Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 78, No 1, 2025

MATHEMATICS
Functional analysis

ON THE SYMMETRIZED DUAL OF SCHLUMPRECHT SPACE

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Received on October 17, 2022
Presented by S. Troyanski, Member of BAS, on April 25, 2023

Abstract

We prove that the symmetric version of the dual of Schlumprecht space contains a subspace isomorphic to l_1.

Key words: Schlumprecht space, dual, subsymmetric basis, symmetric basis, symmetrization, ℓ_1

2020 Mathematics Subject Classification: 46B15, 46B20, 46B45

1. Introduction. A central question of the structure theory of Banach spaces is whether every infinite-dimensional space contains an isomorphic copy of a "nice" infinite-dimensional space. The first question in that direction was if any Banach space contained a subspace isomorphic to c_0 or $l_p, 1 \leq p < \infty$, or, more generally, a symmetric basic sequence. In 1974 Tsirelson [1] constructed a Banach space without any symmetric sequence. He did it by defining the unit ball of his space as the closed convex hull of a carefully chosen set of vectors. Later, Figiel and Johnson [2] found a nice formula for the norm of the dual of that space and named it "Tsirelson space T". Now everywhere in literature the original space constructed by Tsirelson is referred to as the dual T^*.

In 1991 Schlumprecht [3] constructed another Tsirelson-like space which was the first example of an arbitrarily distortable Banach space. It became the foundation for the celebrated space of Gowers and Maurey [4] which has no unconditional basic sequence. The canonical basis of Schlumprecht space S us subsymmetric but not symmetric.

https://doi.org/10.7546/CRABS.2025.01.02

The natural symmetrization $S(T)$ of the Tsirelson space T contains a subspace isomorphic to l_1, while the symmetrization $S(T^*)$ of the original Tsirelson space is reflexive, so it does not have the same property [5]. In an unpublished note Schlumprecht showed that the symmetric version of his space S does contain a subspace isomorphic to l_1 (a similar result can be found in [6]).

In this note we prove that, in contrast to the case of $S(T^*)$, the symmetrization $S(S^*)$ of the dual of Schlumprecht space does contain isomorphically the space l_1.

2. Definitions and main result. A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space is a basic sequence if it is a Schauder basis of its closed linear span. Two normalized basic sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in a Banach space $(X, \|\cdot\|)$ are said to be C-equivalent, $C \geq 1$, provided that

$$\frac{1}{C}\left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \leq C\left\|\sum_{n=1}^{\infty} a_n x_n\right\|$$

for any choice of real numbers $(a_n)_{n=1}^{\infty}$.

The following are standard definitions from [7]. A basic sequence $(x_n)_{n=1}^{\infty}$ is C-unconditional if for any choice of signs $(\epsilon_n)_{n=1}^{\infty}$, the sequence $(\epsilon_n x_n)_{n=1}^{\infty}$ is C-equivalent to $(x_n)_{n=1}^{\infty}$. A Banach space with a C-unconditional basis can be easily renormed so that the basis is 1-unconditional. A basic sequence $(x_n)_{n=1}^{\infty}$ is K-symmetric if the rearranged sequence $(x_{\pi(n)})_{n=1}^{\infty}$ is C-equivalent to $(x_n)_{n=1}^{\infty}$ for any permutation π of the natural numbers. Closely related to symmetry is the notion of subsymmetry. Similarly, a basic sequence $(x_n)_{n=1}^{\infty}$ is said to be subsymmetric if it is unconditional and any subsequence $(x_{n_j})_{j=1}^{\infty}$ is equivalent to $(x_n)_{n=1}^{\infty}$.

Let $(e_i)_{i=1}^{\infty}$ be an unconditional (say 1-unconditional) Schauder basis of a Banach space $(X, \|\cdot\|)$. One can define a natural symmetrization of X, denoted by $S(X)$ [5] by

$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\|_{S(X)} = \sup_{\pi}\left\|\sum_{i=1}^{\infty} a_i e_{\pi(i)}\right\|,$$

where the supremum is taken over all permutations π of $\mathbb{N}$.

Let $(e_i)_{i=1}^{\infty}$ be the canonical basis of c_{00}, the linear space of all finitely supported real-valued sequences. For $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$, $\operatorname{supp} x$ denotes the set $\{i \in \mathbb{N} : a_i \neq 0\}$.

Let E, F be finite nonempty sets of natural numbers. We write $E < F$ provided $\max E < \min F$. For $x, y \in c_{00}$ we say $x < y$ if $\operatorname{supp} x < \operatorname{supp} y$. For $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$ and E a finite subset of $\mathbb{N}$, $Ex = \sum_{i \in E} a_i e_i$.

