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MATHEMATICS

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CYCLIC POLYGONS IN CLASSICAL GEOMETRY, II

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Abstract

Formulas about the side lengths, diagonal lengths or radius of the circumcircle of a cyclic polygon, cyclic ideal polygon and cyclic hyperideal polygon in hyperbolic geometry can be unified.

Key words: cyclic polygon, ideal polygon, hyperideal polygon, hyperbolic geometry, diagonal, radius

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1. Introduction. In [1], the authors proved that a polynomial relation satisfied by the side lengths, diagonal lengths and circumradius of a Euclidean cyclic polygon is also satisfied by those of a hyperbolic or spherical cyclic polygon.

In this paper such a polynomial relation can be generalized to truncated ideal and truncated hyperideal polygons in hyperbolic geometry.

Suppose a hyperbolic polygon in the hyperbolic plane has vertices $P_1, P_2, \ldots, P_n$. The distance between the vertices P_i and P_j is denoted by $|P_iP_j|$ which is the length of either a side or a diagonal of the polygon.

If the points $P_1, P_2, \ldots, P_n$ are on a circle with finite radius, the polygon is a cyclic polygon.

Let $H_1, H_2, \ldots, H_n$ be n mutually disjoint horocycles in the hyperbolic plane. For each pair $\{H_i, H_j\}$, there is a unique geodesic segment perpendicular to both H_i and H_j, and the length of the segment realizes the distance between H_i and H_j. The distance between H_i and H_j is denoted by $|H_iH_j|$. In fact, if H_i and H_j are tangent to the ideal boundary of the hyperbolic plane at points Q_i and Q_j,

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the geodesic determined by Q_i and Q_j is perpendicular to both H_i and H_j. With these geodesic segments, the horocycles $H_1, H_2, \ldots, H_n$ determine a truncated ideal polygon.

If the horocycles $H_1, H_2, \ldots, H_n$ are tangent to a circle with finite radius, the truncated ideal polygon is a cyclic ideal polygon.

Let $G_1, G_2, \ldots, G_n$ be n mutually disjoint geodesics in the hyperbolic plane. For each pair $\{G_i, G_j\}$, there is a unique geodesic segment perpendicular to both G_i and G_j, and the length of the segment realizes the distance between G_i and G_j. The distance between G_i and G_j is denoted by $|G_iG_j|$. With these geodesic segments perpendicular to them, the geodesics $G_1, G_2, \ldots, G_n$ determine a truncated hyperideal polygon.

If the geodesics $G_1, G_2, \ldots, G_n$ are tangent to a circle with finite radius, the truncated hyperideal polygon is a cyclic hyperideal polygon.

2. Triangle. For a triangle in the hyperbolic plane, assume that it has the side lengths a, b, c and the radius of circumcircle r. Then

$$\frac{1}{4}\sinh^2 r = \frac{(\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2})^2}{A_1(A_1-2\sinh\frac{a}{2})(A_1-2\sinh\frac{b}{2})(A_1-2\sinh\frac{c}{2})}, \tag{1}$$

where $A_1 = \sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2}$.

For three mutually disjoint horocycles tangent to a circle with finite radius r in the hyperbolic plane, assume the distances between the three pairs of horocycles are a, b, c. Then

$$\frac{1}{4}e^{2r} = \frac{e^{a+b+c}}{\left(e^{\frac{a}{2}}+e^{\frac{b}{2}}+e^{\frac{c}{2}}\right)\left(-e^{\frac{a}{2}}+e^{\frac{b}{2}}+e^{\frac{c}{2}}\right)\left(e^{\frac{a}{2}}-e^{\frac{b}{2}}+e^{\frac{c}{2}}\right)\left(e^{\frac{a}{2}}+e^{\frac{b}{2}}-e^{\frac{c}{2}}\right)}. \tag{2}$$

For three mutually disjoint geodesics tangent to a circle with finite radius r in the hyperbolic plane, assume the distances between the three pairs of geodesics are a, b, c. Then

$$\frac{1}{4}\cosh^2 r = \frac{(\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2})^2}{A_{-1}(A_{-1}-2\cosh\frac{a}{2})(A_{-1}-2\cosh\frac{b}{2})(A_{-1}-2\cosh\frac{c}{2})}, \tag{3}$$

where $A_{-1} = \cosh\frac{a}{2}+\cosh\frac{b}{2}+\cosh\frac{c}{2}$.

