Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences
Tome 78, No 1, 2025

MATHEMATICS
Functional analysis

A NOTE ON AN ASYMPTOTIC PROPERTY OF THE CONVEXITY OF NORMS

Petar Evgeniev

Received on August 8, 2024
Presented by S. Troyanski, Member of BAS, on December 17, 2024

Abstract

We observe that the recently introduced property local asymptotically midpoint uniform convexity (LAMUC) is isomorphically distinct from AMUC. Specifically, we show that c_0 or T^*, with any locally uniformly rotund norm are strictly locally AMUC. We introduce local AUC in the same spirit and give an example of locally AMUC, not locally AUC and not isomorphic to AMUC space. Following the proof in [10] we observe that these local properties are stable under finite sums with outer space having 1-unconditional basis and uniformly monotone norm.

Key words: asymptotically midpoint uniformly convex spaces, local asymptotic properties

2020 Mathematics Subject Classification: 46B03, 46B04, 46B06, 46B07

1. Introduction. Throughout the paper, $\mathrm{cof}(X)$ will denote the set of closed subspaces of finite codimension in a Banach space X. With S_X we denote its closed unit sphere.

The property of asymptotic uniform convexity (AUC) is due to Milman [1], and in modern terminology is due to [2]. While asymptotic midpoint uniform convexity was defined in [3].

Definition 1.1. A norm of a Banach space $(X, \|\cdot\|)$ is asymptotically uniformly convex AUC if for every $t \in (0, 1]$, there exists $\delta(t) > 0$ such that for all $x \in S_X$ there is a subspace Y of finite codimension so that $\inf\limits_{y \in S_Y} \|x + ty\| \geq 1 + \delta(t)$.

https://doi.org/10.7546/CRABS.2025.01.01

Definition 1.2. A norm of a Banach space $(X, \|\cdot\|)$ is asymptotically midpoint uniformly convex AMUC if for every $t \in (0,1]$, there exists $\delta(t) > 0$ such that for all $x \in S_X$ there is a subspace Y of finite codimension so that $\inf\limits_{y \in S_Y} \max\{\|x+ty\|, \|x-ty\|\} \geq 1 + \delta(t)$.

In this case for each $t \in (0,1]$, the corresponding $\delta(t)$ is called modulus of the corresponding asymptotic property of the norm. By $\tilde{\delta}(t)$ we denote the AMUC modulus

$$\tilde{\delta}_{\|\cdot\|}(t) = \inf_{x \in S_X} \sup_{E \in \mathrm{cof}(X)} \inf_{y \in S_E} [\max\{\|x+ty\|, \|x-ty\|\} - 1].$$

By $\delta(t)$ we denote the AUC modulus

$$\delta_{\|\cdot\|}(t) := \inf_{x \in S_X} \sup_{E \in \mathrm{cof}(X)} \inf_{y \in S_E} \|x+ty\| - 1.$$

These definitions are equivalent to $\tilde{\delta}_{\|\cdot\|}(t) > 0$ (resp. $\delta_{\|\cdot\|}(t) > 0$) for all $t \in (0,1]$.

Recently, in [4] the concept of local AMUC was introduced. In the same paper it is proven that any locally AMUC space Y which has either normalized symmetric basic sequence that is not equivalent to the unit vector basis of ℓ_1, or a normalized sequence with upper p-estimates for some $p > 1$, does not satisfy Lindenstrauss's property for norm-attaining operators, i.e. the norm attaining operators $X \to Y$ are norm-dense in the space $\mathcal{L}(X,Y)$ of all linear bounded operators $X \to Y$ for every Banach space X.

