Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 77, No 12, 2024

MATHEMATICS
Fuzzy sets and logic

BIPOLAR FUZZY SETS ARE INTUITIONISTIC FUZZY SETS

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Received on November 15, 2024
Accepted on November 26, 2024

Abstract

In 1983 the Intuitionistic Fuzzy Sets (IFSs) were introduced as an extension of Zadeh's fuzzy sets. Fifteen years later the Bipolar Fuzzy Sets (BFSs) were introduced as a new extension of fuzzy sets. In the article it is proved that each BFS can be transformed injectively or bijectively to an IFS.
Key words: bipolar fuzzy set, fuzzy set, intuitionistic fuzzy set
2020 Mathematics Subject Classification: 03E72

1. Introduction. In 1965 Zadeh introduced the concept of a Fuzzy Set (FS, see, [1]). In the following years this concept has been an object of various extensions – L-FSs (Goguen, 1967, [2]), Rough FS (Pawlak, 1981, [3]) and Intuitionistic Fuzzy Sets (IFSs, Atanassov, 1983, [4]). In 1993, Vague Sets [5] were introduced and for them Bustince and Burillo in [6] proved that they are a trivial modification of the IFSs.

In the present paper, it is shown that the extension Bipolar Fuzzy Sets (BFSs), proposed by Zhang in 1998 [7], is a modification of the concept of IFSs, too, despite the claims of some authors that BFSs are different from the IFSs, and represent extensions of the IFSs.

The authors are thankful for the support provided under bilateral project No. IC-PL/14/2024-2025 between the Polish Academy of Sciences and the Bulgarian Academy of Sciences.

https://doi.org/10.7546/CRABS.2024.12.02

2. Main result. Let E be a fixed universe and $A \subseteq E$. Then the object

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$$

is called an IFS (see, e.g., [8, 9]), if for each element $x \in E$ the functions $\mu_A(x)$ and $\nu_A(x)$ that represent the degrees of membership and of non-membership, respectively, of $x \in E$ to the set A, satisfy the conditions

$$\mu_A(x), \nu_A(x) \in [0,1], \quad \mu_A(x) + \nu_A(x) \in [0,1].$$

The two degrees generate a new, third degree of uncertainty (indeterminacy) for $x \in E$ to the set A as follows:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

As usual, instead of A^* for brevity we will write below A.

For the same universe E and $B \subseteq E$, the object

$$B^* = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in E\}$$

is called a BFS (see, e.g., [7]), if for each element $x \in E$ the functions $\mu_B(x)$ and $\nu_B(x)$ that represent the degrees of membership and of non-membership of $x \in E$, respectively, to the set B, satisfy the condition

$$\mu_B(x) \in [0,1], \quad \nu_B(x) \in [-1,0].$$

Notably, in [7] the places of functions μ and ν are exchanged.

Again, instead of B^* for brevity we will write below B.

Theorem 1. *Each bipolar fuzzy set is representable by an intuitionistic fuzzy set.*

Proof. Let

$$L^* = \{\langle a, b\rangle \mid a, b \in [0,1] \ \&\ a + b \leq 1\}$$

(see [10]). Let us define function $F\colon [0,1] \times [-1,0] \to [0,1] \times L^*$ by

$$F(m,n) = \left\langle \frac{m}{1-n}, \frac{-n}{1+m} \right\rangle.$$

Obviously, the denominators are higher or equal to 1 and F is a continuous function, for which the equalities

$$F(0,-1) = \langle 0,1\rangle, \quad F(0,0) = \langle 0,0\rangle, \quad F(1,0) = \langle 1,0\rangle, \quad F(1,-1) = \left\langle \tfrac{1}{2}, \tfrac{1}{2} \right\rangle$$

hold. For $m \in [0,1]$ and $n \in [-1,0]$ we directly see that

$$0 \leq \frac{m}{1-n} \leq 1, \quad 0 \leq \frac{-n}{1+m} \leq 1.$$

Let

$$X \equiv \frac{m}{1-n} + \frac{-n}{1+m}.$$

Therefore $X \geq 0$ and

$$X = \frac{m+m^2-n+n^2}{1-n+m-mn}.$$

Let

$$Y \equiv 1-n+m-mn-(m+m^2-n+n^2) = 1-mn-m^2-n^2$$

and $k = -n \geq 0$. Then for $k, m \in [0,1]$ it follows that $k \geq k^2, m \geq m^2$ and

$$Y = 1+mk-m^2-k^2 \geq 1+mk-m-k = (1-m)(1-k) \geq 0.$$

Therefore, $X \leq 1$ and hence $\left\langle \frac{m}{1-n}, \frac{-n}{1+m} \right\rangle$ is an intuitionistic fuzzy pair (see [11]).

Let $m, p \in [0,1]$, $n, q \in [-1,0]$ and $F(m,n) = F(p,q)$. Then

$$\begin{cases} \dfrac{m}{1-n} = \dfrac{p}{1-q} \\ \dfrac{n}{1+m} = \dfrac{q}{1+p} \end{cases}$$

and with a direct check we obtain that $m = p$ and $n = q$, i.e., the function F is injective.