The three cases can be unified using the function introduced in [2]:

$$\tau_s(l) = \frac{1}{2}(e^l - se^{-l}) = \begin{cases} \sinh l & \text{for } s=1, \\ \frac{1}{2}e^l & \text{for } s=0, \\ \cosh l & \text{for } s=-1. \end{cases}$$

Equations (1), (2) and (3) are unified as, for $s \in \{1, 0, -1\}$,

$$\frac{1}{4}\tau_s^2(r) = \frac{\left(\tau_s(\frac{a}{2})\tau_s(\frac{b}{2})\tau_s(\frac{c}{2})\right)^2}{A\left(A-2\tau_s(\frac{a}{2})\right)\left(A-2\tau_s(\frac{b}{2})\right)\left(A-2\tau_s(\frac{c}{2})\right)}, \tag{4}$$

where $A = \tau_s(\frac{a}{2}) + \tau_s(\frac{b}{2}) + \tau_s(\frac{c}{2})$.

In the following, we present a proof of equation (4). This proof serves as an example of the proof of the main result, Theorem 1.

Proof of (4)**.** Let α, β and γ be angles in $(0, 2\pi)$ satisfying $\alpha + \beta + \gamma = 2\pi$. Then

$$\sin\frac{\gamma}{2} = \sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\alpha}{2}.$$

Therefore

$$\left(\sin\frac{\gamma}{2} - \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\right)^2 = \sin^2\frac{\beta}{2}\cos^2\frac{\alpha}{2},$$

or

$$\sin^2\frac{\gamma}{2} - 2\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin^2\frac{\alpha}{2}\left(1 - \sin^2\frac{\beta}{2}\right) = \sin^2\frac{\beta}{2}\left(1 - \sin^2\frac{\alpha}{2}\right)$$

which simplifies to

$$\sin^2\frac{\gamma}{2} + \sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2} = 2\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\cos\frac{\beta}{2}.$$

Squaring both sides yields

$$\left(\sin^2\frac{\gamma}{2} + \sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}\right)^2 = 4\sin^2\frac{\gamma}{2}\sin^2\frac{\alpha}{2}\left(1 - \sin^2\frac{\beta}{2}\right)$$

which is equivalent to

(5)
$$4\left(\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)^2 = \left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)\left(-\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) \times \left(\sin\frac{\alpha}{2} - \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} - \sin\frac{\gamma}{2}\right).$$

Equation (4) will be derived from equation (5).

First, suppose P_1, P_2 and P_3 are vertices of a hyperbolic triangle with circumcentre O and circumradius r. Suppose the side lengths of the triangle are $|P_1P_2| = a$, $|P_2P_3| = b$ and $|P_3P_1| = c$.

In the triangles $\triangle OP_1P_2$, $\triangle OP_2P_3$ and $\triangle OP_3P_1$, by trigonometry

$$\sin\frac{\alpha}{2} = \frac{\sinh\frac{a}{2}}{\sinh r}, \quad \sin\frac{\beta}{2} = \frac{\sinh\frac{b}{2}}{\sinh r}, \quad \sin\frac{\gamma}{2} = \frac{\sinh\frac{c}{2}}{\sinh r}. \tag{6}$$

Second, let H_1, H_2 and H_3 be mutually disjoint horocycles tangent to a circle with centre O and radius r. Suppose $|H_1H_2| = a$, $|H_2H_3| = b$ and $|H_3H_1| = c$. For each $i = 1, 2, 3$, let R_i denote the point of tangency between H_i and C. For each $i = 1, 2, 3$, consider the generalized triangle of type $(0, 0, 1)$ with sides OR_i, OR_{i+1} and the common perpendicular to H_i and H_{i+1}, where $H_4 = H_1$. For more discussions of generalized triangles in hyperbolic plane, see [2].