Definition 1.3. A norm of a Banach space $(X, \|\cdot\|)$ is locally asymptotically midpoint uniformly convex LAMUC if for every $x \in S_X$ its local modulus is positive

$$\tilde{\delta}_{\|\cdot\|}(t; x) := \sup_{E \in \mathrm{cof}(X)} \inf_{y \in S_E} \left[\max_{\pm} \|x \pm ty\| - 1 \right] > 0, \quad \text{for all } t \in (0,1].$$

In the same manner we define

Definition 1.4. A norm of a Banach space $(X, \|\cdot\|)$ is locally asymptotically uniformly convex LAUC if for every $x \in S_X$ its local modulus is positive

$$\delta(t; x) := \sup_{E \in \mathrm{cof}(X)} \inf_{y \in S_E} (\|x+ty\| - 1) > 0, \text{ for all } t \in (0,1].$$

We remark that the local properties generate distinct isomorphic classes. The following assertions are either known or obvious. Note that the arrows with bar indicate that the implication does not hold isomorphically up to an equivalent norm.

$$\begin{array}{ccccc} (\mathrm{UC}) & \xrightarrow[\text{isometrically}]{} & (\mathrm{AUC}) & \xrightarrow[\text{isometrically}]{} & (\mathrm{AMUC}) \\ \uparrow & & \uparrow & \text{isomorphically} & \uparrow \\ (\mathrm{LUR}) & \xrightarrow[\text{isometrically}]{} & (\mathrm{LAUC}) & \xrightarrow[\text{isometrically}]{} & (\mathrm{LAMUC}) \end{array}$$

For convenience we prove the one we use for our main result since it has not been stated formally to the best of our knowledge.

Theorem 1.5. *Every locally uniformly rotund* LUR *norm is locally asymptotically uniformly convex* LAUC.

Proof. Assume $\|\cdot\|$ is LUR but not LAUC. Then there are $x \in S_X, t > 0$ s.t. for all $\delta(t,x) = \delta > 0$ and all $Y \in \mathrm{cof}(X)$ we have $\inf\limits_{y\in S_Y} \|x+ty\| \leq 1+\delta$, i.e. $\inf\limits_{y\in S_Y} \|x+ty\| \leq 1$. Thus for the spaces $Y_0 \in \mathrm{cof}(X)$, on which we have $\inf\limits_{y\in S_{Y_0}} \|x+ty\| \geq 1$ we actually have an equality $\inf\limits_{y\in S_{Y_0}} \|x+ty\| = 1$. Such subspaces are the kernels of supporting functionals at x. Fix one such pair $x_0 \in S_X$ and $t_0 > 0$ and pick a supporting functional $f \in S_{X^*}$ at x_0, i.e. $f(x_0) = 1$. Denote $Y_0 := \ker f$. So for all $y \in Y_0$ we have $\|x_0 + t_0 y\| \geq f(x_0 + t_0 y) = f(x_0) = 1$. Denote the convex function $g_y(u) := \|x_0 + uy\|$ for $1 \leq u \leq t_0$.

Pick a sequence $(y_n)_n \subset S_{Y_0}$ realizing the infimum $\|x_0 + t_0 y_n\| \to 1$ as $n \to \infty$. Now $\|x_0 + (x_0 + t_0 y_n)\|^2 = 4\left\|x_0 + \frac{t_0}{2} y_n\right\|^2$. If $\left\|x_0 + \frac{t_0}{2} y_n\right\| > 1$, then $g_n\left(\frac{t_0}{2}\right) > g_n(0) = 1$ and so by convexity we must have

$$1 \leq g_n\left(\frac{t_0}{2}\right) \leq g_n(t_0) \to 1, \;\; n \to \infty.$$

(Once a convex function starts to increase it cannot strictly decrease). Thus $\left\|x_0 + \frac{t_0}{2} y_n\right\| \to 1$ as $n \to \infty$. And in case $\left\|x_0 + \frac{t_0}{2} y_n\right\| = 1$ for all sufficiently large n there is nothing to prove. In both cases we obtain that

$$\lim_{n\to\infty} 2\|x_0\|^2 + 2\|x_0 + t_0 y_n\|^2 - \|2x_0 + t_0 y_n\|^2 = 0,$$

but $\|x_0 - x_0 - t_0 y_n\| = t_0 > 0$ a contradiction with the analytic characterization of LUR (see [5], Prop. 1, 2). □

2. These local asymptotic properties of convexity of a norm are isomorphically distinct from the corresponding global ones. Note that c_0 with its sup-norm is not AMUC. Trivially for $0 < t \leq 1$ follows from the fact that $\|e_1\|_\infty = 1 = \|e_1 \pm t e_n\|_\infty$ for all $n \geq 2$. Thus its AMUC modulus is zero.

