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Number theory

FIBONACCI NUMBERS THAT ARE η-CONCATENATIONS OF LEONARDO AND LUCAS NUMBERS

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Abstract

Let $\{F_r\}_{r\geq 0}$, $\{L_r\}_{r\geq 0}$ and $\{Le_r\}_{r\geq 0}$ be r-th terms of Fibonacci, Lucas and Leonardo sequences, respectively. In this paper, we determined the effective bounds for the solutions of the Diophantine equation $F_r = \eta^k Le_s + L_t$ in non-negative integers r, s, t, where k represents the number of digits of L_t in base $\eta \geq 2$. In addition, we applied linear forms in logarithms of algebraic numbers and the reduction method based on the continued fraction. In particular, we investigated all solutions of this Diophantine equation for $\eta \in [2, 10]$.

Key words: Diophantine equations, Leonardo numbers, Lucas numbers, Fibonacci numbers, linear forms in logarithms, reduction method

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1. Introduction. The Fibonacci and Lucas sequences $\{F_r\}_{r\geq 0}$ and $\{L_r\}_{r\geq 0}$ are linear recurrence sequences defined by

$$F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_r = F_{r-1} + F_{r-2} \quad \text{for all} \quad r \geq 2 \tag{1.1}$$

and

$$L_0 = 2, \quad L_1 = 1 \quad \text{and} \quad L_r = L_{r-1} + L_{r-2} \quad \text{for all} \quad r \geq 2. \tag{1.2}$$

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In [1] Catarino and Borges defined the Leonardo sequence $\{Le_r\}_{r\geq 0}$ as the following recurrence sequence

$$Le_0 = 1, \quad Le_1 = 1 \quad \text{and} \quad Le_r = Le_{r-1} + Le_{r-2} + 1 \quad \text{for all} \quad r \geq 2. \tag{1.3}$$

The Fibonacci, Lucas, and Leonardo numbers correspond to sequences $A000045$, $A000032$, and $A001595$ in "'OEIS", respectively.

The Diophantine equations involving concatenation of binary linear recurrent sequences represent a generalization of the result by Banks and Luca [2]. They demonstrated that only a finite number of terms in linear recurrent sequences can be expressed as a concatenation of two or more terms from the same sequences. In addition, they found that Fibonacci numbers that are concatenations of two terms of Fibonacci numbers are 13, 12 and 55. Alan [3] determined the concatenations of Fibonacci and Lucas numbers to each other. In addition, Bravo [4] showed the concatenations of Padovan and Perrin numbers. Erduvan [5] presented concatenations of two Padovan and Perrin numbers. Altassan and Alan [6] worked on Fibonacci numbers, which can be written as a mixed concatenation of Fibonacci and Lucas numbers. Later, Adédji and Trebješanin [7] extended the result in [6] for Pell and Pell-Lucas numbers.

Recently, Adédji et al. [8] demonstrated the b-concatenation of Pell and Pell-Lucas numbers. Duman [9] found all Padovan numbers that are concatenations of the Padovan and Perrin numbers. The authors in [10] studied the b-concatenation of Padovan and Perrin numbers. In this paper, we solve the Diophantine equation

$$F_r = \eta^k Le_s + L_t \tag{1.4}$$

in non-negative integers r, s, t and where k is the number of digits of L_t in base $\eta \geq 2$. Thus, we obtain the following results.

