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MULTIDIMENSIONAL QUASI-MONTE CARLO INTEGRATION IN WEIGHTED ANCHORED SOBOLEV SPACES

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Abstract

In this article, an exact formula for the mean square worst-case error of the integration in the weighted anchored Sobolev spaces $H_{\mathrm{Sob},s,\gamma,\mathbf{w}}$ presented in terms of the functions of the system $\Gamma_{\mathcal{B}_s}$ is delivered. The notion of the so-called weighted anchored diaphony is introduced and is shown that it is a quantitative measure for the irregularity of the distribution of sequences. The relationship that exists between the mean square worst-case error and this type of the diaphony is established.

Key words: weighted anchored Sobolev spaces, the function system $\Gamma_{\mathcal{B}_s}$, mean square worst-case error, weighted anchored diaphony

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1. Introduction. Let $s \geq 1$ be a fixed integer which will denote the dimension throughout the article. Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence in the unit cube $[0,1)^s$. For an arbitrary integer $N \geq 1$ and a subinterval $J \subseteq [0,1)^s$ let us denote $A(J;N) = \#\{n\colon 0 \leq n \leq N-1, \mathbf{x}_n \in J\}$. Let λ_s denote the Lebesgue measure on $[0,1)^s$. Following KUIPERS and NIEDERREITER [1], we recall that the sequence ξ is called uniformly distributed in $[0,1)^s$ if the limit equality $\lim\limits_{N\to\infty} A(J;N)/N = \lambda_s(J)$ holds for each subinterval J of $[0,1)^s$.

To assess the quality of the distribution of the points of sequences, various special quantitative measures, including different types of the discrepancy and the

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diaphony, are used. It is worth noting that ZINTERHOF [2] introduced the concept of the so-called classical diaphony.

Some classes of complete orthonormal function systems have been successfully used as a tool for investigation of the uniformly distributed sequences. These systems are the trigonometric system, the systems of the Walsh and the Haar functions of base b, the systems of functions constructed over finite groups, the system of the b-adic functions.

We note the fact that in the last few years exists an increasing interest to use functions constructed in Cantor systems in the theory of the uniformly distributed sequences and the quasi-Monte Carlo integration in Hilbert spaces.

More information about the construction of these function systems and their applications the reader can find in the monograph of BAYCHEVA and GROZDANOV [3].

2. The function system $\Gamma_{\mathcal{B}_s}$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the set of the non-negative integers. The constructive principle of the Cantor systems, which are natural generalizations of the ordinary b-adic number system, is given as follows: Let $B = \{b_0, b_1, \dots : b_i \geq 2 \text{ for } i \geq 0\}$ be a given sequence of integers. The so-called generalized powers are defined as: $B_0 = 1$ and for $i \geq 0$ we put $B_{i+1} = B_i \cdot b_i$. Thus, to the sequence B of bases corresponds the sequence $\{B_0, B_1, \dots\}$ of generalized powers.

Every nonnegative integer k has a unique representation in the B-adic Cantor system of the form $k = \sum_{i=0}^{\nu} k_i B_i$, where for $0 \leq i \leq \nu$ $k_i \in \{0, 1, \dots, b_i - 1\}$ and $k_\nu \neq 0$. An arbitrary real $x \in [0, 1)$ has a representation in this system of the form $x = \sum_{i=0}^{\infty} x_i / B_{i+1}$, where for $i \geq 0$ $x_i \in \{0, 1, \dots, b_i - 1\}$. Let us assume that for infinitely many i we have that $x_i \neq b_i - 1$. Under this assumption the B-adic representation of x is also unique. To the end of the article the B-adic representations of integer and real numbers will be subordinated to this supposition.

The multidimensional Cantor systems are constructed in the following manner: For $1 \leq j \leq s$ let $B_j = \{b_0^{(j)}, b_1^{(j)}, \dots : b_i^{(j)} \geq 2 \text{ for } i \geq 0\}$ be given sequences of bases. Let us denote $\mathcal{B}_s = (B_1, \dots, B_s)$. PETROVA [4] proposed the concept of a new function system constructed in $\mathcal{B}_s$-adic Cantor systems.