In the three generalized triangles of type $(0, 0, 1)$, by trigonometry [2], we obtain

$$\sin\frac{\alpha}{2} = \frac{e^{\frac{a}{2}}}{e^r}, \quad \sin\frac{\beta}{2} = \frac{e^{\frac{b}{2}}}{e^r}, \quad \sin\frac{\gamma}{2} = \frac{e^{\frac{c}{2}}}{e^r}. \tag{7}$$

Third, let G_1, G_2 and G_3 be mutually disjoint geodesics tangent to a circle with centre O and radius r. Suppose $|G_1G_2| = a$, $|G_2G_3| = b$ and $|G_3G_1| = c$. For each $i = 1, 2, 3$, let R_i denote the point of tangency between G_i and C. For each $i = 1, 2, 3$, consider the generalized triangle of type $(-1, -1, 1)$ with sides OR_i, OR_{i+1} and the common perpendicular to G_i and G_{i+1}, where $G_4 = G_1$.

In the three generalized triangles of type $(-1, -1, 1)$, by trigonometry [2], we obtain

$$\sin\frac{\alpha}{2} = \frac{\cosh\frac{a}{2}}{\cosh r}, \quad \sin\frac{\beta}{2} = \frac{\cosh\frac{b}{2}}{\cosh r}, \quad \sin\frac{\gamma}{2} = \frac{\cosh\frac{c}{2}}{\cosh r}. \tag{8}$$

The equations in (6), (7) and (8) can be unified as

$$\sin\frac{\alpha}{2} = \frac{\tau_s(\frac{a}{2})}{\tau_s(r)}, \quad \sin\frac{\beta}{2} = \frac{\tau_s(\frac{b}{2})}{\tau_s(r)}, \quad \sin\frac{\gamma}{2} = \frac{\tau_s(\frac{c}{2})}{\tau_s(r)}. \tag{9}$$

Substituting the expressions of $\sin\frac{\alpha}{2}, \sin\frac{\beta}{2}$ and $\sin\frac{\gamma}{2}$ in equation (9) into equation (5), we get

$$\begin{aligned} 4\left(\tau_s(\tfrac{a}{2})\tau_s(\tfrac{b}{2})\tau_s(\tfrac{c}{2})\right)^2 = \tau_s^2(r) &\left(\tau_s(\tfrac{a}{2}) + \tau_s(\tfrac{b}{2}) + \tau_s(\tfrac{c}{2})\right)\left(-\tau_s(\tfrac{a}{2}) + \tau_s(\tfrac{b}{2}) + \tau_s(\tfrac{c}{2})\right) \\ &\times \left(\tau_s(\tfrac{a}{2}) - \tau_s(\tfrac{b}{2}) + \tau_s(\tfrac{c}{2})\right)\left(\tau_s(\tfrac{a}{2}) + \tau_s(\tfrac{b}{2}) - \tau_s(\tfrac{c}{2})\right) \end{aligned}$$

which is equivalent to equation (4). □

3. Cyclic quadrilateral. For a convex quadrilateral with vertices P_1, P_2, P_3, P_4 which lie on a circle with finite radius r in the hyperbolic plane, suppose the distance between the vertices are $|P_1P_2| = a$, $|P_2P_3| = b$, $|P_3P_4| = c$, $|P_4P_1| = d$, $|P_1P_3| = e$, $|P_2P_4| = f$.

For four mutually disjoint horocycles H_1, H_2, H_3, H_4 which are tangent to a circle with finite radius r in the hyperbolic plane, suppose the distance between the vertices are $|H_1H_2| = a$, $|H_2H_3| = b$, $|H_3H_4| = c$, $|H_4H_1| = d$, $|H_1H_3| = e$, $|H_2H_4| = f$.

For mutually disjoint geodesics G_1, G_2, G_3, G_4 which are tangent to a circle with finite radius r in the hyperbolic plane, suppose the distance between the vertices are $|G_1G_2| = a$, $|G_2G_3| = b$, $|G_3G_4| = c$, $|G_4G_1| = d$, $|G_1G_3| = e$, $|G_2G_4| = f$.