Theorem 1.1. *Let $\eta \geq 2$ be an integer. Then, all non-negative integer solutions of Diophantine equation* (1.4) *satisfy the following inequality*

$$r < 2 \cdot 10^{28} (\log \eta)^3. \tag{1.5}$$

Corollary 1.2. *Let $\eta \in [2, 10]$ be an integer. Only the Fibonacci numbers which satisfy the Diophantine equation* (1.4) *are given in Table 1.*

2. Preliminaries. ***2.1. Properties of Fibonacci and Lucas sequences.*** The characteristic polynomial of both sequences is defined by

$$f(x) = x^2 - x - 1$$

with two roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

The Binet formulas for the Fibonacci and Lucas sequences are defined by

$$F_r = \frac{\alpha^r - \beta^r}{\alpha - \beta}, \quad L_r = \alpha^r + \beta^r \quad \text{for all} \quad r \geq 0. \tag{2.1}$$

Table 1

The only non-negative integer solutions of the Diophantine equation (1.4)

r	s	t	η	k	F_r	Le_s	L_t	r	s	t	η	k	F_r	Le_s	L_t
4	0	1	2	1	3	1	1	4	1	1	2	1	3	1	1
5	0	0	3	1	5	1	2	5	0	1	4	1	5	1	1
5	1	0	3	1	5	1	2	5	1	1	4	1	5	1	1
6	0	0	6	1	8	1	2	6	0	1	7	1	8	1	1
6	0	2	5	1	8	1	3	6	1	0	6	1	8	1	2
6	1	1	7	1	8	1	1	6	1	2	5	1	8	1	3
7	0	2	10	1	13	1	3	7	0	3	3	2	13	1	4
7	0	3	9	1	13	1	4	7	1	2	10	1	13	1	3
7	1	3	3	2	13	1	4	7	1	3	9	1	13	1	4
7	2	1	4	1	13	3	1	8	2	2	6	1	21	3	3
8	3	1	4	1	21	5	1	9	2	3	10	1	34	3	4
9	2	4	3	2	34	3	7	9	2	4	9	1	34	3	7
9	3	3	6	1	34	5	4	10	2	4	4	2	55	3	7
10	4	1	6	1	55	9	1	15	8	4	3	2	610	67	7
15	8	4	9	1	610	67	7	17	10	3	3	2	1597	177	4
17	10	3	9	1	1597	177	4	18	11	1	9	1	2584	287	1
22	10	5	10	2	17711	177	11								

Furthermore, the following inequalities

$$\alpha^{r-2} \leq F_r \leq \alpha^{r-1} \quad \text{for all} \quad r \geq 1, \tag{2.2}$$

$$\alpha^{r-1} \leq L_r \leq 2\alpha^{r} \quad \text{for all} \quad r \geq 0 \tag{2.3}$$

hold.

2.2. Properties of Leonardo sequence. The characteristic polynomial of Leonardo sequence is given by

$$f(x) = x^3 - 2x^2 + 1$$

with two roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

The Binet formula for Leonardo sequence is defined by

$$Le_r = \frac{2\alpha^{r+1} - 2\beta^{r+1} - \alpha + \beta}{\alpha - \beta} \quad \text{for all} \quad r \geq 0. \tag{2.4}$$

Thus, we have the following inequality

$$\alpha^{r} \leq Le_r \leq \alpha^{r+1} \quad \text{for all} \quad r \geq 2. \tag{2.5}$$

2.3. Linear forms in logarithms.

Definition 2.1. The absolute logarithmic height is denoted by $h(\Upsilon)$ and is defined by

$$h(\Upsilon) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d} \log\max\left\{\left|\Upsilon^{(i)}\right|, 1\right\}\right), \tag{2.6}$$

where $\Upsilon^{(i)}$ denotes the conjugates of the algebraic number Υ of degree d, and $c_0 > 0$ is the leading coefficient of the minimal polynomial of Υ given by $f(X) = c_0 \prod_{i=1}^{d} \left(X - \Upsilon^{(i)}\right) \in \mathbb{Z}[X]$.

The absolute logarithmic height of a rational number $\Upsilon = a/b$ is given by $h(\Upsilon) = \log(\max\{|a|, b\})$, where $b > 0$ and $\gcd(a, b) = 1$.

