m}$, and so $P = \mathfrak{m}$, a contradiction. Hence, $P\mathfrak{m}$ is prime ideal of R, and so $P\mathfrak{m} = P$. Since R is Noetherian, by Nakayama's Lemma, we get $P = (0)$. Thus, R is a domain and $\mathfrak{m}$ is the unique nonzero prime ideal of R. □

REFERENCES

[1] Almahdi F., M. Tamekkante, A. Mamouni (2020) Rings over which every semi-primary ideal is 1-absorbing primary, Commun. Algebra, **48**(9), 3838–3845.

[2] Anderson D. D., E. Smith (2003) Weakly prime ideals, Houston J. Math., **29**(4), 831–840.

[3] Anderson D. F., D. E. Dobbs (1995) Zero-Dimensional Commutative Rings, Lecture Notes in Pure and Applied Mathematics, **171**, CRC Press, 400 pp.

[4] Badawi A., U. Tekir, E. Yetkin (2014) On 2-absorbing primary ideals in commutative rings, Bull. Korean Math. Soc., **51**(4), 1163–1173.

[5] Badawi A. (2007) On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc., **75**(3), 417–429.

[6] Badawi A., E. Yetkin (2020) On 1-absorbing primary ideals of commutative rings, J. Algebra Appl., **19**(06), 2050111.

[7] Badawi A. (1999) On divided commutative rings, Commun. Algebra, **27**(3), 1465–1474.

[8] Beddani C., W. Messirdi (2016) 2-prime ideals and their applications, J. Algebra Appl., **15**(03), 1650051.

[9] Bennis D., B. Fahid (2018) Rings in which every 2-absorbing ideal is prime, Beitr. Algebra Geom., **59**(2), 391–396.

[10] Călugăreanu G. (2016) UN-rings, J. Algebra Appl., **15**(10), 1650182.

[11] El Khalfi A., M. Issoual, N. Mahdou, A. Reinhart (2021) Commutative rings with one-absorbing factorization, Commun. Algebra, **49**(6), 2689–2703.

[12] Gilmer R. W. (1962) Rings in which semi-primary ideals are primary, Pac. J. Math., **12**(4), 1273–1276.

[13] Yassine A., M. J. Nikmehr, R. Nikandish (2021) On 1-absorbing prime ideals of commutative rings, J. Algebra Appl., **20**(10), 2150175.

[14] Nikandish R., M. J. Nikmehr, A. Yassine (2020) More on the 2-prime ideals of commutative rings, Bull. Korean Math. Soc., **57**(1), 117–126.

[15] Redmond S. P. (2003) An ideal-based zero-divisor graph of a commutative ring, Commun. Algebra, **31**(9), 4425–4443.

[16] Tekir Ü., S. Koç, K. H. Oral (2017) n-ideals of commutative rings, Filomat, **31**(10), 2933–2941.

[17] Tekir Ü., G. Ulucak, S. Koç (2020) On divided modules, Iran. J. Sci. Technol. Trans. A Sci., **44**(1), 265–272.
[18] Von Neumann J. (1936) On regular rings, PNAS, **22**(12), 707–713.
[19] Wang F., H. Kim (2016) Foundations of Commutative Rings and Their Modules, Singapore, Springer Nature Singapore Pte Ltd.
[20] Yildiz E., B. A. Ersoy, Ü. Tekir, S. Koç (2021) On S-Zariski topology, Commun. Algebra, **49**(3), 1212–1224.

Team of Modeling and Scientific Computing
Mathematics Department
Pluridisciplinary Faculty
Mohammed First University
B.P. 300, Selouane, Nador 62700, Morocco
e-mail: mehdi8bouba@hotmail.fr

*Department of Mathematics
Faculty of Science
University Moulay Ismail Meknes
Box 11201, Zitoune, Morocco
e-mail: tamekkante@yahoo.fr

**Department of Mathematics
Marmara University
Istanbul, Turkey
e-mail: utekir@marmara.edu.tr

***Department of Mathematics
Istanbul Medeniyet University
Istanbul, Turkey
e-mail: suat.koc@medeniyet.edu.tr