In the three cases, the formulas representing diagonal lengths in terms of side lengths are

$$\tau_s^2(\tfrac{e}{2}) = \big(\tau_s(\tfrac{a}{2})\tau_s(\tfrac{c}{2}) + \tau_s(\tfrac{b}{2})\tau_s(\tfrac{d}{2})\big)\,\frac{\tau_s(\frac{a}{2})\tau_s(\frac{d}{2}) + \tau_s(\frac{b}{2})\tau_s(\frac{c}{2})}{\tau_s(\frac{a}{2})\tau_s(\frac{b}{2}) + \tau_s(\frac{c}{2})\tau_s(\frac{d}{2})}, \tag{10}$$

$$\tau_s^2(\tfrac{f}{2}) = \big(\tau_s(\tfrac{a}{2})\tau_s(\tfrac{c}{2}) + \tau_s(\tfrac{b}{2})\tau_s(\tfrac{d}{2})\big)\,\frac{\tau_s(\frac{a}{2})\tau_s(\frac{b}{2}) + \tau_s(\frac{c}{2})\tau_s(\frac{d}{2})}{\tau_s(\frac{a}{2})\tau_s(\frac{d}{2}) + \tau_s(\frac{b}{2})\tau_s(\frac{c}{2})}. \tag{11}$$

The multiplication of the two equations implies Ptolemy's theorem

$$\tau_s(\tfrac{e}{2})\tau_s(\tfrac{f}{2}) = \tau_s(\tfrac{a}{2})\tau_s(\tfrac{c}{2}) + \tau_s(\tfrac{b}{2})\tau_s(\tfrac{d}{2}).$$

When $s = 0$, it becomes

$$e^{\frac{1}{2}(e+f)} = e^{\frac{1}{2}(a+c)} + e^{\frac{1}{2}(b+d)}.$$

Penner [3] proved that this formula holds for any decorated ideal quadrilateral and the distances a, b, c, d, e, f are allowed to be negative.

In fact, for example, if the horocycle H_1 is shrunken away from the tangent circle by a distance $x > 0$, the distance $|H_1H_2|$ becomes $a+x$, the distance $|H_1H_3|$ becomes $e+x$, and the distance $|H_4H_1|$ becomes $d+x$. Then Ptolemy's theorem still holds

$$e^{\frac{1}{2}(e+x+f)} = e^{\frac{1}{2}(a+x+c)} + e^{\frac{1}{2}(b+d+x)}.$$

For more discussion on this formula and its applications in theory of Teichmüller space and mathematical physics, see [4].

Proof of (10)**,** (11)**.** Let α, β, γ and δ be angles in $(0, 2\pi)$ satisfying $\alpha+\beta+\gamma+\delta = 2\pi$.

We claim that

$$\sin^2\frac{\alpha+\beta}{2} = \left(\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} + \sin\frac{\beta}{2}\sin\frac{\delta}{2}\right)\frac{\sin\frac{\alpha}{2}\sin\frac{\delta}{2} + \sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\sin\frac{\delta}{2}}, \tag{12}$$

$$\sin^2\frac{\alpha+\delta}{2} = \left(\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} + \sin\frac{\beta}{2}\sin\frac{\delta}{2}\right)\frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\sin\frac{\delta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\delta}{2} + \sin\frac{\beta}{2}\sin\frac{\gamma}{2}}. \tag{13}$$

To verify them, we start with

$$\cos\frac{\alpha+\beta}{2} + \cos\frac{\gamma+\delta}{2} = 0,$$

or

$$\left(\sin\frac{\alpha}{2}\sin\frac{\beta}{2} - \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right) + \left(\sin\frac{\gamma}{2}\sin\frac{\delta}{2} - \cos\frac{\gamma}{2}\cos\frac{\delta}{2}\right) = 0.$$