In the subsequent sections of this paper, the following properties of the logarithmic height function will be used

$$h(\Upsilon \pm \Phi) \leq h(\Upsilon) + h(\Phi) + \log 2, \quad h\left(\Upsilon \Phi^{\pm 1}\right) \leq h(\Upsilon) + h(\Phi), \quad h\left(\Upsilon^{t}\right) = |t| h(\Upsilon).$$

The following theorem, which is a modified version of MATVEEV's result [11], was presented by BUGEAUD et al. [12, Theorem 9.4].

Theorem 2.2. *Let $\Upsilon_1, \ldots, \Upsilon_t$ be positive real algebraic number in the number field $\mathbb{L}$ of degree D over $\mathbb{Q}$, and let $b_1, b_2, \ldots, b_t$ be nonzero integers. Let A_i be a positive real number that satisfies*

$$A_i \geq \max\left\{D h\left(\Upsilon_i\right), \left|\log\left(\Upsilon_i\right)\right|, 0.16\right\}, \quad 1 \leq i \leq t,$$

and

$$B := \max\left\{\left|b_1\right|, \ldots, \left|b_t\right|\right\}.$$

If $\Lambda := \Upsilon_1^{b_1} \cdots \Upsilon_t^{b_t} - 1 \neq 0$, then

$$\log|\Lambda| > (-1.4)\left(30^{t+3}\right)\left(t^{4.5}\right)\left(D^2\right)\left(A_1 \cdots A_t\right)(1 + \log D)(1 + \log B).$$

2.4. The de Weger reduction method. We present a variation of Baker and Davenport's reduction method developed by DE WEGER [13] to reduce the upper bound. Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given and $x_1, x_2 \in \mathbb{Z}$ be unknowns.

Let

$$\Lambda = \beta + x_1 \vartheta_1 + x_2 \vartheta_2. \tag{2.7}$$

Let c, δ be positive constants. We set $X = \max\{|x_1|, |x_2|\}$ and let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{2.8}$$

$$Y \leq X \leq X_0. \tag{2.9}$$

Case 1. If $\beta = 0$ in Eq. (2.7), then $\Lambda = x_1 \vartheta_1 + x_2 \vartheta_2$, where $\vartheta = -\vartheta_1/\vartheta_2$. Let us assume that x_1 and x_2 are relatively primes. The continued fraction expansion of ϑ is represented by $[a_0, a_1, a_2, \ldots]$. The k-th convergent of ϑ is denoted by p_k/q_k, where $k = 0, 1, 2, \ldots$. Without loss of generality, we may assume that $|\vartheta_1| < |\vartheta_2|$ and $x_1 > 0$. We obtain the following lemma.

Lemma 2.3 ([13, Lemma 3.2]). *Let* $A = \max_{0 \le k \le Y_0} a_{k+1}$, *where*

$$Y_0 = -1 + \frac{\log\left(\sqrt{5}X_0 + 1\right)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

If (2.8) *and* (2.9) *hold for* x_1, x_2 *and* $\beta = 0$, *then* $Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right)$.

Case 2. If $\beta \neq 0$ in Eq. (2.7), then we get $\Lambda/\vartheta_2 = \psi - x_1\vartheta + x_2$, where $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Let p_k/q_k be k-convergent of ϑ with $q > X_0$. The distance between the real number m and the nearest integer is denoted by $\|m\| = \min\{|m-n|\colon n \in \mathbb{Z}\}$. The following lemma is presented.

Lemma 2.4 ([13, Lemma 3.3]). *Suppose that* $\|q\psi\| > 2X_0/q$. *Then the solutions of* (2.8) *and* (2.9) *satisfy* $Y < \frac{1}{\delta}\log\left(\frac{q^2c}{|\vartheta_2|X_0}\right)$.

In the subsequent, we present lemmas that are fundamental to our computations.

Lemma 2.5 ([14, Lemma 7]). *If* $c \ge 1$, $S > (4c^2)^c$, *and* $L/(\log L)^c < S$, *then* $L < 2^c S(\log S)^c$.