We shall now define Schlumprecht space S. Let $f : [1,\infty) \to [1,\infty)$ be the function $f(x) = \log_2(x+1)$. Then $(S, \|\cdot\|)$ is the completion of c_{00} with respect to the norm $\|\cdot\|$ that satisfies the following implicit equation:

$$\|x\| = \max\left\{\|x\|_\infty, \sup_{E_1<E_2<\cdots<E_n} \frac{1}{f(n)} \sum_{i=1}^{n} \|E_i x\|\right\}.$$

Clearly, $(e_i)_{i=1}^{\infty}$ is a 1-unconditional, 1-subsymmetric basis of S, and thus so is the basis $(e_i^*)_{i=1}^{\infty}$ of the dual space S^*.

Like in the case of Tsirelson space one can give an alternative definition through the unit ball of S^*, see [6].

Let $K_0 = \{\pm e_n^* : n \in \mathbb{N}\}$. Define inductively for every $j \geq 0$

$$K_{j+1}{=}K_j \bigcup \left\{\frac{1}{f(n)}\sum_{i=1}^{d} x_i^* : x_i^* \in K_i, d \leq n \in \mathbb{N}, n \geq 2, \operatorname{supp} x_1^* < \cdots < \operatorname{supp} x_d^*\right\}.$$

Finally, let $K = \bigcup_{j=0}^{\infty} K_j$. The unit ball B_{S^*} of S^* is the closed convex hull of K.

Then for $x \in c_{00}$, $\|x\| = \|x\|_S = \sup\langle x^*, x\rangle$. We shall denote the dual norm again by $\|\cdot\|$, that is, $(S^*, \|\cdot\|)$.

Fact 1. *For every* $n \geq 2$,

$$\left\|\frac{1}{f(n)} \sum_{i=1}^{n} e_i^*\right\| = 1.$$

Indeed, $\frac{1}{f(n)} \sum_{i=1}^{n} e_i^* \in K_1 \subseteq B_{S^*}$, so $\left\|\frac{1}{f(n)} \sum_{i=1}^{n} e_i^*\right\| \leq 1$. On the other hand, from [3], $\left\|\frac{f(n)}{n} \sum_{i=1}^{n} e_i\right\| = 1$, and

$$\left\langle \frac{1}{f(n)} \sum_{i=1}^{n} e_i^*, \frac{f(n)}{n} \sum_{i=1}^{n} e_i \right\rangle = 1.$$

We prove now our main result.

Theorem 2. *The symmetrized version of the dual of Schlumprecht space,* $S(S^*)$*, contains a subspace isomorphic to* l_1*.*

Proof. Recall that for $v^* \in S(S^*)$, $v^* = \sum_{i=1}^{\infty} a_i e_i^*$,

$$\|v^*\|_{S(S^*)} = \sup_{\pi} \left\|\sum_{1}^{\infty} a_i e_{\pi(i)}^*\right\|_{S^*},$$

where the supremum is taken over all permutations π of $\mathbb{N}$.

We shall say that two vectors x and y in S have the same distribution if $x = \sum_{j=1}^{k} a_j e_{n_j}$ and $y = \sum_{j=1}^{k} a_j e_{m_j}$, where $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_k$. Since the canonical basis of S is 1-subsymmetric, $\|x\| = \|y\|$. The situation is the same for vectors with the same distribution in S^* and $S(S^*)$, since the basis $(e_i^*)_{i=1}^{\infty}$ in S^* is also 1-subsymmetric and $(e_i^*)_{i=1}^{\infty}$ in $S(S^*)$ is similarly 1-subsymmetric.

The main tool in our proof is the "yardstick" construction from [8], Theorem 3, which gives for arbitrary $n \in \mathbb{N}$ a set of disjointly – but not successively – supported vectors (with respect to the canonical basis), which set is uniformly equivalent to the unit vector basis of l_∞^n. More precisely, there exists an increasing sequence $(p_k)_{k=0}^{\infty}$ of natural numbers, with $p_0 = 1$, and for $k \geq 0$, $p_{k+1} = p_k d_k$, $d_k \in \mathbb{N}$ and $(d_k)_{k=0}^{\infty}$ rapidly increasing, with the following properties. For every $n \in \mathbb{N}$, there exist n disjointly supported vectors $v_j \in S$, $1 \leq j \leq n$, such that for all $1 \leq k \leq n$, v_k has the same distribution as

$$u_{p_k} = \frac{f(p_k)}{p_k} \sum_{i=1}^{p_k} e_i$$

and

$$\left\| \sum_{j=1}^{n} v_j \right\| \leq 2.$$

By [3], $\|v_j\| = 1$. The yardstick construction is defined as follows.