Proposition 2.1. *The space c_0 does not admit an equivalent norm which is* AMUC.

Proof. One direct proof utilizes that c_0 in its supremum norm is not AMUC and James's distortion theorem. Suppose $\|\cdot\|$ is an equivalent norm on c_0 and fix arbitary $t > 0, \varepsilon > 0$ and $Y \in \mathrm{cof}(c_0)$. Then there exists a block-basic sequence $y_1, \ldots, y_k, \ldots$ of the standard basis (dependent on ε and $\|\cdot\|$) s.t. $\|y_k\| = 1$ for all

integers k and

$$(1-\varepsilon)\max_k |a_k| \leq \left\| \sum_{k=1}^{\infty} a_k y_k \right\| \leq (1+\varepsilon)\max_k |a_k|, \ \ \forall (a_k)_k \in c_0.$$

In particular $y_k \xrightarrow{w} 0$ is weakly null as a block-basic sequence in c_0.

We claim that $\operatorname{dist}(y_k, Y) \to 0$ as $k \to \infty$. There are vectors of norm one $p_1, \ldots, p_s \notin Y$ s.t. $c_0 = [p_1, \ldots, p_s] \oplus Y$. Equivalently every $x \in c_0$ has representation of the form $x = \lambda_1 p_1 + \ldots \lambda_s p_s + y$, where $y \in Y$. From Hahn–Banach theorem there are functionals $\sum_{j=1}^{\infty} b_j^i e_j^* = f_i \in \ell_1$ s.t. $f_i(p_k) = \delta_{ik}$ and $f_i(Y) = \{0\}$, for $i = 1, \ldots, s$ and $k \in \mathbb{N}$. Thus for each $k \in \mathbb{N}$ there is $v_k \in Y$ s.t. $y_k = \sum_{j=1}^{s} \lambda_j^k p_j + v_k$. Thus $f_i(y_k) = \lambda_i^k$. Combined with the weak nullity of $(y_k)_k$ yields $\lambda_i^k = f_i(y_k) \to 0$ as $k \to \infty$ for each $i = 1, \ldots, s$. In particular

$$\|y_k - v_k\| = \left\| \sum_{j=1}^{s} \lambda_j^k e_{n_j} \right\| \to 0 \ \text{ as } \ k \to \infty.$$

So $(v_k)_k \subset Y$ is a seqence minimizing the distance from $(y_k)_k$ to Y. Denoting $z_k := v_k / \|v_k\|$ we obtain norm one vectors minimizing the distance. Now finally we calculate the local AMUC modulus of y_1 as

$$\tilde{\delta}_{(c_0,\|\cdot\|)}(t; Y; y_1) = \inf_{z \in S_Y} (\max_{\pm} \|y_1 \pm tz\| - 1) \leq \max_{\pm} \|y_1 \pm t z_k\| - 1, \ \ \forall k \in \mathbb{N}.$$

Estimating that

$$\|y_1 \pm t z_k\| = \|y_1 \pm t z_k + t y_k - t y_k\| = \|y_1 + t(z_k \mp y_k) \pm t y_k\| \leq$$

$$\leq \|y_1 \pm t y_k \pm t(z_k - y_k)\| \leq (1+\varepsilon)\max\{1, \pm t\} + t\,\|z_k - y_k\|\,.$$

Hence

$$\tilde{\delta}_{(c_0,\|\cdot\|)}(t; Y; y_1) \leq (1+\varepsilon).1 - 1 + t\,\|z_k - y_k\| \to (1+\varepsilon) - 1 \ \text{ as } \ k \to \infty.$$