Lemma 2.6 ([13, Lemma 2.2]). *Let* v, $x \in \mathbb{R}$ *and* $0 < v < 1$. *If* $|x| < v$, *then* $|\log(1+x)| < \frac{-\log(1-v)}{v}|x|$.

3. Proof of Theorem 1.1. First, we assume that Eq. (1.4) holds with the given value k representing the number of digits of L_t. Thus, $k = \lfloor \log_\eta L_t \rfloor + 1$. From (2.5), we can get

$$k = \lfloor \log_\eta L_t \rfloor + 1 \le \log_\eta L_t + 1 \le 1 + \log_\eta 2\alpha^t = 1 + \frac{\log 2}{\log \eta} + t\frac{\log \alpha}{\log \eta} < t + 2$$

and

$$k = \lfloor \log_\eta L_t \rfloor + 1 > \log_\eta L_t \ge \log_\eta \alpha^{t-1} = (t-1)\frac{\log \alpha}{\log \eta}.$$

Consequently, we obtain

$$(t-1)\frac{\log \alpha}{\log \eta} < k < t + 2. \tag{3.1}$$

To find the relation between r, s, t, combining Eq. (1.4) with (2.2) and (2.5), we obtain

$$\alpha^{r-1} \ge F_r = \eta^k Le_s + L_t > L_t Le_s + L_t > L_t Le_s \ge \alpha^{s+t-1}$$

and

$$\alpha^{r-2} \le F_r = \eta^k Le_s + L_t < \eta L_t Le_s + L_t < (\eta+1)Le_s L_t \le \alpha^{s+t+1+\log_\alpha(2\eta+2)}.$$

We used the fact that $L_t = \eta^{\log_\eta L_t} < \eta^k \leq \eta^{1+\log_\eta L_t} = \eta L_t$. So, we conclude that

$$s + t < r < s + t + 3 + \frac{\log(2\eta + 2)}{\log \alpha}. \tag{3.2}$$

In this proof we consider $r - t \geq 2$, since we have

$$\alpha^{r-1} \geq F_r = \eta^k Le_s + L_t \geq \eta^k + L_t > 2L_t \geq 2\alpha^{t-1} > \alpha^t.$$

Combining (2.1) and (2.4) with Eq. (1.4), we get

$$\frac{\alpha^r}{\sqrt{5}} - \frac{2\eta^k \alpha^{s+1}}{\sqrt{5}} = \frac{\beta^r}{\sqrt{5}} - \frac{2\eta^k \beta^{s+1}}{\sqrt{5}} - \frac{\eta^k \alpha}{\sqrt{5}} + \frac{\eta^k \beta}{\sqrt{5}} + L_t.$$

Using $\beta = -\alpha^{-1}$ and taking the absolute value, we obtain that

$$\left| \frac{\alpha^r}{\sqrt{5}} - \frac{2\eta^k \alpha^{s+1}}{\sqrt{5}} \right| < \frac{1}{\alpha^r \sqrt{5}} + \frac{2\eta^k}{\alpha^{s+1}\sqrt{5}} + \frac{\eta^k \alpha}{\sqrt{5}} + \frac{\eta^k}{\alpha\sqrt{5}} + L_t.$$

Dividing both sides by $(2\eta^k \alpha^{s+1})/\sqrt{5}$ and using $L_t < \eta^k$, we conclude that

$$\begin{aligned} \left| \frac{\alpha^{r-s-1}}{2\eta^k} - 1 \right| &< \frac{1}{2\eta^k \alpha^{r+s+1}} + \frac{1}{\alpha^{2s+2}} + \frac{1}{2\alpha^s} + \frac{1}{2\alpha^{s+2}} + \frac{\sqrt{5}}{2\alpha^{s+1}} \\ &< \frac{1}{\alpha^s} \left(\frac{1}{2\eta^k \alpha^{r+1}} + \frac{1}{\alpha^{s+2}} + \frac{1}{2} + \frac{1}{2\alpha^2} + \frac{\sqrt{5}}{2\alpha} \right) < \frac{2}{\alpha^s}. \end{aligned} \tag{3.3}$$

