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MOMENT GENERATING FUNCTION OF THE AVERAGED LOG-RETURNS IN THE HESTON'S STOCHASTIC VOLATILITY MODEL

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Abstract

The aim of this short note is to investigate the domain of the moment generating function of the log-returns in the Heston's stochastic volatility model averaging the initial volatility value through its stationary distribution. This way we deal with the problem that arises from the fact that the volatility is a hidden market object and it is hard to be extracted.

Key words: Heston model, stochastic volatility, moment generating function, moment finiteness

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1. Introduction. The Heston [1] stochastic volatility model generalized the famous Black and Scholes [2] one assuming that the volatility is driven by a Cox-Ingersoll-Ross (CIR) mean reverting process, first used in Cox et al. [3] for modelling interest rates. Note that the CIR process admits a stationary distribution which is a Gamma one. The wider usage of the Heston's model is constrained by the lack of market data about the current volatility. Its extraction is a hard task and there is no unified method for this. A possible workaround is to average its values through the stationary distribution – this approach is first suggested by Drăgulescu and Yakovenko [4]. Later, Zaevski et al. [5] use

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this method for option pricing whereas Zaevski and Nedeltchev [6] apply it to risk-management.

We obtain in this article the abscissas of convergence of the moment generation function (MGF, hereafter) of the log-returns after averaging. Also, we position these values w.r.t. the points of analytical continuation for the original Heston's domain. Like every approximation, volatility averaging comes at certain costs and entails information loss: it shrinks the domain of the MGF finiteness – for the original log-returns see del Baño Rollin et al. [7]. The possible applications of the obtained results are in several directions – moment explosions [8], volatility smiles [9], option pricing [10], entropic VaR [11], etc.

2. Characteristic and moment generating functions of the Heston's log-returns. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a filtered probability space and B_t and $\tilde{B}_t$ be two correlated Brownian motions with coefficient ρ. The Heston's stochastic volatility model is specified via the following two-dimensional stochastic differential equation for the asset price S_t and the volatility V_t

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dB_t \\ dV_t &= \xi \left(\eta - V_t\right) dt + \theta \sqrt{V_t} d\tilde{B}_t. \end{aligned} \tag{2.1}$$

We can assume without loss of generality that the initial asset price is one, $S_0 = 1$. The initial volatility $V_0 = v$ is a hidden parameter, not directly observable at the market. The variable ξ is the speed of reversion to the long-term value η and θ is the volatility of the volatility. Thus the Heston's model is driven by five parameters μ, ξ, η, θ, and ρ. If $2\xi\eta \geq \theta^2$, then the process V_t never touches the zero. On the contrary, this is possible when $2\xi\eta < \theta^2$. For more detail, see Feller [12] or Pitman and Yor [13]. Let X_t be the centred log-returns (i.e. without drift), $X_t = \ln S_t - \mu t$. The Itô's differential rule leads to

$$\begin{aligned} dX_t &= -\frac{V_t}{2} dt + \sqrt{V_t} dB_t \\ X_0 &= 0. \end{aligned} \tag{2.2}$$

For a fixed t, the characteristic function of the log-price X_t can be written as

$$\psi\left(u\right) = e^{C(u) + vD(u)}, \tag{2.3}$$

where

$$\begin{aligned} C\left(u\right) &= \frac{\xi\eta}{\theta^2}\left[\left(\xi - \rho\theta ui + h\right)t - 2\log\left(\frac{1 - g\exp\left(ht\right)}{1 - g}\right)\right] \\ D\left(u\right) &= \frac{\xi - \rho\theta ui + h}{\theta^2}\frac{1 - \exp\left(ht\right)}{1 - g\exp\left(ht\right)} \\ g &= \frac{\xi - \rho\theta ui + h}{\xi - \rho\theta ui - h} \\ h &= \sqrt{\left(\rho\theta ui - \xi\right)^2 + \theta^2\left(ui + u^2\right)}. \end{aligned} \tag{2.4}$$

It is well known that differently to the characteristic function, the MGF does not always exist. The following result, which can be found as Theorem 3.1 from del Baño Rollin et al. [7], provides its domain as well as its form:

Proposition 2.1. *Let the quadratic function $p(x)$ be defined as*

$$p(x) = (\xi - \rho\theta x)^2 + \theta^2 \left(x - x^2\right) \equiv -x^2\theta^2 \left(1 - \rho^2\right) + x\theta\left(\theta - 2\rho\xi\right) + \xi^2. \tag{2.5}$$

It has two real roots, possibly infinitely small (large), $-\infty \le x_1 < 0 < 1 < x_2 \le \infty$:

$$x_{1,2} = \frac{\theta - 2\rho\xi \pm \sqrt{\theta^2 - 4\rho\theta\xi + 4\xi^2}}{2\theta\left(1 - \rho^2\right)}. \tag{2.6}$$

Note that $p(x) > 0$ when $x \in (x_1, x_2)$, and $p(x) < 0$ if $x < x_1$ or $x > x_2$. Let $P(x) = \sqrt{|p(x)|}$ and $f(\cdot)$ be defined as

$$\begin{aligned} f(x) &= f_1(x) + (\xi - \rho\theta x) f_2(x) \\ f_1(x) &= \cosh\left(\frac{P(x)t}{2}\right) I_{x\in(x_1,x_2)} + \cos\left(\frac{P(x)t}{2}\right) I_{x\notin(x_1,x_2)} \\ f_2(x) &= \frac{1}{P(x)}\left[\sinh\left(\frac{P(x)t}{2}\right) I_{x\in(x_1,x_2)} + \sin\left(\frac{P(x)t}{2}\right) I_{x\notin(x_1,x_2)}\right]. \end{aligned} \tag{2.7}$$

For a fixed time value t, the MGF of X_t can be written as $M(x) = e^{c(x)+vd(x)}$, where

$$\begin{aligned} c(x) &= \frac{2\xi\eta}{\theta^2}\left((\xi - \rho\theta x)\frac{t}{2} - \ln f(x)\right) \\ d(x) &= \left(x^2 - x\right)\frac{f_2(x)}{f(x)}. \end{aligned} \tag{2.8}$$

If we need to mark the dependence on t, we shall use the notations $c(x;t)$ and $d(x;t)$ or $c(\cdot;t)$ and $d(\cdot;t)$. Let us denote the domain of the MGF by $\left(x^-, x^+\right)$. The left abscissa of convergence x^- is the largest root of the equation $f(x) = 0$ smaller than x_1. The right abscissa x^+ can be obtained through the following statements.

1. *If $\xi \ge \rho\theta$, then x^+ is the smallest root of $f(x)$ larger than x_2.*

2. *If $\rho = 1$ and $2\xi = \theta$, then*

$$x^+ = \frac{1}{1 - e^{-\xi t}}. \tag{2.9}$$

3. *Let us consider $\xi < \rho\theta$ excluding the case $\rho = 1$ and $2\xi = \theta$. Let $\bar{t}$ be defined as*

$$\bar{t} = \frac{2}{\rho\theta x_2 - \xi}. \tag{2.10}$$

(*a*) *If $t < \bar{t}$, then x^+ is the smallest zero of $f(x)$ larger than x_2.*
(*b*) *If $t \geq \bar{t}$, then x^+ is the unique root of $f(x)$ in the interval $(1, x_2]$.*

We can observe the following immediate corollary:

Corollary 2.1. *The domain of the MGF of the Heston's log-returns is independent of the initial value of the volatility.*

3. Averaging over the volatility. It is well known that the CIR process driving the volatility in SDE (2.2) admits a stationary distribution which is a Gamma one with parameters $\alpha = \frac{2\xi}{\theta^2}$ and $\beta = \frac{2\xi\eta}{\theta^2}$. Its density is

$$g_\gamma(u) = \frac{\alpha^\beta}{\Gamma(\beta)} u^{\beta-1} \exp(-\alpha u) I_{u \geq 0}, \tag{3.1}$$

where $\Gamma(\cdot)$ is the Gamma function. Before we continue, we need the following lemma:

Lemma 3.1. *The real part of the function $D(u)$ from the characteristic function* (2.3) *is non-positive.*