But $Y \in \operatorname{cof}(c_0)$ was arbitrary so $\tilde{\delta}_{(c_0,\|\cdot\|)}(t; y_1) \leq (1+\varepsilon) - 1$. And finally $\tilde{\delta}_{(c_0,\|\cdot\|)}(t) = \inf_{\|x\|=1} \tilde{\delta}_{(c_0,\|\cdot\|)}(t; x) \leq \tilde{\delta}_{(c_0,\|\cdot\|)}(t; y_1) \leq (1+\varepsilon) - 1$. And since $\varepsilon > 0$ is arbitary we get that that for each $t \in (0,1)$ the AMUC modulus of $(c_0, \|\cdot\|)$ is zero. □

Theorem 2.2. *The properties* AMUC *and* LAMUC*, respectively,* LAUC *and* AUC*, generate distinct isomorphic classes.*

Proof. An easy corollary of Proposition 2.1. Indeed suppose $\|\cdot\|$ is an equivalent locally uniformly rotund (LUR) norm on c_0. (For example Day's norm.) Since

$$\text{LUR} \Rightarrow \text{LAUC} \Rightarrow \text{LAMUC} \quad \text{isometrically}$$

this norm is in particular LAMUC. But by Proposition 2.1 it cannot be AMUC. So any LUR norm on c_0 is locally asymptotically midpoint uniformly convex but not globally so. □

3. Reflexive example of strictly locally AMUC space. We combine known results to obtain reflexive example of LAMUC space which has no equivalent AMUC norm.

Definition 3.1 ([6])**.** Let X be a Banach space. For $n \in \mathbb{N}$ suppose E is an n dimensional Banach space with normalized basis $(e_j)_{j=1}^n$. We say E is in the n-th asymptotic structure of X and write $E \in \{X\}_n$, if for every $\varepsilon > 0$ and all $X_k \in \text{cof}(X)$ there are $x_k \in S_{X_k}$ for $k \in \{1, \dots, n\}$, so that $(x_j)_{j=1}^n$ is $(1+\varepsilon)$-equivalent to $(e_j)_{j=1}^n$.

Theorem A in [7] states that if for all $n \in \mathbb{N}$ the space ℓ_∞^n is in the n-th asymptotic structure of a Banach space Y, then countably branching diamonds of arbitrary height can be equi-bi-Lipschitzly embedded into Y.

The original Tsirelson [8] space T^* in the notation of Figiel and Johnson [9] is a reflexive space whose canonical basis $(e_j)_{j=1}^\infty$ is asymptotic ℓ_∞. That is there is $C > 0$ s.t. for every $n \in \mathbb{N}$ there is $M(n) > 0$ s.t. for any sequence $(x_i)_{i=1}^n$ of normalized block-basis of $(e_j)_{j=M(n)}^\infty$ is C-equivalent to the unit vector basis of ℓ_∞^n. Thus T^* contains ℓ_∞^n in its asymptotic structure for all $n \in \mathbb{N}$ by the distortion theorem of James (Lemma 3.1 [7]).

Therefore by Theorem A, countably branching diamonds of arbitrary height equi-bi-Lipschitzly embed in T^*. On the other hand, it was proved in [7] that such graphs do not equi-bi-Lipschitzly embed in any Banach space with an equivalent AMUC norm. Therefore T^* has no equivalent AMUC norm. On the other hand, every separable Banach space can be LUR renormed. So T^* with any LUR norm is locally AUC and in particular strictly locally AMUC Banach space, yielding a reflexive example and thus proving the following:

Theorem 3.2. *The Tsirelson space T^* does not admit an equivalent* AMUC *norm, while any locally uniformly rotund norm on it is locally* AMUC.

4. Finite sums of locally AMUC spaces.

Definition 4.1. A space $(E, \|\cdot\|)$ with 1-unconditional basis (e_k) is *uniformly monotone* if for every $\varepsilon > 0$ there is $M(\varepsilon) > 0$ so that for all $a = \sum_{n=1}^\infty a_n e_n$ with $a_n \geq 0$ for all n, so that $\|a\| = 1$ and all $b = \sum_{n=1}^\infty b_n e_n$ with $b_n \geq 0$ for all n, so that $\|b\| \geq \varepsilon$ it holds $\|a+b\| \geq 1 + M(\varepsilon)$. On the other hand, the norm is *strictly monotone* if $\|a+b\| > 1$.

