

MATHEMATICS

Complex analysis

A LOWER BOUND ESTIMATE FOR THE UNIFORM NORM OF A LINEAR COMBINATION OF COMPLEX POLYNOMIALS

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Abstract

In this paper, we present lower bound estimate of the linear combination of the complex polynomials in terms of the moduli of two coefficients of the linear combination of the polynomials provided their indices $s < t$ obey a suitable restriction and we also show that this assumption cannot be removed or weakened.

Key words: polynomial, uniform norm, coefficient

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1. Introduction. The well-known Russian mathematician Chebyshev (1821–1894) studied some properties of polynomials with the least deviation from a given continuous function and introduced the concept of best approximation in mathematical analysis. In the geometric function theory, there are various studies which deal with the dependence between the norm of polynomials and its coefficients. More recently, the maximum norms of polynomials of lower degree over various domains (e.g., sector, triangle) have been investigated (see, for example, [1–3], etc).

Let $p(z) = \sum_{j=0}^d a_j z^j$ be a polynomial of degree d in the complex plane and we shall denote the uniform norm, $\max_{|z|=1} |p(z)| = \|p\|_\infty$ throughout this paper. For

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polynomial of degree 1, simply $p(z) = a_0 + a_1z$ and hence, we can easily evaluate $\|p\|_\infty = |a_0| + |a_1|$. So, we need not find the estimate of $\|p\|_\infty$ as its exact value is readily obtainable, and in the theorems proved in this paper, we particularly mean polynomials of degree ≥ 2 .

2. Main results. In this paper, we first prove the following interesting result which gives the lower bound estimates of the uniform norm of the linear combination $F(z)$ of two complex polynomials in terms of two arbitrary coefficients say A_s and A_t of $F(z)$ under certain assumption on the pair $s < t$ which is a generalization of a result due to DUBICKAS [4]. In fact, we prove

Theorem 2.1. Let $T(z) = p(z) + \lambda q(z) = \sum_{j=0}^d A_j z^j$, where

$$(2.1) \quad A_j = \begin{cases} a_j + \lambda b_j, & 0 \leq j \leq d' \\ a_j, & d' < j \leq d, \end{cases}$$

$p(z) = \sum_{j=0}^d a_j z^j$ and $q(z) = \sum_{j=0}^{d'} b_j z^j$ are polynomials of degree d and d' , respectively, with $d \geq d'$, and λ is a complex number. Let also s, t be two integers satisfying $0 \leq s < t \leq d$ and

$$(2.2) \quad s \leq \min \left(\frac{t-1}{2}, 2t-d-1 \right).$$

Then for $r > 0$

$$(2.3) \quad r^{\frac{(t-2s)}{n}} \|T(r^{1/n}z)\|_\infty \geq r|A_s| + r^2|A_t|,$$

where $n = t - s$ throughout the paper.

Remark 2.1. Theorem 2.1 can be readily extended for the linear combination of a finite number of polynomials.

Remark 2.2. Taking $r = 1$, we have the following interesting result.

Corollary 2.1. Let $T(z) = p(z) + \lambda q(z) = \sum_{j=0}^d A_j z^j$, where

$$(2.4) \quad A_j = \begin{cases} a_j + \lambda b_j, & 0 \leq j \leq d' \\ a_j, & d' < j \leq d, \end{cases}$$

$p(z) = \sum_{j=0}^d a_j z^j$ and $q(z) = \sum_{j=0}^{d'} b_j z^j$ are polynomials of degree d and d' , respectively, with $d \geq d'$, and λ is a complex number. Let also s, t be two integers satisfying $0 \leq s < t \leq d$ and

$$(2.5) \quad s \leq \min \left(\frac{t-1}{2}, 2t-d-1 \right).$$

Then

$$(2.6) \quad \|T\|_{\infty} \geq |A_s| + |A_t|.$$

Remark 2.3. Letting $\lambda = 0$ in Theorem 2.1, we get a generalization of a result due to Dubickas [4].