Multiplying $2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sin\frac{\delta}{2}$ on both sides of the above equation yields

$$\left(2\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}-2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right)\sin\frac{\gamma}{2}\sin\frac{\delta}{2}$$
$$+\left(2\sin^2\frac{\gamma}{2}\sin^2\frac{\delta}{2}-2\sin\frac{\gamma}{2}\sin\frac{\delta}{2}\cos\frac{\gamma}{2}\cos\frac{\delta}{2}\right)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}=0,$$

which is equivalent to

$$\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}-\sin^2\frac{\alpha+\beta}{2}\right)\sin\frac{\gamma}{2}\sin\frac{\delta}{2}$$
$$+\left(\sin^2\frac{\gamma}{2}+\sin^2\frac{\delta}{2}-\sin^2\frac{\gamma+\delta}{2}\right)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}=0.$$

Rearranging the terms yields

$$\sin^2\frac{\alpha+\beta}{2}\sin\frac{\gamma}{2}\sin\frac{\delta}{2}+\sin^2\frac{\gamma+\delta}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$
$$=\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}\right)\sin\frac{\gamma}{2}\sin\frac{\delta}{2}+\left(\sin^2\frac{\gamma}{2}+\sin^2\frac{\delta}{2}\right)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Using the fact $\sin\frac{\alpha+\beta}{2}=\sin\frac{\gamma+\delta}{2}$ to the left-hand side and factorizing the right-hand side of the above equation, we have

$$\sin^2\frac{\alpha+\beta}{2}\left(\sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\sin\frac{\delta}{2}\right)$$
$$=\left(\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\delta}{2}\right)\left(\sin\frac{\alpha}{2}\sin\frac{\delta}{2}+\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right),$$

which is equivalent to (12).

Equation (13) is derived by switching β and δ in (12).

Using similar construction as in the case of triangles, for a cyclic quadrilateral, a cyclic truncated ideal quadrilateral or a cyclic truncated hyperideal quadrilateral with side lengths a, b, c, d and lengths of diagonals e, f, by trigonometry, we obtain

$$\begin{gathered}\sin\frac{\alpha}{2}=\frac{\tau_s(\frac{a}{2})}{\tau_s(r)},\quad \sin\frac{\beta}{2}=\frac{\tau_s(\frac{b}{2})}{\tau_s(r)},\quad \sin\frac{\gamma}{2}=\frac{\tau_s(\frac{c}{2})}{\tau_s(r)},\quad \sin\frac{\delta}{2}=\frac{\tau_s(\frac{d}{2})}{\tau_s(r)},\\ \sin\frac{\alpha+\beta}{2}=\frac{\tau_s(\frac{e}{2})}{\tau_s(r)},\quad \sin\frac{\alpha+\delta}{2}=\frac{\tau_s(\frac{f}{2})}{\tau_s(r)},\end{gathered}\tag{14}$$

where the angles α, β, γ, δ have vertex O, the centre of the circumcircle, and are opposite to the sides of lengths a, b, c, d, respectively.

Then equations (10) and (11) are obtained by substituting (14) into equations (12) and (13). □

4. Cyclic polygons. Recall that, in [1], $\mathcal{H}_n$ is defined as the set of polynomials h of $\frac{n(n-1)}{2}+1$ variables[1] such that

$$h\left(\sinh\frac{|P_iP_j|}{2}, \frac{1}{2}\sinh r\right)=0$$

for any n points $P_1, P_2, \ldots, P_n$ on a circle of radius r in the hyperbolic plane.

Now we define $\mathcal{H}_n^1$ to be the set of polynomials f of $\frac{n(n-1)}{2}+1$ variables such that

$$f\left(\sinh\frac{|P_iP_j|}{2}, \sinh r\right)=0$$

for any n points $P_1, P_2, \ldots, P_n$ on a circle of radius r in the hyperbolic plane.

There exists a bijection between $\mathcal{H}_n$ and $\mathcal{H}_n^1$. A polynomial $h(y_{12}, y_{13}, \ldots, y_{n-1,n}, z)$ belongs to $\mathcal{H}_n$ if and only if the polynomial

$$f(y_{12}, y_{13}, \ldots, y_{n-1,n}, z)=h\left(y_{12}, y_{13}, \ldots, y_{n-1,n}, \frac{1}{2}z\right)$$

belongs to $\mathcal{H}_n^1$.