Let $\Lambda_1 := \frac{\alpha^{r-s-1}}{2\eta^k} - 1$. Now, assuming $\Lambda_1 = 0$ implies that $2\eta^k = \alpha^{r-s-1}$. Applying $\mathbb{Q}$-automorphism of the Galois extension and taking the absolute values yields $2\eta^k < 1$, which is impossible. Thus $\Lambda_1 \neq 0$. By applying Theorem 2.2, putting

$$t := 3, \quad (\Upsilon_1, b_1) := (\alpha, r - s - 1), \quad (\Upsilon_2, b_2) := (\eta, -k), \quad (\Upsilon_3, b_3) := (2, -1).$$

It is clear that $\Upsilon_1, \Upsilon_2, \Upsilon_2 \in \mathbb{L} = \mathbb{Q}(\sqrt{5})$ and $D := 2$. Therefore, the properties of absolute logarithmic height have been applied to estimate the following

$$h(\Upsilon_1) = \log \alpha/2, \quad h(\Upsilon_2) = \log \eta, \quad h(\Upsilon_3) = \log 2.$$

Thus

$$A_1 := \log \alpha, \quad A_2 := 2\log \eta, \quad A_3 := 2\log 2.$$

To determine the value of B, we have

$$\eta^k < \eta^k + \frac{L_t}{Le_s} = \frac{F_r}{Le_s} \leq \frac{\alpha^{r-1}}{\alpha^s} = \alpha^{r-s-1} \implies k < (r-s-1)\frac{\log \alpha}{\log \eta} < r - s - 1 < r.$$

We obtain that

$$B := \max\{|b_1|, |b_2|, |b_3|\} = \max\{1, k, r - s - 1\} < r.$$

Using Theorem 2.2 leads to

$$\frac{2}{\alpha^s} > |\Lambda_1| > \exp\{-W(1+\log(r))(\log\alpha)(2\log\eta)(2\log 2)\},$$

where $W = 1.4\left(30^6\right)\left(3^{4.5}\right)(2^2)(1+\log 2)$ and we used $1+\log(r) < 3\log r$ for all $r \geq 2$. After some calculations, we get $s < 8.1 \cdot 10^{12}\log\eta\log r$.

Again, we repeat the same procedure as in the previous case for

$$\frac{\alpha^r}{\sqrt{5}} - \alpha^t - \eta^k Le_s = \frac{\beta^r}{\sqrt{5}} + \beta^t \implies \frac{\alpha^r}{\sqrt{5}}(1-\sqrt{5}\alpha^{t-r}) - \eta^k Le_s = \frac{\beta^r}{\sqrt{5}} + \beta^t.$$

First, using $\beta = -\alpha^{-1}$, taking the absolute value for both sides and then dividing by $\alpha^r(1-2\alpha^{t-r+1})/\sqrt{5}$, we obtain

(3.4)
$$\left|1 - \frac{\sqrt{5}\eta^k Le_s\alpha^{-r}}{(1-\sqrt{5}\alpha^{t-r})}\right| < \frac{1}{1-\sqrt{5}\alpha^{t-r}}\left(\frac{1}{\alpha^{2r}} + \frac{\sqrt{5}}{\alpha^{r+t}}\right) < \frac{6.9}{\alpha^r}\left(\frac{1}{\alpha^r} + \frac{\sqrt{5}}{\alpha^t}\right) < \frac{18}{\alpha^r}.$$