Proof. The Jensen's inequality leads to

$$|\psi(u)|^2 = (E[\cos(X_t u)])^2 + (E[\sin(X_t u)])^2 \leq 1. \tag{3.2}$$

Suppose that for some u, $\Re D(u) > 0$. Using formula (2.3), we see that

$$|\psi(u)|^2 = e^{\Re C(u) + v\Re D(u)} \tag{3.3}$$

is larger than one for large enough values of v. This contradicts to inequality (3.2). □

As we mentioned above, the initial volatility v in characteristic function (2.3) is unobservable and thus a natural approach to deal with this problem is to average via the stationary distribution. The following proposition allows this:

Proposition 3.1. *There exists a random variable $\widetilde{V}$ with characteristic function*

$$\begin{aligned} \widetilde{\Psi}(u) &= \int_0^\infty \psi(u; v) g_\gamma(v) dv \\ &= \int_0^\infty e^{C(u)+D(u)v} g_\gamma(v) dv \\ &= e^{C(u)} \psi_\gamma(-iD(u)), \end{aligned} \tag{3.4}$$

where $\psi_\gamma(\cdot)$ is the characteristic function of the Gamma distribution

$$\psi_\gamma(u) = \left(1 - \frac{iu}{\alpha}\right)^{-\beta} \equiv \left(1 - \frac{i\theta^2 u}{2\xi}\right)^{-\frac{2\xi\eta}{\theta^2}}. \tag{3.5}$$

Proof. The proof is based on the Bochner's theorem. To use it, we need $\Re D(u) < \alpha$ which is true due to Lemma 3.1. Note that the MGF of the Gamma distribution is well-defined for all values less than α. □

We are ready to investigate the main task of this article, namely the domain of the MGF of the random variable $\widetilde{V}$. Let us denote it by $\mathcal{D} \equiv \left(\widetilde{x}^-, \widetilde{x}^+\right)$. We shall prove first that $\mathcal{D}$ is a subset of the MGF's domain of the original Heston's log-returns.

Lemma 3.2. *We have the inclusion $\mathcal{D} \subset \left(x^-,\ x^+\right)$.*

Proof. Suppose that $x \in \mathcal{D}$ but $x \notin \left(x^-,\ x^+\right)$. Using formula (3.4), we see that the MGF of $\widetilde{V}$ at the point x can be written as

$$\widetilde{M}(x) = \int_0^\infty \psi(-ix; v)\, g_\gamma(v)\, dv. \tag{3.6}$$

On the other hand, Corollary 2.1 gives us that the MGF of the original Heston's log-returns $\psi(-ix; v)$ diverges for all initial values of the volatility. Hence, integral (3.6) diverges too. □

Lemma 3.3. *For a fixed t, the function $d(x)$ decreases in the interval $\left(x^-, 0\right)$ from $+\infty$ to zero and increases for $x \in \left(1, x^+\right)$ from zero to $+\infty$.*

Proof. Using the log-convexity of the MGF, we see that $c''(x) + vd''(x) > 0$ for all positive values of v and thus $d''(x)$ is always positive. Hence, the function $d(\cdot)$ is convex in the interval $x \in \left(x^-,\ x^+\right)$ and therefore it has at most two roots in this interval. Something more, the function decreases before the smaller root and increases after the larger one. We finish the proof mentioning that $d(0) = d(1) = 0$ and $d\left(x^-\right) = d\left(x^+\right) = +\infty$ due to $f\left(x^-\right) = f\left(x^+\right) = 0$ and formula (2.8). □

As a corollary, we can state our first result:

Theorem 3.1. *The abscissas of convergence $\widetilde{x}^-$ and $\widetilde{x}^+$ for the random variable $\widetilde{V}$ are the unique roots of the equation $d(x) = \dfrac{2\xi}{\theta^2}$ in the intervals $\left(x^-, 0\right)$ and $x \in \left(1, x^+\right)$, respectively. For every $x \in \left(\widetilde{x}^-, \widetilde{x}^+\right)$, the MGF is*

$$\widetilde{M}(x) = e^{c(x)} \left(1 - \frac{\theta^2 d(x)}{2\xi}\right)^{-\frac{2\xi\eta}{\theta^2}}. \tag{3.7}$$

Proof. The theorem holds because $d(x) < \dfrac{2\xi}{\theta^2}$ for $x \in \left(\widetilde{x}^-, \widetilde{x}^+\right)$ and the domain of the Gamma MGF consists of all points below $\alpha = \dfrac{2\xi}{\theta^2}$. □

Next we discuss the position of $\widetilde{x}^-$ and $\widetilde{x}^+$ w.r.t. x_1 and x_2, where x_1 and x_2 are the roots of function (2.5) and are given by formulas (2.6). We shall investigate separately the cases $|\rho| < 1$ and $|\rho| = 1$.