Definition 1. For an arbitrary integer $k \geq 0$ and a real number $x \in [0, 1)$, which in the B-adic system have the representations $k = \sum_{i=0}^{\nu} k_i B_i$ and $x = \sum_{i=0}^{\infty} x_i / B_{i+1}$, the k-th function ${}_B\gamma_k \colon [0, 1) \to \mathbb{C}$ is defined as

$$ {}_B\gamma_k(x) = e^{2\pi \mathbf{i} \left(\frac{k_0}{B_1} + \frac{k_1}{B_2} + \cdots + \frac{k_\nu}{B_{\nu+1}} \right) (x_0 + x_1 B_1 + \cdots + x_\nu B_\nu)}. $$

For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}} \colon [0,1)^s \to \mathbb{C}$ is defined as ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\gamma_{k_j}(x_j)$, $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$.

The set $\Gamma_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) \colon \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ we will call $\mathcal{B}_s$-adic function system.

For arbitrary real numbers $x = \sum_{i=0}^{\infty} x_i/B_{i+1}$ and $y = \sum_{i=0}^{\infty} y_i/B_{i+1}$ we define the operation $x \oplus_B^{[0,1)} y = \sum_{i=0}^{\infty} z_i/B_{i+1}$, where the sequence $\{z_0, z_1, z_2, \dots\}$ is defined by the following rules:

$$
\begin{gathered}
x_0 + y_0 = t_0 \cdot b_0 + z_0, \quad t_0 \in \{0,1\}, \quad z_0 \in \{0,1,\dots,b_0-1\},\\
x_1 + y_1 + t_0 = t_1 \cdot b_1 + z_1, \quad t_1 \in \{0,1\}, \quad z_1 \in \{0,1,\dots,b_1-1\}\\
\dots .
\end{gathered}
$$

Let $\ominus_B^{[0,1)}$ be the opposite operation to the operation $\oplus_B^{[0,1)}$.

For arbitrary vectors $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$ and $\mathbf{y} = (y_1, \dots, y_s) \in [0,1)^s$ we define $\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \mathbf{y} = (x_1 \oplus_{B_1}^{[0,1)} y_1, \dots, x_s \oplus_{B_s}^{[0,1)} y_s)$ and $\mathbf{x} \ominus_{\mathcal{B}_s}^{[0,1)^s} \mathbf{y} = (x_1 \ominus_{B_1}^{[0,1)} y_1, \dots, x_s \ominus_{B_s}^{[0,1)} y_s)$. The functions of the system $\Gamma_{\mathcal{B}_s}$ with respect to the above operations satisfy the equalities

$$
{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \mathbf{y}) = {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) \cdot {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{y}) \quad \text{and} \quad {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x} \ominus_{\mathcal{B}_s}^{[0,1)^s} \mathbf{y}) = {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) \cdot {}_{\mathcal{B}_s}\overline{\gamma}_{\mathbf{k}}(\mathbf{y})
$$

for all $\mathbf{k} \in \mathbb{N}_0^s$.

3. Quasi-Monte Carlo integration in Hilbert spaces. We will present some details of the Quasi-Monte Carlo integration in reproducing kernel Hilbert spaces. Thus, following Aronszajn [5] we will remind the notion of a reproducing kernel for a Hilbert space. Let F be a class of functions defined on E forming Hilbert space with an inner product $\langle \cdot \rangle$ and a norm $\| \cdot \|$. The function $K(x,y)$ of $x, y \in E$ is called a reproducing kernel for the space F if the following properties hold:

1. For every fixed $y \in E$ $K(x,y)$ considered a function of x belongs to F;

2. (*reproducing property*) For every function $f \in F$ and every $y \in E$ the equality $f(y) = \langle f(x), K(x,y) \rangle_x$ holds. Here, the subscript x indicates that the inner product is given with respect to the variable x.