We used the fact that $1 < \frac{1}{1-\sqrt{5}\alpha^{t-r}} < 6.9$ for $r - t \geq 2$. Let $\Lambda_2 := 1 - \frac{\sqrt{5}\eta^k Le_s\alpha^{-r}}{(1-\sqrt{5}\alpha^{t-r})}$. It is clear that $\Lambda_2 \neq 0$ is the same as Λ_1 in the previous case. By Theorem 2.2, we take the parameters as

$$t := 3, \quad (\Upsilon_1, b_1) := (\alpha, -r), \quad (\Upsilon_2, b_2) := (\eta, k), \quad (\Upsilon_3, b_3) := \left(\frac{\sqrt{5}Le_s}{(1-\sqrt{5}\alpha^{t-r})}, 1\right).$$

Compared with the previous case, we have

$$D := 2, \quad B := r, \quad A_1 := \log\alpha, \quad A_2 := 2\log\eta.$$

Subsequently, we estimate the value of $h(\Upsilon_3)$ as

$$\begin{aligned} h(\Upsilon_3) &\leq h(\sqrt{5}) + h(Le_s) + (r-t+1)h(\alpha) + \log 2 \\ &< \frac{\log 5}{2} + (s+1)\frac{\log\alpha}{2} + (r-t)\frac{\log\alpha}{2} + \frac{\log 5}{2} + \log 2 \\ &< 2.3 + (2s+4)\frac{\log\alpha}{2} + \frac{\log(2\eta+2)}{2} < 2.3 + (2s+4)\frac{\log\alpha}{2} + 2\log\eta. \end{aligned}$$

In the above inequality, we used (3.2). Therefore, we can get $A_3 := 4.6 + (2s+4)\log\alpha + 4\log(\eta)$. Applying Theorem 2.2, we obtain

(3.5)
$$\frac{18}{\alpha^r} > |\Lambda_2| > \exp\{-W(1+\log(r))(\log\alpha)(2\log\eta)(4.6+(2s+4)\log\alpha + 4\log(\eta))\},$$

where $W = 1.4\left(30^6\right)\left(3^{4.5}\right)(2^2)(1+\log 2)$. After some computations, we get

$$r < 5.85 \cdot 10^{25}(\log r)^2(\log\eta)^2.$$

By applying Lemma 2.5 with parameters $S := 5.85 \cdot 10^{25}(\log\eta)^2$, $L := r$ and $c := 2$, we deduce that

$$\begin{aligned} r &< 2^2(5.85 \cdot 10^{25}(\log\eta)^2)(\log(5.85 \cdot 10^{25}(\log\eta)^2)) \\ &< 4(5.85 \cdot 10^{25}(\log\eta)^2)(59.34 + 2\log\log\eta) \\ &< 4(5.85 \cdot 10^{25}(\log\eta)^2)(85\log\eta) < 2 \cdot 10^{28}(\log\eta)^3. \end{aligned}$$

Hence, the proof of Theorem 1.1 is complete.

In the following result, Theorem 1.1 is applied to obtain the upper bound of r, which is very large. To reduce that upper bound, we apply the de Weger reduction method which includes Lemmas 2.3 and 2.4.

4. Proof of Corollary 1.2. For each $\eta \in [2, 10]$, from (1.5), we obtain $r < 2.5 \cdot 10^{29}$. First, we computed Eq. (1.4) for $r \in [0, 250]$ and $s, t \geq 0$ with the help of Maple program revealed the list of solutions presented in Corollary 1.2. For the remaining possibility, we assume that $r > 250$.

Let $\Gamma_1 := (r - s - 1)\log\alpha - k\log\eta - \log 2$. If $s \geq 2$ in (3.3), thus $|\Lambda_1| < 0.77$. Applying Lemma 2.5, we get

$$|\Gamma_1| = -\frac{\log(1 - 0.77)}{0.77} \cdot \frac{2}{\alpha^s} < \frac{3.9}{\alpha^s}.$$

It follows that

$$0 < |(r - s - 1)\log\alpha - k\log\eta - \log 2| < 3.9\exp(-(s)\log\alpha). \tag{4.1}$$

Now, we consider the following cases.