3.1. Main case, $|\rho| < 1$. We need the following lemma:

Lemma 3.4. *The inequality $\xi - \rho\theta x_1 > 0$ holds. Something more, if $\xi \geq \rho\theta$, then $\xi - \rho\theta x_2 > 0$ too.*

Proof. First we shall prove the result for the lower root x_1. If $\rho \geq 0$, then the inequality holds since $x_1 < 0$. Suppose that $\rho < 0$. Hence, the desired inequality is equivalent to the following ones

$$\begin{aligned}
&\xi > \rho\frac{\theta - 2\rho\xi - \sqrt{\theta^2 - 4\rho\theta\xi + 4\xi^2}}{2\left(1-\rho^2\right)} \quad \text{due to (2.6)} \\
&\sqrt{\theta^2 - 4\rho\theta\xi + 4\xi^2} < -2\frac{\xi}{\rho} + \theta \\
&\theta^2 - 4\rho\theta\xi + 4\xi^2 < \left(2\frac{\xi}{\rho} - \theta\right)^2 \\
&0 < \left(1-\rho^2\right)\left(\xi - \theta\rho\right).
\end{aligned} \tag{3.8}$$

The last equation is true again due to $\rho < 0$.

Let us turn to the statement related to x_2 supposing $\xi \geq \rho\theta$. If $\rho \leq 0$, then the desired result holds since $x_2 > 0$. Assume now that $\rho > 0$. The inequality $\xi - \rho\theta x_2 > 0$ is equivalent to

$$\sqrt{\theta^2 - 4\rho\theta\xi + 4\xi^2} < 2\frac{\xi}{\rho} - \theta. \tag{3.9}$$

Having in mind that the right hand-side of inequality (3.9) is positive (due to $\xi \geq \rho\theta$), we find that it is equivalent to $0 < \left(1-\rho^2\right)\left(\xi - \theta\rho\right)$ which is true. □

We are ready now to prove the next main results for the position of $\widetilde{x}^-$ and $\widetilde{x}^+$ w.r.t. x_1 and x_2. Let the constants $y_{1,2}$ and $t_{1,2}$ be defined as

$$\begin{aligned}
y_{1,2} &= \pm\frac{2\rho^2 - 1}{\rho} \\
t_{1,2} &= \frac{4\xi}{\theta^2\left(x_{1,2}^2 - x_{1,2}\right) - 2\xi\left(\xi - \theta\rho x_{1,2}\right)}.
\end{aligned} \tag{3.10}$$

Theorem 3.2. *We have $\widetilde{x}^- \in \left(x^-, x_1\right)$ in the following cases: $\left\{\rho \leq \frac{1}{2}\right\}$, $\left\{\rho > \frac{1}{2}, \frac{\theta}{\xi} \geq y_1 + 2\rho\right\}$, and $\left\{\rho > \frac{1}{2}, \frac{\theta}{\xi} < y_1 + 2\rho, t < t_1\right\}$. If $\left\{\rho > \frac{1}{2}, \frac{\theta}{\xi} < y_1 + 2\rho, t > t_1\right\}$, then $\widetilde{x}^- \in (x_1, 0)$. The equality $\widetilde{x}^- = x_1$ holds when $\left\{\rho > \frac{1}{2}, \frac{\theta}{\xi} < y_1 + 2\rho, t = t_1\right\}$.*

Proof. We shall investigate when $\widetilde{x}^- \in (x^-, x_1)$. This happens when the following equivalent inequalities hold:

$$
\begin{aligned}
&d(x_1;t) < \frac{2\xi}{\theta^2} \\
(3.11) \qquad &\left(x_1^2 - x_1\right)\frac{f_2(x_1)}{f(x_1)} < \frac{2\xi}{\theta^2} \text{ due to (2.8)} \\
&\frac{x_1^2 - x_1}{\frac{2}{t} + \xi - \theta\rho x_1} < \frac{2\xi}{\theta^2} \text{ due to } \lim_{x\to 0} x\cot x = 1.
\end{aligned}
$$