Note that in finite dimensional space E strict monotonicity and uniform monotonicity coincide. The following sufficient condition is essentially due to [10] but holds true for finite sums of LAMUC and LAUC spaces with slightly modified proof, which we provide for completeness.

Theorem 4.2. *Suppose* $(X_n, \|\cdot\|_n)_{n=1}^N$ *are locally* AMUC *infinite dimensional Banach spaces and* $(E, \|\cdot\|_E)$ *is a Banach space with* 1*-unconditional basis* (e_n) *and uniformly monotone norm* $\|\cdot\|_E$*. Then their sum* $X := \left(\sum_{n=1}^N X_n\right)_E$ *is also locally* AMUC.

Proof. Pick arbitrary $\varepsilon > 0$ and $x = (x_n)_{n=1}^N$ so that $\|x\| = 1$. We claim that

$$\tilde{\delta}_X(\varepsilon; x) \geq \min\left(\frac{1}{4}M\left(\frac{\varepsilon}{4}\right); M\left(\tilde{\Delta}\left(\frac{\varepsilon}{2}; x\right). \min\left\{\frac{1}{4}; \frac{1}{8}M\left(\frac{\varepsilon}{4}\right)\right\}\right)\right) > 0,$$

where $\tilde{\Delta}(t; x) < \min\limits_{1\leq n\leq N, x_n\neq 0} \tilde{\delta}_{X_n}\left(t; \dfrac{x_n}{\|x_n\|_n}\right) > 0$ is smaller than minimal local modulus of its normed nonzero coordinates in the respective spaces X_n, and $M(\varepsilon)$ is a constant obeying the definition of uniform monotonicity.

(a) For each integer $1 \leq n \leq N$ pick a supporting functional $f_n \in S_{X_n^*}$, i.e. $f_n(x_n) = \|x_n\|_n$ and denote by

$$V_n := \{y_n \in X_n : f_n(y_n) = 0\} = \ker f_n$$

its kernel. It is subspace of codimension one in X_n. Then the following holds $\|x_n \pm y_n\|_n \geq f_n(x_n \pm y_n) = f_n(x_n) = \|x_n\|_n$, $1 \leq n \leq N$. We can choose $U_n \in \mathrm{cof}(X_n)$ in V_n so that $y_n \in U_n$ s.t. $\|y_n\|_n \geq \dfrac{\varepsilon}{2}\|x_n\|_n$ implies

$$\max_{\pm} \|x_n \pm y_n\|_n \geq \|x_n\|_n . \left(1 + \tilde{\Delta}\left(\frac{\varepsilon}{2}; x\right)\right) \text{ for all } 1 \leq n \leq N.$$

Indeed, since

$$\|x_n\|_n \left(1 + \tilde{\Delta}\left(\frac{\varepsilon}{2}; x\right)\right) \leq \sup_{Q_n\in\mathrm{cof}(X_n)} \inf_{\substack{z_n\in Q_n \\ \|z_n\|_n\geq\varepsilon/2}} \max_{\pm} \|x_n \pm z_n \|x_n\|_n\|_n ,$$

for all $1 \leq n \leq N$, there is $Z_n \in \mathrm{cof}(X_n)$ such that

$$\|x_n\|_n \left(1 + \tilde{\Delta}\left(\frac{\varepsilon}{2}; x\right)\right) < \inf_{\substack{z_n\in Z_n \\ \|z_n\|_n\geq\varepsilon/2}} \max_{\pm} \|x_n \pm z_n \|x_n\|_n\|_n .$$

Now we can take $U_n := V_n \cap Z_n$ for each $n \in \{1, \ldots, N\}$ and we get the desired spaces of finite codimension.