Let $H_s(K)$ be a Hilbert space generated by the reproducing kernel K with an inner product $\langle \cdot \rangle_{H_s(K)}$, which engenders the norm $\| \cdot \|_{H_s(K)}$.

The technique of the numerical integration in reproducing kernel Hilbert spaces is given as follows: The integral $I_s(f) = \int_{[0,1]^s} f(\mathbf{x})\, d\mathbf{x}$, $f \in H_s(K)$ is approximated by a Quasi-Monte Carlo algorithm $Q_s(f;P_N) = \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)$, where $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ is a deterministic sample point net in $[0,1)^s$.

Definition 2. The worst-case error of the integration in the space $H_s(K)$ by using the net of nodes P_N is defined as

$$e(H_s(K);P_N) = \sup_{f\in H_s(K),\ \|f\|_{H_s(K)}\leq 1} |I_s(f) - Q_s(f;P_N)|.$$

The main advantage of the Quasi-Monte Carlo algorithm in reproducing kernel Hilbert spaces is that there is a formula in explicit form for the worst-case error of the integration. Thus, following Sloan and Woźniakowski [6], the worst-case error of the integration in the space $H_s(K)$ by using the net of nodes P_N satisfies the presentation

$$(1)\quad e^2(H_s(K);P_N) = \int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} - \frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1]^s} K(\mathbf{x}_n,\mathbf{y})\,d\mathbf{y} + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} K(\mathbf{x}_n,\mathbf{x}_m).$$

Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of N points in $[0,1)^s$. Let $\sigma \in [0,1)^s$ be an arbitrary and fixed vector and define the net $P_N(\sigma) = \left\{\mathbf{x}_0 \oplus_{B_s}^{[0,1)^s} \sigma, \ldots, \mathbf{x}_{N-1} \oplus_{B_s}^{[0,1)^s} \sigma\right\}$, which we will call a *$B_s$-adic digitally shifted net.*

Definition 3. The mean square worst-case error of the integration in the space $H_s(K)$ by using the net P_N is defined as

$$\tilde{e}^2(H_s(K);P_N) = \int_{[0,1]^s} e^2(H_s(K);P_N(\sigma))\,d\sigma.$$

Definition 4. For an arbitrary reproducing kernel K we define the associated digitally shifted kernel as

$$K_{ds}(\mathbf{x},\mathbf{y}) = \int_{[0,1]^s} K\left(\mathbf{x} \oplus_{B_s}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{B_s}^{[0,1)^s} \sigma\right) d\sigma, \quad \forall\, \mathbf{x},\mathbf{y} \in [0,1)^s.$$

In the following lemma, we will show the special form of the Fourier's expansion of the digitally shifted kernel associated with an arbitrary reproducing kernel.

Lemma 1. *Let K be an arbitrary reproducing kernel. Then, the associated digitally shifted kernel K_{ds} has a Fourier's expansion with respect to the system $\Gamma_{\mathcal{B}_s}$ of the form*

$$K_{ds}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}\in\mathbb{N}_0^s} \widehat{K}(\mathbf{k},\mathbf{k})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x})_{\mathcal{B}_s}\overline{\gamma}(\mathbf{y}), \quad \forall\, \mathbf{x},\mathbf{y} \in [0,1)^s,$$

where for each vector $\mathbf{k} \in \mathbb{N}_0^s$ *we have that*

$$\widehat{K}(\mathbf{k},\mathbf{k}) = \int_{[0,1]^s}\int_{[0,1]^s} K(\mathbf{x},\mathbf{y})_{\mathcal{B}_s}\overline{\gamma}(\mathbf{x})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}.$$

We will prove the following theorem:

Theorem 1. *Let* $H_s(K)$ *be an arbitrary Hilbert space generated by the reproducing kernel* K *and* P_N *be an arbitrary net of points in* $[0,1)^s$*. Then, the mean square worst-case error of the integration in the space* $H_s(K)$*, by using the net* P_N*, satisfies the equality*

$$\tilde{e}(H_s(K);P_N) = e(H_s(K_{ds});P_N).$$

Proof. By using Definition 3 and formula (1) we consecutively obtain that

$$\tilde{e}^2(H_s(K);P_N) = \int_{[0,1]^s} e^2(H_s(K);P_N(\sigma))\,d\sigma \tag{2}$$

$$= \int_{[0,1]^s}\left[\int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}\right]d\sigma - \frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1]^s}\left[\int_{[0,1]^s} K(\mathbf{x}_n \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma,\mathbf{y})\,d\mathbf{y}\right]d\sigma$$
$$+ \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\int_{[0,1]^s} K\left(\mathbf{x}_n \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{x}_m \oplus_{\mathcal{B}_s}^{[0,1]^s} \sigma\right)d\sigma.$$

We will use the following property: For each integrable over $[0,1]^s$ function f and an arbitrary and fixed vector σ the equality

$$\int_{[0,1]^s} f(\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma)\,d\mathbf{x} = \int_{[0,1]^s} f(\mathbf{x})\,d\mathbf{x}$$

holds. This property was announced by Chrestenson [7] and it is not difficult to adapt it to our operation $\oplus_{\mathcal{B}_s}^{[0,1)^s}$. Now, we will consider the expressions in equality (2). Consecutively, we obtain that

$$\int_{[0,1]^s}\left[\int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}\right]d\sigma = \int_{[0,1]^s}\left[\int_{[0,1]^{2s}} K\left(\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}_s}^{[0,1]^s} \sigma\right)d\mathbf{x}\,d\mathbf{y}\right]d\sigma \tag{3}$$
$$= \int_{[0,1]^{2s}}\left[\int_{[0,1]^s} K(\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma)\,d\sigma\right]d\mathbf{x}\,d\mathbf{y} = \int_{[0,1]^{2s}} K_{ds}(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y};$$

$$\int_{[0,1]^s}\left[\int_{[0,1]^s} K\left(\mathbf{x}_n \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{y}\right)d\mathbf{y}\right]d\sigma = \int_{[0,1]^s}\left[\int_{[0,1]^s} K\left(\mathbf{x}_n \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma\right)d\mathbf{y}\right]d\sigma \tag{4}$$
$$= \int_{[0,1]^s}\left[\int_{[0,1]^s} K\left(\mathbf{x}_n \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma\right)d\sigma\right]d\mathbf{y} = \int_{[0,1]^s} K_{ds}(\mathbf{x}_n,\mathbf{y})\,d\mathbf{y};$$

$$(5)\qquad \int_{[0,1]^s} K\left(\mathbf{x}_n \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{x}_m \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma\right) d\sigma = K_{ds}(\mathbf{x}_n, \mathbf{x}_m).$$

Equalities (2)–(5) prove the statement of the theorem. □

4. The weighted anchored Sobolev space $H_{\mathrm{Sob},s,\gamma,\mathbf{w}}$. Let $\gamma=(\gamma_1,\ldots,\gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_s > 0$ be an arbitrary vector of coordinate weights. The weighted Sobolev spaces are tensor product of Hilbert space of univariate functions. Let $H_{s,\gamma} = H_{1,\gamma_1} \otimes \cdots \otimes H_{1,\gamma_s}$, where H_{1,γ_j}, $1 \leq j \leq s$, be Sobolev spaces of absolutely continuous real functions defined on $[0,1)$ whose derivatives belong to $L_2([0,1))$. According to the tensor product structure of the Sobolev space $H_{s,\gamma}$, for each vectors $\mathbf{x} = (x_1,\ldots,x_s)$, $\mathbf{y} = (y_1,\ldots,y_s) \in [0,1)^s$ its reproducing kernel has the product form $K_{s,\gamma}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} K_{1,\gamma_j}(x_j,y_j)$, where K_{1,γ_j}, $1 \leq j \leq s$, is the reproducing kernel of the space H_{1,γ_j}.