Let $\{w_{m_1,m_2,\ldots,m_j} : j \leq n, m_i \leq d_{i-1} \ \forall i \leq j\}$ be any sequence of natural numbers such that

$$w_{m_1,m_2,\ldots,m_j} < w_{r_1,r_2,\ldots,r_l}$$

if one of the following conditions holds:

- there exists $i \leq \max\{j, l\}$ such that $m_s = r_s$ for all $s < i$, and $m_i < r_i$;
- $j < l$ and $m_s = r_s$ for all $s \leq j$.

Then we define for each $1 \leq j \leq n$,

$$v_j = \frac{f(p_j)}{p_j} \sum_{i_1=1}^{d_0} \sum_{i_2=1}^{d_1} \cdots \sum_{i_j=1}^{d_{j-1}} e_{w_{i_1,i_2,\ldots,i_j}}. \tag{1}$$

By [8] we have $\| \sum_{j=1}^{n} v_j \|_S \leq 2$.

Now, we define inductively vectors $(v_j^*)_{j=1}^{\infty}$ such that for all $j \in \mathbb{N}$, v_j^* has the same distribution as

$$\frac{1}{f(p_j)} \sum_{i=1}^{p_j} e_i^*$$

and their supports are consecutive with respect to $(e_i^*)_{i=1}^{\infty}$, that is to say,

$$\operatorname{supp} v_1^* < \operatorname{supp} v_2^* < \cdots < \operatorname{supp} v_j^* < \cdots$$

Let $n \in \mathbb{N}$ and $(a_j)_{j=1}^n$ be any sequence of n real numbers. Since $(e_i^*)_{i=1}^{\infty}$ is 1-unconditional in both S^* and $S(S^*)$, without loss of generality we may assume for convenience that all $a_j \geq 0$. We want to estimate $\| \sum_{j=1}^n a_j v_j^* \|_{S(S^*)}$.

First, by the fact presented in the beginning, $\|v_j^*\|_{S^*} = 1$. Since for each j, all the nonzero coefficients of v_j^* are equal, v_j^* will have the same distribution under any permutation π of $\mathbb{N}$. Since $(e_i^*)_{i=1}^{\infty}$ is 1-subsymmetric in S^*, it follows from the definition of the norm in $S(S^*)$ that $\|v_j^*\|_{S(S^*)} = 1$ for all $j \in \mathbb{N}$.

Now choose a permutation π of $\mathbb{N}$, depending on n, such that π sends v_j^*, $j = 1, 2, \ldots, n$, to a vector u_j^* such that $\operatorname{supp} u_j^* = \operatorname{supp} v_j$, described in 1 for this particular n. We can do this since $\operatorname{supp} v_i^* \cap \operatorname{supp} v_j^* = \varnothing$ for $i \neq j$. Consider $u_n = \sum_{j=1}^n v_j$. By [8], $\|u_n\|_S \leq 2$. We shall compute $\left\langle \sum_{j=1}^n a_j u_j^*, u_n \right\rangle$. Remark that $\sum_{j=1}^n a_j u_j^*$ is the image of $\sum_{j=1}^n a_j v_j^*$ under the permutation π chosen above. Then

$$\left\langle \sum_{j=1}^n a_j u_j^*, u_n \right\rangle = \sum_{j=1}^n a_j \langle u_j^*, v_j \rangle = \sum_{j=1}^n a_j \frac{f(p_j)}{p_j} \frac{1}{f(p_j)} p_j = \sum_{j=1}^n a_j.$$

Since $\|u_n\|_S \leq 2$, we obtain that

$$\left\| \sum_{j=1}^n a_j u_j^* \right\|_{S^*} \geq \frac{1}{2} \sum_{j=1}^n a_j.$$

By the definition of the norm of $S(S^*)$, we also have

$$\left\| \sum_{j=1}^n a_j v_j^* \right\| \geq \frac{1}{2} \sum_{j=1}^n a_j.$$

Therefore, the sequence $(v_j^*)_{j=1}^{\infty}$ in $S(S^*)$ is 2-equivalent to the unit vector basis of l_1. □

Acknowledgement. The author thanks Professor Denka Kutzarova for posing the question and for helpful suggestions.

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