Now, for the universe E we construct the set

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\} = \left\{\left\langle x, \frac{\mu_B(x)}{1-\nu_B(x)}, \frac{-\nu_B(x)}{1+\mu_B(x)} \right\rangle \mid x \in E\right\}$$

and, as we have shown above, it is an IFS.

Therefore to an arbitrary BFS we can assign an IFS. □

We can give another, more general assertion.

Theorem 2. *Intuitionistic fuzzy sets and bipolar fuzzy sets are equipolent.*

Proof. Let us transform (bijectively) the set B to the set

$$C = \{\langle x, \mu_B(x), -\nu_B(x)\rangle \mid x \in E\} = \{\langle x, \mu_C(x), \nu_C(x)\rangle \mid x \in E\},$$

where $\mu_C(x) = \mu_B(x)$ and $\nu_C(x) = -\nu_B(x)$ for each $x \in E$. Therefore, $\mu_C(x)$, $\nu_C(x) \in [0,1]$. Now, following each of the two algorithms from [9], we can transform (bijectively) the element degrees of set C in L^*. For example, we can use the transformation formulas

$$G(x,y) = \begin{cases} \left\langle x - \dfrac{y}{2}, \dfrac{y}{2} \right\rangle, & \text{if } x, y \in [0,1] \text{ and } x \geq y, \\ \left\langle \dfrac{x}{2}, y - \dfrac{x}{2} \right\rangle, & \text{if } x, y \in [0,1] \text{ and } x \leq y. \end{cases}$$

In [9] it is proved that the function G is bijective.

Therefore, the result of this transformation will be an IFS (see the two cases illustrated in Fig. 1). □

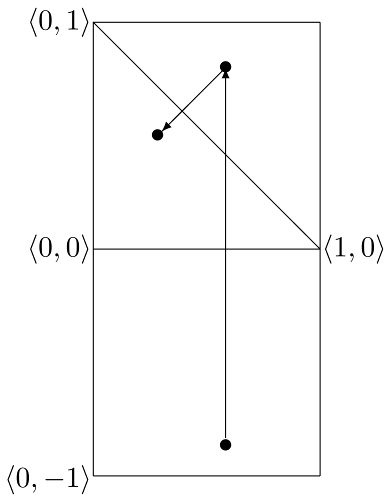

Fig. 1

3. Discussion and concluding remarks. Of course, following the Roman principle "*Prior tempore, potior iure*", by analogy with Bustince and Burillo's assertion that vague sets are a trivial modification of IFSs, we can assert that the bipolar fuzzy sets are a trivial modification of the intuitionistic fuzzy sets. But more important is the fact that in the research over BFSs there are operations only related to the set-theoretical operations of first-order logic "union" and "intersection" as well as others on the same level, while in IFS theory there are modal operators and various extensions of theirs. The reason is that, as we mentioned above, for each element $x \in E$ in the IFS A there is a degree of uncertainty $\pi_A(x)$, while in BFS theory such a degree is not discussed. Probably, the reason is that the uncertainty degree will have an essentially more complex form: for an element x with degrees $\langle m, n \rangle$ it will be $\dfrac{1 - mn - m^2 - n^2}{1 + m - n - mn}$. Now, it is clear that the object "bipolar intuitionistic fuzzy set", discussed by some authors, is non-legitimate.

REFERENCES

[1] ZADEH L. (1965) Fuzzy sets, Information and Control, **8**, 338–353.

[2] GOGUEN J. (1967) L-fuzzy sets, J. Math. Anal. Appl., **18**(1), 145–174.

[3] PAWLAK Z. (1981) Rough functions, ICS, PAS Report 467.

[4] ATANASSOV K. (1983) Intuitionistic fuzzy sets. In: VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[5] GAU W.-L., D. J. BUEHRER (1993) Vague sets, IEEE Transactions on Systems, Man, and Cybernetics, **23**(2), 610–614.

[6] Bustince H., P. Burillo (1996) Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems, **79**(3), 403–405.
[7] Zhang W.-R. (1998) Bipolar fuzzy sets. In: Proc. of IEEE Int. Conf. on Fuzzy Systems, Anchorage, AK, USA, 04-09 May 1998, Vol. 1, 835–840.
[8] Atanassov K. (1999) Intuitionistic Fuzzy Sets, Springer, Heidelberg.
[9] Atanassov K. (2012) On Intuitionistic Fuzzy Sets Theory, Springer, Berlin.
[10] Deschrijver G., E. Kerre (2003) On the composition of intuitionistic fuzzy relations, Fuzzy Sets and Systems, **136**(3), 333–361.
[11] Atanassov K., E. Szmidt, J. Kacprzyk (2013) On intuitionistic fuzzy pairs, Notes on Intuitionistic Fuzzy Sets, **19**(3), 1–13.

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