Hence, to complete the definition of the s-dimensional space $H_{s,\gamma}$ it is enough to specify the one-dimensional spaces H_{1,γ_j} or equivalently their kernels K_{1,γ_j}. In our work, we will consider the so-called *anchored* case. Following Hickernell [8], see also Novak et al. [9], we briefly will remind the details. Let $\mathbf{w} = (w_1,\ldots,w_s) \in [0,1]^s$ be a fixed vector, which we will call *anchor*. For an arbitrary subset $u \subseteq \{1,\ldots,s\}$ let $|u|$ denote its cardinality. For an arbitrary vector $\mathbf{x} = (x_1,\ldots,x_s) \in [0,1)^s$ and a subset u the notation $(\mathbf{x}_u, \mathbf{w}_{-u})$ will denote the vector from $[0,1)^s$ with coordinate x_j when $j \in u$ and w_j when $j \in \{1,\ldots,s\} \setminus u$. For two functions f, g their inner product is defined as

$$\langle f,g\rangle_{s,\gamma,\mathbf{w}} = \sum_{u\subseteq\{1,\ldots,s\}} \gamma_u^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|}}{\partial \mathbf{x}_u} f(\mathbf{x}_u,\mathbf{w}_{-u}) \frac{\partial^{|u|}}{\partial \mathbf{x}_u} g(\mathbf{x}_u,\mathbf{w}_{-u})\, d\mathbf{x}_u.$$

Then, the space $H_{\mathrm{Sob},s,\gamma,\mathbf{w}}$ is defined as $H_{\mathrm{Sob},s,\gamma,\mathbf{w}} = \{f\colon \|f\|_{s,\gamma,\mathbf{w}} < +\infty\}$.

For arbitrary reals $\gamma > 0$ and $w \in [0,1]$ the function

$$K_{1,\gamma,w}(x,y) = 1 + \frac{\gamma}{2}(|x-w| + |y-w| - |x-y|), \quad x,y \in [0,1)$$

is the reproducing kernel of the space $H_{\mathrm{Sob},1,\gamma,w}$. For arbitrary vectors γ and $\mathbf{w}$ the function $K_{s,\gamma,\mathbf{w}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} K_{1,\gamma_j,w_j}(x_j,y_j)$, $\mathbf{x} = (x_1,\ldots,x_s)$, $\mathbf{y} = (y_1,\ldots,y_s) \in [0,1)^s$ is the reproducing kernel of the space $H_{\mathrm{Sob},s,\gamma,\mathbf{w}}$.

For $1 \leq j \leq s$ let $K_{ds,1,\gamma_j,w_j}$ be the digitally shifted kernel associated with the function K_{1,γ_j,w_j}. Then, the digitally shifted kernel $K_{ds,s,\gamma,\mathbf{w}}$ associated with the function $K_{s,\gamma,\mathbf{w}}$ is defined as $K_{ds,s,\gamma,\mathbf{w}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} K_{ds,1,\gamma_j,w_j}(x_j,y_j)$, $\mathbf{x} = (x_1,\ldots,x_s)$, $\mathbf{y} = (y_1,\ldots,y_s) \in [0,1)^s$.