Corollary 2.2. Let $p(z) = \sum_{j=0}^d a_j z^j$ be a polynomial of degree d and let s, t be two integers satisfying $0 \leq s < t \leq d$ and inequality (2.2). Then for $r > 0$

$$(2.7) \quad r^{\frac{(t-2s)}{n}} \|p(r^{1/n} z)\|_{\infty} \geq r|a_s| + r^2|a_t|.$$

Remark 2.4. Setting $r = 1$, $s = 0$ and $t = d$ in Corollary 2.2, we get the following celebrated result due to VISSER [5, Theorem 3].

Corollary 2.3. Let $p(z) = \sum_{j=0}^d a_j z^j$ be a polynomial of degree d . Then

$$(2.8) \quad \|p\|_{\infty} \geq |a_0| + |a_d|.$$

Remark 2.5. Furthermore, an interesting result concerning the bound of the derivative of a self-inversive polynomial $p(z)$ of degree d could be obtained in terms of two arbitrary coefficients with certain restriction, as an application of Corollary 2.2:

For a self-inversive polynomial $p(z) = \sum_{j=0}^d a_j z^j$, we have

$$(2.9) \quad \|p'\|_{\infty} = \frac{d}{2} \|p\|_{\infty}.$$

If s, t are two integers satisfying $0 \leq s < t \leq d$ and $s \leq \min\left(\frac{t-1}{2}, 2t-d-1\right)$, then applying inequality (2.7) of Corollary 2.2 for $r = 1$ in (2.9), we have

$$(2.10) \quad \|p'\|_{\infty} \geq \frac{d}{2} (|a_s| + |a_t|).$$

Further, we are able to prove the following result by reversing the assumption on s and t given by inequality (2.5) of Corollary 2.1 and it turns out that condition (2.5) on $s < t$ cannot be removed or even weakened. More precisely, in the next theorem, under assumption of reverse inequality, for any non-zero $A_s, A_t \in \mathbb{C}$, we will construct polynomials for which (2.5) does not hold.

Theorem 2.2. For any integers s, t and $d \geq 1$ satisfying $0 \leq s < t \leq d$ and

$$(2.11) \quad s > \min\left(\frac{t-1}{2}, 2t-d-1\right)$$

and any non-zero complex numbers A_s, A_t , there exists a polynomial $Q(z) = \sum_{j=0}^d B_j z^j$ and $(B_s, B_t) = (A_s, A_t)$, such that

$$(2.12) \quad \|Q\|_{\infty} < |A_s| + |A_t|.$$

3. Proof of the Theorems. Proof of Theorem 2.1. Let $T(z) = p(z) +$

$\lambda q(z) = \sum_{j=0}^d A_j z^j$, where

$$(3.1) \quad A_j = \begin{cases} a_j + \lambda b_j, & 0 \leq j \leq d' \\ a_j, & d' < j \leq d, \end{cases}$$

$p(z)$ and $q(z)$ are polynomials of degree d and d' , respectively, with $d \geq d'$, and λ is a complex number. Let s and t be two integers satisfying $0 \leq s < t \leq d$ and inequality (2.2). Let $n = t - s$ and $m = t - 2s$. Suppose $\psi_k = r^{1/n} \xi_k$, where $\xi_k = e^{2\pi i(k-1)/n}$ for $k = 1, 2, \dots, n$. Take $\theta = e^{ix_0}$, where

$$(3.2) \quad x_0 n = x_0(t - s) = \arg \left(\frac{A_s}{A_t} \right).$$

Consider

$$(3.3) \quad S = \sum_{k=1}^n \psi_k^m T(\theta \psi_k) = \sum_{k=1}^n r^{\frac{m+j}{n}} \xi_k^m \sum_{j=1}^d A_j \theta^j \xi_k^j = \sum_{j=1}^d r^{\frac{t-2s+j}{n}} A_j \theta^j \sum_{k=1}^n \xi_k^{t-2s+j}.$$

Note that

$$\sum_{k=1}^n \xi_k^l = \begin{cases} n & \text{if } n \text{ divides } l \\ 0 & \text{if } n \text{ does not divide } l, \end{cases}$$

which is the well-known property of roots of unity.