As an analogue, $\mathcal{H}_n^0$ is defined as the set of polynomials f of $\frac{n(n-1)}{2}+1$ variables such that

$$f\left(\frac{1}{2}e^{\frac{|H_iH_j|}{2}}, \frac{1}{2}e^r\right)=0$$

for any n mutually disjoint horocycles $H_1, H_2, \ldots, H_n$ tangent to a circle of radius r in the hyperbolic plane.

And $\mathcal{H}_n^{-1}$ is defined as the set of polynomials f of $\frac{n(n-1)}{2}+1$ variables such that

$$f\left(\cosh\frac{|G_iG_j|}{2}, \cosh r\right)=0$$

for any n mutually disjoint geodesics $G_1, G_2, \ldots, G_n$ tangent to a circle of radius r in the hyperbolic plane.

Theorem 1. $\mathcal{H}_n^1=\mathcal{H}_n^0=\mathcal{H}_n^{-1}$.

In fact, a polynomial in the set $\mathcal{H}_n^1=\mathcal{H}_n^0=\mathcal{H}_n^{-1}$ can be expressed uniformly as

$$f\left(\tau_s(\frac{l_{ij}}{2}), \tau_s(r)\right)=0,$$

where l_{ij} is the distance between two points, two disjoint horocycles or two disjoint geodesics.

[1]The error in [1] regarding the use of “$\frac{n(n+1)}{2}$ variables” is corrected here.

5. Homogeneity. To prove Theorem 1, we need the following lemma.

Lemma 2. *If* $f \in \mathcal{H}_n^1 \cup \mathcal{H}_n^0 \cup \mathcal{H}_n^{-1}$*, then* f *is homogeneous.*

Proof. It is proved in [1] that any $f \in \mathcal{H}_n$ is homogeneous. The bijection between $\mathcal{H}_n$ and $\mathcal{H}_n^1$ implies that any $f \in \mathcal{H}_n^1$ is also homogeneous.

Assume $f \in \mathcal{H}_n^0$. Then

$$f\left(\frac{1}{2}e^{\frac{|H_iH_j|}{2}}, \frac{1}{2}e^r\right) = 0$$

for n mutually disjoint horocycles $H_1, H_2, \ldots, H_n$ tangent to a circle C of radius r in the hyperbolic plane. Let C' be the circle with the same centre of circle C and radius $r+x$ with $x>0$. Let H_i' be the horocycle which is tangent to the ideal boundary of the Poincaré disk at the same point as H_i and is tangent to the circle C'. Then $|H_i'H_j'| = |H_iH_j| + 2x$. So the relation is also satisfied for the horocycles H_i''s and the circle C':

$$f\left(\frac{1}{2}e^{\frac{|H_iH_j|+2x}{2}}, \frac{1}{2}e^{r+x}\right) = 0.$$

Therefore f is homogeneous since the above equation holds for any $x>0$.

Assume $f \in \mathcal{H}_n^{-1}$. Then

$$f\left(\cosh\frac{|G_iG_j|}{2}, \cosh r\right) = 0$$

for n mutually disjoint geodesics $G_1, G_2, \ldots, G_n$, each of which is tangent to a circle C with centre O and radius r in the hyperbolic plane. Suppose O is the origin. Let R_i denote the point of tangency between G_i and C, for each $i = 1, 2, \ldots, n$.

A hyperbolic geodesic G_i is a portion of a Euclidean circle with centre O_i and perpendicular to the ideal boundary of the Poincaré disk. Let l be a straight passing through O and O_i. Then l is perpendicular to G_i and passes through the point R_i.

Consider the generalized triangle of type $(-1,-1,1)$ with sides OR_i, OR_j and the common perpendicular to G_i and G_j. It has side lengths r, r and $|G_iG_j|$. Half of this triangle forms a right triangle, to which we apply the sine law for a generalized hyperbolic triangle [2], yielding

$$\frac{\sin\frac{\angle R_iOR_j}{2}}{\cosh\frac{|G_iG_j|}{2}} = \frac{\sin\frac{\pi}{2}}{\cosh r}.$$

Let C' be the circle with centre O and radius $r+x$ with $x>0$. For each i, let G_i' be the hyperbolic geodesic tangent to the circle C', and be a portion of Euclidean circle with centre $O_i' \in l$ and perpendicular to the ideal boundary of the Poincaré disk. Let R'_i denote the point of tangency between G'_i and C. Then l is perpendicular to G'_i and passes through the point R'_i.