(b) Take $y := (y_n)_{n=1}^N, y_n \in U_n, \|y\| \geq \varepsilon$ and note that the space $U := \left\{(y_n)_{n=1}^N : y_n \in U_n\right\} \in \mathrm{cof}(X)$ is of finite co-dimension in X. Denote sets $A := \left\{1 \leq n \leq N : \|y_n\|_n < \frac{\varepsilon}{2}\|x_n\|_n\right\}$, $B := \left\{1 \leq n \leq N : \|y_n\|_n \geq \frac{\varepsilon}{2}\|x_n\|_n\right\}$.

By 1-unconditionality of (e_n) we have $\left\|\sum_{n\in A} y_n e_n\right\| < \frac{\varepsilon}{2}\left\|\sum_{n\in A} x_n e_n\right\| \leq \frac{\varepsilon}{2}\|x\| = \frac{\varepsilon}{2}$. But since $\varepsilon \leq \|y\| \leq \left\|\sum_{n\in A} y_n e_n\right\| + \left\|\sum_{n\in B} y_n e_n\right\|$ we must have $\left\|\sum_{n\in B} y_n e_n\right\| \geq \frac{\varepsilon}{2}$.

(c) *Case 1:* Assume first $\left\|\sum_{n\in B} x_n e_n\right\| < \min\left(\frac{1}{2}, \frac{1}{4}M\left(\frac{\varepsilon}{4}\right)\right)$. Then estimate for $\theta \in \{\pm 1\}$ and $y \in U$ with $\|y\| \geq \varepsilon$ so that

$$\max_{\pm}\|x \pm y\| = \|x + \theta y\| = \left\|\sum_{n\in A}(x_n + \theta y_n)e_n + \sum_{n\in B}(x_n + \theta y_n)e_n\right\| \geq$$

$$\geq \left\|\sum_{n\in A} x_n e_n + \sum_{n\in B}(x_n + \theta y_n)e_n\right\| =$$

$$= \left\|\sum_{n\in A} x_n e_n\right\| . \left\|\frac{\sum_{n\in A} x_n e_n}{\left\|\sum_{n\in A} x_n e_n\right\|} + \frac{\sum_{n\in B}(x_n + \theta y_n)e_n}{\left\|\sum_{n\in A} x_n e_n\right\|}\right\| \geq$$

$$\geq \left\|\sum_{n\in A} x_n e_n\right\| . \left(1 + M\left(\frac{\varepsilon}{4}\right)\right) \geq \left\|\sum_{n\in A} x_n e_n\right\| + \frac{1}{2}M\left(\frac{\varepsilon}{4}\right)$$

$$\geq \left\|\sum_{n=1}^N x_n e_n\right\| + \frac{1}{2}M\left(\frac{\varepsilon}{4}\right) - \left\|\sum_{n\in B} x_n\right\| \geq \|x\| + \frac{1}{2}M(\tfrac{\varepsilon}{4}) - \frac{1}{4}M\left(\frac{\varepsilon}{4}\right)$$

$$\geq 1 + \frac{1}{4}M\left(\frac{\varepsilon}{4}\right).$$

Here we used that $\|x_n + \theta y_n\|_n \geq \|x_n\|$, 1-unconditionality, the reverse triangle inequality, $\left\|\sum_{n\in A} x_n e_n\right\| \geq \frac{1}{2}$, and that $\left\|\frac{\sum_{n\in B}(x_n + \theta y_n)e_n}{\left\|\sum_{n\in A} x_n e_n\right\|}\right\| \geq \frac{\varepsilon}{4}$. For the latter observe that since for each $n \in \{1, \ldots, N\}$ the function $h_n(t) := \|tx_n + y_n\|_n$ is convex hence $h(0) \leq \max(h(-1); h(1))$ meaning $\|y_n\|_n \leq \max_{\pm}\|x_n \pm y_n\|_n$. Pick $\theta_n \in \{\pm 1\}$ so that $\|x_n + \theta_n y_n\|_n = \max_{\pm}\|x_n \pm y_n\|_n$. By 1-unconditionality $\left\|\sum_{n\in B}(x_n + \theta_n y_n)e_n\right\| \geq \left\|\sum_{n\in B} y_n e_n\right\| \geq \frac{\varepsilon}{2}$ as noted in (b). Now partition $B =$