From inequality (2.2), it is noted that $t - 2s > 0$ and $t - 2s + d < 3n$. Therefore, out of $d + 1$ consecutive integers $t - 2s + j$, where $j = 0, 1, 2, \dots, d$, there are only two multiples of $n = t - s$, namely n and $2n$ corresponding to $j = s$ and $j = t$ and as a result of this, (3.3) becomes

$$S = n \{ r A_s \theta^s + r^2 A_t \theta^t \} = n \theta^s A_t \left\{ r \frac{A_s}{A_t} + r^2 \theta^{t-s} \right\},$$

from which

$$(3.4) \quad |S| = n |A_t| \left| r \frac{A_s}{A_t} + r^2 \theta^{t-s} \right|.$$

Because of inequality (3.2), the numbers $r \frac{A_s}{A_t}$ and $r^2 \theta^{t-s}$ have the same arguments and hence

$$(3.5) \quad \left| r \frac{A_s}{A_t} + r^2 \theta^{t-s} \right| = r \left| \frac{A_s}{A_t} \right| + r^2.$$

Using (3.5), (3.4) yields

$$(3.6) \quad |S| = n \{r|A_s| + r^2|A_t|\}.$$

From (3.3),

$$|S| = \left| \sum_{k=1}^n \psi_k^m T(\theta \psi_k) \right| \leq \sum_{k=1}^n |r^{\frac{t-2s}{n}}| \left| T(r^{\frac{1}{n}} \theta \xi_k) \right| \leq n r^{\frac{t-2s}{n}} \|T(r^{\frac{1}{n}} \theta \xi_k)\|_{\infty},$$

which in conjunction with (3.6), yields

$$(3.7) \quad r^{\frac{t-2s}{n}} \|T(r^{\frac{1}{n}} z)\|_{\infty} \geq r|A_s| + r^2|A_t|,$$

which completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. Let s and t be two integers satisfying $0 \leq s < t \leq d$ and

$$(3.8) \quad s > \min \left(\frac{t-1}{2}, 2t-d-1 \right).$$

From condition (3.8), we have either $2s \geq t$ in case $2t-d-1 \geq \frac{t-1}{2}$ or $s \geq 2t-d$ in case $2t-d-1 \leq \frac{t-1}{2}$. When $2s \geq t$, we have

$$(3.9) \quad 0 \leq 2s-t < s < t \leq d,$$

and for $s \geq 2t-d$,

$$(3.10) \quad 0 \leq s < t < 2t-s \leq d.$$

If possible, suppose inequality (3.10) is true. In order to construct the polynomial

$Q(z) = \sum_{j=0}^d B_j z^j$ satisfying property (2.12), we make $B_s = A_s$, $B_t = A_t$.

$$(3.11) \quad B_{2t-s} = \frac{-\delta}{(4+\delta)} e^{2i\phi} A_s,$$

where

$$(3.12) \quad \frac{A_t}{A_s} = \delta e^{i\phi}$$

with $\phi \in [0, 2\pi)$ and take other B_j , $j \notin \{s, t, 2t-s\}$, arbitrarily in the disc $|z| \leq \epsilon/d$, where ϵ is a fixed positive number.

On $|z| = 1$ for each Q , we have

$$\begin{aligned} |Q(z)| &\leq |A_s z^s + A_t z^t + B_{2t-s} z^{2t-s}| + |B_0| + |B_1 z| + \cdots + |B_d z^d| \\ &< |A_s z^s + A_t z^t + B_{2t-s} z^{2t-s}| + \frac{d\epsilon}{d}, \end{aligned}$$

from which, we have

$$(3.13) \quad \|Q\|_{\infty} < \|A_s z^s + A_t z^t + B_{2t-s} z^{2t-s}\|_{\infty} + \epsilon.$$

Now,

$$(3.14) \quad \|A_s z^s + A_t z^t + B_{2t-s} z^{2t-s}\|_{\infty} \\ = |A_s| \left\| 1 + \frac{A_t}{A_s} z^{t-s} + \frac{B_{2t-s}}{A_s} z^{2t-2s} \right\|_{\infty} = |A_s| \left\| 1 + \frac{A_t}{A_s} z + \frac{B_{2t-s}}{A_s} z^2 \right\|_{\infty}.$$

Using (3.11) and (3.12) to (3.14), we have

$$(3.15) \quad \|A_s z^s + A_t z^t + b_{2t-s} z^{2t-s}\|_{\infty} = |A_s| \left\| 1 + \delta e^{i\phi} z - \frac{\delta}{4+\delta} e^{2i\phi} z^2 \right\|_{\infty}.$$