Note that the function

$$
(3.12) \qquad H(t) = \frac{x_1^2 - x_1}{\frac{2}{t} + \xi - \theta\rho x_1}
$$

increases from $H(0) = 0$ to $H(\infty) = \dfrac{x_1^2 - x_1}{\xi - \theta\rho x_1} > 0$ due to Lemma 3.4. We shall check when $H(\infty) \le \dfrac{2\xi}{\theta^2}$ or equivalently $-\rho\theta x_1 \le \xi$ due to definition (2.5). If $\rho \le 0$, then the inequality holds since its left hand-side is non-positive. Suppose now that $\rho > 0$. Using formula (2.6) for x_1, we see that $-\rho\theta x_1 \le \xi$ is equivalent to

$$
(3.13) \qquad h\left(\frac{\theta}{\xi} - 2\rho\right) \le 2\frac{1-\rho^2}{\rho},
$$

where the function $h(\cdot)$ is defined as

$$
(3.14) \qquad h(y) = \sqrt{y^2 + 4(1-\rho^2)} - y.
$$

It decreases from $+\infty$ to zero since its derivative $h'(y) = \dfrac{y}{\sqrt{y^2 + 4(1-\rho^2)}} - 1$ is always negative. Hence, the equation $h(y) = 2\dfrac{1-\rho^2}{\rho}$ has unique root which is namely y_1. Also, $y_1 + 2\rho > 0$ when $\rho > \dfrac{1}{2}$. Hence, inequality (3.13) holds when $\rho \le \dfrac{1}{2}$ or when $\left\{\rho > \dfrac{1}{2}, \ \dfrac{\theta}{\xi} \ge y_1 + 2\rho\right\}$.

Suppose that $\left\{\rho > \dfrac{1}{2}, \ \dfrac{\theta}{\xi} < y_1 + 2\rho\right\}$. Therefore, $H(\infty) > \dfrac{2\xi}{\theta^2}$ and thus the equation $H(t) = \dfrac{2\xi}{\theta^2}$ has a unique positive root and it is t_1. Thus we conclude that inequalities (3.11) hold when $t < t_1$. The opposite inequalities are true for $t > t_1$. This finishes the proof. □

We turn now to the position of x^+.

Theorem 3.3. *The following statements characterize the position of* $\widetilde{x}^{+}$ *w.r.t.* x_2.

1. *If* $\xi \geq \rho\theta$, *then*

   (a) $\widetilde{x}^{+} \in \left(x_2, x^{+}\right)$ *when* $\{\rho \geq 0\}$, $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} \leq 2\rho - y_2\right\}$, $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} > 2\rho - y_2, t < t_2\right\}$, *or* $\{\rho \leq -0.5, t < t_2\}$.

   (b) $\widetilde{x}^{+} \in (1, x_2)$ *when* $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} > 2\rho - y_2, t > t_2\right\}$ *or* $\{\rho \leq -0.5, t > t_2\}$.

   (c) $\widetilde{x}^{+} = x_2$ *when* $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} > 2\rho - y_2, t = t_2\right\}$ *or* $\{\rho \leq -0.5, t = t_2\}$.

2. *Suppose that* $\xi < \rho\theta$. *Note that* $t_2 < \bar{t}$ *(*$\bar{t}$ *is defined by formula* (2.10)*) because the function* $d(x_2; t)$ *increases w.r.t.* t *and it is infinity for* $\bar{t}$.

   (a) *If* $t < t_2$, *then* $\widetilde{x}^{+} \in \left(x_2, x^{+}\right)$.

   (b) *If* $t = t_2$, *then* $\widetilde{x} = x_2$.

   (c) *If* $t_2 < t$, *then* $1 < \widetilde{x}^{+} < x_2 < x^{+}$.