$B_+ \cup B_-$ where $B_+ = \{n \in B : \theta_n = 1\}$ and $B_- = \{n \in B : \theta_n = -1\}$. Then

$$\frac{\varepsilon}{2} \le \left\| \sum_{n\in B} (x_n + \theta_n y_n) e_n \right\| \le \left\| \sum_{n\in B_+} (x_n + y_n) e_n \right\| + \left\| \sum_{n\in B_-} (x_n - y_n) e_n \right\|.$$

Thus they cannot both be strictly smaller than $\frac{\varepsilon}{4}$, i.e. either for $\theta = 1$ holds by 1-unconditionality $\left\| \sum_{n\in B} (x_n + y_n) e_n \right\| \ge \left\| \sum_{n\in B_+} (x_n + y_n) e_n \right\| \ge \frac{\varepsilon}{4}$ or for $\theta = -1$ we have $\left\| \sum_{n\in B} (x_n - y_n) e_n \right\| \ge \left\| \sum_{n\in B_-} (x_n - y_n) e_n \right\| \ge \frac{\varepsilon}{4}$. Thus for $\theta \in \{\pm 1\}$ so that $\|x + \theta y\| = \max\limits_{\pm} \|x \pm y\|$ we must have $\left\| \sum_{n\in B} (x_n + \theta y_n) e_n \right\| \ge \frac{\varepsilon}{4}$.

(d) Now let us consider *Case 2:* $\left\| \sum_{n\in B} x_n e_n \right\| \ge \min\left(\frac{1}{2}; \frac{1}{4} M(\frac{\varepsilon}{4})\right)$. Estimating

$$\left\| \sum_{n=1}^{N} (x_n + y_n) e_n \right\| = \left\| \sum_{n=1}^{N} \|x_n\|_n e_n + \sum_{n=1}^{N} (\|x_n + y_n\|_n - \|x_n\|_n) e_n \right\|_E$$

$$\ge \left\| \sum_{n=1}^{N} \|x_n\|_n e_n + \sum_{n\in B_+} \|x_n\|_n \tilde{\Delta}(\frac{\varepsilon}{2}; x) e_n \right\|_E \ge 1 + M\left(\tilde{\Delta}(\frac{\varepsilon}{2}; x) \left\| \sum_{n\in B_+} \|x_n\| e_n \right\| \right),$$

where we used for $n \in B_+$ the inequalities $\|x_n + y_n\|_n - \|x_n\|_n \ge \|x_n\|_n \tilde{\Delta}(\frac{\varepsilon}{2}; x)$ while for $n \in \{1 \ldots, N\} \setminus B_+$ simply used $\|x_n + \theta_n y_n\|_n - \|x_n\|_n \ge 0$ and 1-unconditionaility of the basis (e_n) of E.

Similarly $\left\| \sum_{n=1}^{N} (x_n - y_n) e_n \right\| \ge 1 + M\left(\tilde{\Delta}(\frac{\varepsilon}{2}; x) \left\| \sum_{n\in B_-} x_n e_n \right\| \right)$. From Case 2, $\min\left(\frac{1}{2}; \frac{1}{4} M\left(\frac{\varepsilon}{4}\right)\right) \le \left\| \sum_{n\in B} x_n e_n \right\| \le \left\| \sum_{n\in B_+} x_n e_n \right\| + \left\| \sum_{n\in B_-} x_n e_n \right\|$ yields $\max\limits_{\pm} \left\| \sum_{n\in B_\pm} x_n e_n \right\| \ge \min\left(\frac{1}{4}; \frac{1}{8} M\left(\frac{\varepsilon}{4}\right)\right)$. Finally by 1-unconditionality we have $\max\limits_{\theta\in\{\pm1\}} \left\| \sum_{n=1}^{N} (x_n + \theta y_n) e_n \right\| \ge 1 + M\left(\tilde{\Delta}\left(\frac{\varepsilon}{2}; x\right) . \min\left(\frac{1}{4}; \frac{1}{8} M\left(\frac{\varepsilon}{4}\right)\right)\right)$. □

Note that the necessary condition for finite sums is clearly true for any finite sum of LAMUC or LAUC spaces, i.e. if the sum of finitely many LAMUC (resp. LAUC spaces) is LAMUC (resp. LAUC), then each of the summands is necessarily LAMUC (resp. LAUC).