Let $F_c(z) = 1 + \delta e^{i\phi} z - c e^{2i\phi} z^2$, where $c = \frac{\delta}{4+\delta}$. For $z = e^{iy}$, where $y \in [0, 2\pi)$, we have

$$(3.16) \quad |F_c(e^{iy})|^2 = 1 + \delta^2 + c^2 + 2\delta(1-c)\cos(\phi+y) - 2c\cos(2\phi+2y).$$

Denote $w = \cos(\phi+y)$ and $\Psi(w) = |F_c(e^{iy})|^2$, we have

$$\Psi(w) = 1 + \delta^2 + c^2 + 2\delta(1-c)w - 2c(2w^2 - 1) = \delta^2 + \left(\frac{4+2\delta}{4+\delta}\right)^2 + \frac{8\delta}{4+\delta}w - \frac{4\delta}{4+\delta}w^2,$$

which is a quadratic polynomial in w with $c = \frac{\delta}{4+\delta}$.

Using second derivative method, it is obvious that $\Psi(w)$ attains its maximum on $\mathbb{R}$ at the unique point $w = 1$ and hence

$$\max_{-1 \leq w \leq 1} \Psi(w) = \Psi(1) = \delta^2 + \left(\frac{4+2\delta}{4+\delta}\right)^2 + \frac{4\delta}{4+\delta} = \left(\delta + \frac{4}{4+\delta}\right)^2,$$

which implies that

$$(3.17) \quad \max_{|z|=1} |F_c(z)| = \delta + \frac{4}{4+\delta}.$$

Using (3.17) to (3.15), we have

$$(3.18) \quad \|A_s z^s + A_t z^t + b_{2t-s} z^{2t-s}\|_{\infty} = |A_s| \left(\delta + \frac{4}{4+\delta} \right) = |A_t| + \frac{4}{4+\delta} |A_s|.$$

Combining (3.13) and (3.18), we obtain

$$\|Q\|_{\infty} < \epsilon + |A_t| + \frac{4}{4+\delta} |A_s|.$$

Now, choosing $\epsilon < \delta|A_s|/(4 + \delta)$, the above inequality becomes

$$(3.19) \quad \|Q\|_\infty < |A_s| + |A_t|$$

and this proves (2.12) for the second case satisfying (3.10).

The proof of the first case, when (3.9) holds, is completely that of the second case considered and we omit the details. This completes the proof of Theorem 2.2. $\square$

Remark 3.1. It is worth noting that we do not have the knowledge of the estimate of the lower bound of the norm of the derivative $p'(z)$ of a polynomial $p(z)$ except the trivial zero bound. However, for instance, if a polynomial $p(z)$ of degree d has all its zeros in $|z| \leq 1$, in view of TURÁN's inequality [6], we have,

$$\|p'\|_\infty \geq \frac{d}{2} \|p\|_\infty.$$

It would be interesting to see that for some particular problem, if we apply Corollary 2.3 for the derivative of a polynomial $p(z)$ satisfying the assumption of the above Turán's result, the lower bound of $\|p'\|_\infty$ so obtained is better than that given by Turán's inequality.

Example 3.1. Let $p(z) = 10z^4 + 2z^2 + 5z + 1$ be a polynomial having all its zeros in $|z| \leq 1$ that holds Corollary 2.3 attaining $\|p\|_\infty$ and $\|p'\|_\infty$ at $z = 1$.

Applying Corollary 2.3 to $p'(z)$, we have

$$(3.20) \quad \|p'\|_\infty \geq 45,$$

whereas by Turán's inequality, we get

$$(3.21) \quad \|p'\|_\infty \geq 36$$

showing that our result (Corollary 2.3) gives better lower bound estimate of $\|p'\|_\infty$ than that of Turán's inequality.

4. Conclusion. This paper studies the estimate of the uniform norm of a polynomial obtained as a linear combination of two polynomials as a lower bound in terms of two restricted arbitrary coefficients of the polynomial. As a consequence of our result, we also give a lower bound estimate of $\|p'\|_\infty$ of a self-inversive polynomial $p(z)$ in terms of its two restricted arbitrary coefficients.

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