In fact, G_i and G'_i determine a hyperbolic pencil of geodesics perpendicular to l. For more discussion on hyperbolic pencils, see [5, p. 72] and [6, p. 168].

Similarly, by the sine law for a generalized hyperbolic triangle, we have

$$\frac{\sin\frac{\angle R'_iOR'_j}{2}}{\cosh\frac{|G'_iG'_j|}{2}} = \frac{\sin\frac{\pi}{2}}{\cosh(r+x)}.$$

Since $\angle R'_iOR'_j = \angle R_iOR_j$, we have

$$\cosh\frac{|G_iG_j|}{2}\frac{\cosh(r+x)}{\cosh r} = \cosh\frac{|G'_iG'_j|}{2}.$$

Since the relation holds also for the geodesics G'_i's and circle C':

$$f\left(\cosh\frac{|G'_iG'_j|}{2}, \cosh(r+x)\right) = 0.$$

Then

$$0 = f\left(\cosh\frac{|G_iG_j|}{2}\frac{\cosh(r+x)}{\cosh r}, \cosh(r+x)\right) = f\left(y\cosh\frac{|G_iG_j|}{2}, y\cosh r\right)$$

for an arbitrary $y = \frac{\cosh(r+x)}{\cosh r}$. Therefore f is homogeneous. □

6. Relations of angles. Let O be a point in the hyperbolic plane, and let $l_1, \dots, l_n$ be a family of geodesic rays emanating from O, with each ray l_{i+1} obtained from l_i by a counterclockwise rotation, for $i = 1, \dots, n$, and $l_{n+1} = l_1$. Let $\theta_{ij} \in (0, 2\pi)$ denote the angle between l_i and l_j. The ambiguity between θ_{ij} and $2\pi - \theta_{ij}$ is irrelevant, as only $\sin\frac{\theta_{ij}}{2}$ will be used.

Let $\mathcal{A}_n$ be the set of all polynomials g in the $\frac{n(n-1)}{2}$ variables x_{ij}, where $1 \le i < j \le n$, such that $g\left(\sin\frac{\theta_{12}}{2}, \sin\frac{\theta_{13}}{2}, \dots, \sin\frac{\theta_{n-1,n}}{2}\right) = 0$.

For a polynomial $g(x_{12}, x_{13}, \dots, x_{n-1,n}) \in \mathcal{A}_n$ of degree m, the homogenization of g produces a homogeneous polynomial g^*, defined by the relation

$$g^*(y_{12}, y_{13}, \dots, y_{n-1,n}, z) = z^m g\left(\frac{y_{12}}{z}, \frac{y_{13}}{z}, \dots, \frac{y_{n-1,n}}{z}\right).$$

To prove Theorem 1, we will show that each of $\mathcal{H}_n^1$, $\mathcal{H}_n^0$ and $\mathcal{H}_n^{-1}$ equals $\mathcal{A}_n^* = \{g^* \colon g \in \mathcal{A}_n\}$.

Proof of Theorem 1. First, let $P_1, \dots, P_n$ be n-points in a circle with centre O and radius r in the hyperbolic plane. Let l_i be the geodesic ray from O to P_i.

In the triangle $\triangle OP_iP_j$, by trigonometry, we have

$$\frac{\sin\frac{\theta_{ij}}{2}}{\sinh\frac{|P_iP_j|}{2}} = \frac{\sin\frac{\pi}{2}}{\sinh r}$$

then

$$\sin\frac{\theta_{ij}}{2} = \frac{\sinh\frac{|P_iP_j|}{2}}{\sinh r}.$$

Second, let $H_1, \dots, H_n$ be n mutually disjoint horocycles tangent to a circle with centre O and radius r in the hyperbolic plane. Let l_i be the geodesic ray from O to the tangent point in H_i.