Proof. Suppose first that $\xi \geq \rho\theta$. The proof of the desired results can be obtained in a similar manner as in Theorem 3.2. First, we shall find when the function $d(x; t)$ is less than $\frac{2\xi}{\theta^2}$. Note that $d(x; t)$ increases w.r.t. t between $H(0) = 0$ and $H(+\infty) = \frac{x_2^2 - x_2}{\xi - \theta\rho x_2}$, where the function $H(\cdot)$ is defined as

$$H(t) = \frac{x_2^2 - x_2}{\frac{2}{t} + \xi - \theta\rho x_2}. \tag{3.15}$$

Hence, we need to find when $H(+\infty)$ is less than $\frac{2\xi}{\theta^2}$ or equivalently when $-\rho\theta x_2 < \xi$. We can immediately see that this inequality always holds when $\rho \geq 0$. On the other hand, if $\rho < 0$, then we have to find when $h\left(2\rho - \frac{\theta}{\xi}\right) < -2\frac{1-\rho^2}{\rho}$ – the function $h(\cdot)$ is defined in (3.14). The desired results follow the fact that the term $2\rho - \frac{\theta}{\xi}$ varies between $-\infty$ and 2ρ and thus $h\left(2\rho - \frac{\theta}{\xi}\right)$ is between $2(1-\rho)$ and $+\infty$.

Let us turn to the case $\xi < \rho\theta$. The first part in the second point is obvious. The remaining cases follow the fact that function (3.15) increases from zero to infinity for $t \in \left(0, \bar{t}\right)$. Note that $\xi - \theta\rho x_2 < 0$, because $\xi < \theta\rho$ and $x_2 > 1$. □

3.2. Limiting case, $|\rho| = 1$. We shall discuss now the cases when the Brownian motions are perfectly correlated. Suppose first that $\rho = 1$ and $2\xi \neq \theta$. Thus the function (2.5) has unique root

$$\overline{x} = \frac{\xi^2}{\theta\,(2\xi - \theta)}. \tag{3.16}$$

If $2\xi < \theta$, then $x_1 = \overline{x}$ and $x_2 = \infty$. We can easily check that Theorem 3.2 holds having in mind that the constants y_1 and t_1 turn into $y_1 = 1$ and

$$t_1 = 4\frac{(2\xi - \theta)^2}{\xi\,(\theta - \xi)\,(3\xi - \theta)}. \tag{3.17}$$

Note that the important limit for $\frac{\theta}{\xi}$ is the number $y_1 + 2\rho = 3$. Also, $x_2 = \infty$ implies $t_2 = 0$ and thus Theorem 3.3 can be generalized in this sense.

If $2\xi > \theta$, then $x_1 = -\infty$ and $x_2 = \overline{x}$. We can easily check that $2\xi > \theta$ leads to $x_2 > 1$. Theorem 3.2 holds again in a generalized sense since t_1 turns to zero. Theorem 3.3 holds too since the critical value for the time t_2 turns into (3.17). Note that in Theorem 3.3, the cases $\xi > \theta$ and $\xi < \theta$ lead to different results. We have to mention that $y_2 = -1$ and thus the important boundary for $\frac{\theta}{\xi}$ is again the number $2\rho - y_2 = 3$.

The results for the case $2\xi = \theta$ are provided in the following theorem:

Theorem 3.4. *If $\rho = 1$ and $2\xi = \theta$,*[1] *then*

$$\widetilde{x}^{-,+} = \mp\sqrt{\frac{1}{2}\left(\coth\left(\frac{\xi t}{2}\right) + 1\right)}. \tag{3.18}$$

Proof. When $\rho = 1$ and $2\xi = \theta$, the function $P(\cdot)$ turns into the constant $P(x) = \xi$. Thus the equation $d(x) = \frac{2\xi}{\theta^2}$ turns into a quadratic one which roots are given by formula (3.18). □

Let us discuss now briefly the case $\rho = -1$. The root of function (2.5) is $-\frac{\xi^2}{\theta\,(2\xi + \theta)}$. Note that it is negative and thus this is the smaller root x_1. Also, $x_2 = +\infty$. We can easily check Theorems 3.2 and 3.3 having in mind $d(x_1) < \frac{2\xi}{\theta^2}$ and $t_2 = 0$.

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[1] Let us mention that $x_1 = -\infty$ and $x_2 = \infty$ in this case.

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