Corollary 4.3. *Suppose* $(E, \|\cdot\|_E)$ *is a Banach space with* 1*-unconditional basis* (e_k) *and uniformly monotone norm. Let* $Y := (\ell_2, \|\cdot\|)$ *be the renorming of* ℓ_2 *in* [3]*, where the norm is defined as*

$$(B_{\ell_2}, \|\cdot\|) := \overline{\mathrm{conv}}^{\|\cdot\|_2}((B_{\ell_2}, \|\cdot\|_2) \cup \{\pm(e_1 + e_n)_{n\geq 2}\}).$$

It is constructed as an example of AMUC *and not* AUC *space. Then the space* $X \in \{c_0 \oplus_E Y, T^* \oplus_E Y\}$*, where* c_0 *and* T^* *are renormed with any* LUR *norm, is* LAMUC*, not* LAUC *and not isomorphic to* AMUC *space.*

Proof. It is clear from Theorem 4.2 that X is LAMUC space. Also it is not LAUC and is never AMUC since one of the summands is not. Note that $T^* \oplus_E Y$ is reflexive example. □

REFERENCES

[1] Milman V. D. (1971) Geometric theory of Banach spaces. Part II. Geometry of the unit ball, Uspehi Mat. Nauk, **26**(6), 73–149 (in Russian); English translation in Russian Math. Surveys **26**(6), 79–163.

[2] Johnson W. B., J. Lindenstrauss, D. Preiss, G. Schechtman (2002) Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces, Proc. London Math. Soc. (3), **84**(3), 711–746.

[3] Dilworth S. J., D. Kutzarova, N. Randrianarivony et al. (2016) Lenses and asymptotic midpoint uniform convexity, J. Math. Anal. Appl., **436**(2), 810–821.

[4] Fovelle A. (2024) Norm attaining operators into locally asymptotically midpoint uniformly convex Banach spaces, `https://doi.org/10.48550/arXiv.2402.19067`.

[5] Deville R., G. Godefroy, V. Zizler (1993) Smoothness and renormings in Banach spaces, Pitman Monogr. Surveys Pure Appl. Math., **64**, Harlow, Longman Scientific & Technical.

[6] Maurey B., V. Milman, N. Tomczak-Jaegermann (1995) Asymptotic infinite dimensional theory of Banach spaces, Oper. Theory Adv. Appl., **77**, 149–175, Basel, Birkhäuser Verlag.

[7] Baudier F., R. Causey, S. Dilworth et al. (2017) On the geometry of the countably branching diamond graphs, J. Funct. Anal., **273**(10), 3150–3199.

[8] Tsirelson B. S. (1974) It is impossible to imbed ℓ_p or c_0 into an arbitrary Banach space, Funkcional. Anal. i Priložen., **8**(2), 57–60 (in Russian); English translation: Not every Banach space contains an imbedding of ℓ_p or c_0, Functional Anal. Appl., **8**, 138–141.

[9] Figiel T., W. B. Johnson (1974) A uniformly convex Banach space which contains no ℓ_p, Compositio Math., **29**, 179–190.

[10] Dilworth S. J., D. Kutzarova, N. Randrianarivony, M. Romney (2017) Sums of asymptotically midpoint uniformly convex spaces, Bull. Belg. Math. Soc. Simon Stevin, **24**(3), 439–446.

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd, 1164 Sofia, Bulgaria
e-mail: psevgeniev@uni-sofia.bg