For each i, let R_i denote the point of tangency between H_i and the circle. In the generalized triangle of type $(0,0,1)$ with sides OR_i, OR_j and the common perpendicular to H_i and H_j, by trigonometry [2], we have

$$\sin\frac{\theta_{ij}}{2} = \frac{\frac{1}{2}e^{\frac{|H_iH_j|}{2}}}{\frac{1}{2}e^r}.$$

Third, let $G_1, \dots, G_n$ be n mutually disjoint geodesics tangent to a circle with centre O and radius r in the hyperbolic plane. Let l_i be the geodesic ray from O to the tangent point in G_i.

For each i, let R_i denote the point of tangency between G_i and the circle. In the generalized triangle of type $(-1,-1,1)$ with sides OR_i, OR_j and the common perpendicular to G_i and G_j, by trigonometry [2], we have

$$\sin\frac{\theta_{ij}}{2} = \frac{\cosh\frac{|G_iG_j|}{2}}{\cosh r}.$$

The three cases can be unified as

$$\sin\frac{\theta_{ij}}{2} = \frac{\tau_s\left(\frac{l_{ij}}{2}\right)}{\tau_s(r)},$$

where l_{ij} is the distance between two points, two disjoint horocycles or two disjoint geodesics.

If a polynomial g in $\frac{n(n-1)}{2}$ variables belongs to $\mathcal{A}_n$, then

$$g\left(\sin\frac{\theta_{12}}{2}, \sin\frac{\theta_{13}}{2}, \dots, \sin\frac{\theta_{n-1,n}}{2}\right) = 0,$$

which implies

$$g\left(\frac{\tau_s\left(\frac{l_{12}}{2}\right)}{\tau_s(r)}, \frac{\tau_s\left(\frac{l_{13}}{2}\right)}{\tau_s(r)}, \dots, \frac{\tau_s\left(\frac{l_{n-1,n}}{2}\right)}{\tau_s(r)}\right) = 0.$$

Let m be the degree of g, multiplying $\tau_s^m(r)$ on both sides of the above equation which yields

$$g^*\left(\tau_s\left(\frac{l_{12}}{2}\right), \tau_s\left(\frac{l_{13}}{2}\right), \ldots, \tau_s\left(\frac{l_{n-1,n}}{2}\right), \tau_s(r)\right) = 0.$$

This establishes that $g^* \in \mathcal{H}_n^s$ for $s \in \{1, 0, -1\}$.

On the other hand, if a polynomial f in $\frac{n(n-1)}{2} + 1$ variables belongs to $\mathcal{H}_n^s$ for $s \in \{1, 0, -1\}$, then

$$f\left(\tau_s\left(\frac{l_{12}}{2}\right), \tau_s\left(\frac{l_{13}}{2}\right), \ldots, \tau_s\left(\frac{l_{n-1,n}}{2}\right), \tau_s(r)\right) = 0.$$

Since f is homogeneous, we have

$$f\left(\frac{\tau_s\left(\frac{l_{12}}{2}\right)}{\tau_s(r)}, \frac{\tau_s\left(\frac{l_{13}}{2}\right)}{\tau_s(r)}, \ldots, \frac{\tau_s\left(\frac{l_{n-1,n}}{2}\right)}{\tau_s(r)}, 1\right) = 0.$$

It becomes

$$f\left(\sin\frac{\theta_{12}}{2}, \sin\frac{\theta_{13}}{2}, \ldots, \sin\frac{\theta_{n-1,n}}{2}, 1\right) = 0.$$

Rewrite $g(x_{12}, x_{13}, \ldots, x_{n-1,n}) = f(x_{12}, x_{13}, \ldots, x_{n-1,n}, 1)$. Then g is a polynomial in $\mathcal{A}_n$ and $f = g^*$. □

Remark. This proof also works for Euclidean and spherical geometry. So there is a one-to-one correspondence between $\mathcal{E}_n$ (or $\mathcal{S}_n$) and $\mathcal{A}_n^*$. This provides another proof of Theorem 1 in [1].

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