Case 1. If $\eta \in [3, 10]$, it can be seen that $\beta \neq 0$ in (4.1), applying Lemma 2.4 and putting

$$c := 3.9, \quad \delta := \log\alpha, \quad \psi := \frac{\log 2}{\log\eta}, \quad \vartheta := \frac{\log\alpha}{\log\eta},$$
$$\vartheta_1 := \log\alpha, \quad \vartheta_2 := -\log\eta, \quad \beta := -\log 2.$$

Let $X_0 := 2.5 \cdot 10^{29}$. By using the Maple program to compute Lemma 2.4, we present the results in Table 2.

Table 2

Reducing the upper bound on s

η	3	4	5	6	7	8	9	10
q_k	q_{63}	q_{65}	q_{60}	q_{59}	q_{67}	q_{61}	q_{58}	q_{61}
s	168.25	148.65	155.1	158.89	148.64	160.34	154.15	157.95

Case 2. If $\eta = 2$, then (4.1) becomes $|(r - s - 1)\log\alpha - (k + 1)\log 2| < 3.9\exp(-(s)\log\alpha)$. It is clear that $\beta = 0$. We have $X_0 = 2.5 \cdot 10^{29}$, applying

Lemma 2.3, we compute the values of Y_0 and A as

$$Y_0 = -1 + \frac{\log(\sqrt{5} \cdot 2.5 \cdot 10^{29} + 1)}{\log\left(\frac{1+\sqrt{5}}{2}\right)} = 141.34, \quad A = \max_{0 \le k \le 141} a_{k+1} = 134.$$

We conclude that

$$s < \frac{1}{\log \alpha} \cdot \left(\frac{3.9(134+2)2.5 \cdot 10^{29}}{|\log 2|} \right) = 154.47.$$

In all cases we obtain that $s < 168$. From inequality (3.5), we get $r < 1.5 \cdot 10^{17} (\log \eta)^3$.

Let $\Gamma_2 := -(r) \log \alpha + k \log \eta + \log\left(\frac{\sqrt{5} L e_s}{(1-\sqrt{5}\alpha^{t-r})}\right)$. If $r \ge 7$ in (3.4), we obtain $|\Lambda_2| < 0.62$. By Lemma 2.5, we get $|\Gamma_2| < 28.1$. Therefore, we conclude that

$$0 < \left| -(r) \log \alpha + k \log \eta + \log\left(\frac{\sqrt{5} L e_s}{(1-\sqrt{5}\alpha^{t-r})} \right) \right| < 28.1 \exp(-(r) \log \alpha).$$

Clearly, $\beta \neq 0$ in the inequality above. Thus, applying Lemma 2.4 and choosing

$$c := 28.1, \quad \delta := \log \alpha, \quad \psi := \log\left(\frac{\sqrt{5} L e_s}{(1-\sqrt{5}\alpha^{t-r})} \right) / \log \eta,$$

$$\vartheta := \frac{\log \alpha}{\log \eta}, \quad \vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log \eta, \quad \beta := \log\left(\frac{\sqrt{5} L e_s}{(1-\sqrt{5}\alpha^{t-r})} \right).$$

Let $X_0 := 1.84 \cdot 10^{18}$ for all $\eta \in [2, 10]$. Thus, we have $2 \le r - t \le s + 3 + \frac{\log(2\eta+2)}{\log \alpha}$, which implies that $2 \le r - t \le 177$ for each $\eta \in [2, 10]$. Then, Table 3 was calculated with the help of a Maple program.

T a b l e 3

Reducing the upper bound on r

η	2	3	4	5	6	7	8	9	10
q_k	q_{60}	q_{56}	q_{58}	q_{53}	q_{52}	q_{61}	q_{52}	q_{51}	q_{54}
r	179.48	183.85	178.04	183.9	178.03	184.72	177.2	178.2	179.1

Hence, $r \le 184$, which contradicts our assumption. Thus, the proof of Corollary 1.2 is complete.

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