The statement of Lemma 1, applied to the reproducing kernel $K_{s,\gamma,\mathbf{w}}$, gives us the following result:

Lemma 2. *For an arbitrary integer $k \geq 0$ let the Fourier coefficient $\widehat{K}_{1,\gamma,w}(k,k)$ be as in the condition of Lemma 1. Then, the equalities hold*

$$\widehat{K}_{1,\gamma,w}(k,k) = \begin{cases} 1+\gamma\left(w^2 - w + \dfrac{1}{3}\right), & \text{if } k=0, \\ \dfrac{\gamma}{2}\left(\dfrac{1}{\sin^2 \pi \frac{k_g}{b_g}} - \dfrac{1}{3}\right)\dfrac{1}{B_{g+1}^2}, & \text{if } k_g B_g \leq k \leq (k_g+1)B_g - 1, \\ & \quad g \geq 0,\ k_g \in \{1,\dots,b_g-1\}. \end{cases}$$

For an arbitrary vector $\mathbf{k} = (k_1,\dots,k_s) \in \mathbb{N}_0^s$ let the Fourier coefficient $\widehat{K}_{s,\gamma,\mathbf{w}}(\mathbf{k},\mathbf{k})$ be as in the condition of Lemma 1. Then, the equality $\widehat{K}_{s,\gamma,\mathbf{w}}(\mathbf{k},\mathbf{k}) = \prod_{j=1}^{s} \widehat{K}_{1,\gamma_j,w_j}(k_j,k_j)$ holds and the kernel $K_{ds,s,\gamma,\mathbf{w}}$ has an expansion of the form

$$K_{ds,s,\gamma,\mathbf{w}}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}\in\mathbb{N}_0^s} \widehat{K}_{s,\gamma,\mathbf{w}}(\mathbf{k},\mathbf{k})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x})_{\mathcal{B}_s}\overline{\gamma}(\mathbf{y}), \quad \forall\, \mathbf{x},\mathbf{y} \in [0,1)^s.$$

In Theorem 2, we will present the formula for the mean square worst-case error of the integration in the space $H_{\mathrm{Sob},s,\gamma,\mathbf{w}}$.

Theorem 2. *Let $P_N = \{\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$. Then, the mean square worst-case error of the integration in the space $H_{\mathrm{Sob},s,\gamma,\mathbf{w}}$ by using the net P_N satisfies the equality*

$$\tilde{e}^2(H_{\mathrm{Sob},s,\gamma,\mathbf{w}};P_N) = -\prod_{j=1}^{s}\left[1+\gamma_j\left(w_j^2 - w_j + \tfrac{1}{3}\right)\right] + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} K_{ds,s,\gamma,\mathbf{w}}(\mathbf{x}_n,\mathbf{x}_m).$$

5. The weighted anchored $(\boldsymbol{\Gamma}_{\mathcal{B}_s};\boldsymbol{\gamma};\mathbf{w})$-diaphony. For arbitrary vectors $\gamma = (\gamma_1,\dots,\gamma_s)$, $\mathbf{w} = (w_1,\dots,w_s)$ and $\mathbf{k} = (k_1,\dots,k_s) \in \mathbb{N}_0^s$ let us denote

$$\rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k}) = \prod_{j=1}^{s} \widehat{K}_{1,\gamma_j,w_j}(k_j,k_j), \tag{6}$$

where the coefficients $\widehat{K}_{1,\gamma_j,w_j}(k_j,k_j)$ are determined in the condition of Lemma 2 and to put $C(\gamma;\mathbf{w};\mathcal{B}_s) = \sum_{\mathbf{k}\in\mathbb{N}^s\setminus\{\mathbf{0}\}} \rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k})$. Then, the quantity $C(\gamma;\mathbf{w};\mathcal{B}_s)$ satisfies the presentation

$$C(\gamma;\mathbf{w};\mathcal{B}_s) = \prod_{j=1}^{s}\left[1+\gamma_j\left(w_j^2 - w_j + \frac{1}{3}\right) + \frac{\gamma_j}{6}\sum_{g=0}^{\infty}\frac{b_g^{(j)}-1}{B_{g+1}^{(j)}}\right] - \prod_{j=1}^{s}\left[1+\gamma_j\left(w_j^2 - w_j + \frac{1}{3}\right)\right]. \tag{7}$$

In the following definition, we will introduce the concept of the so-called weighted anchored $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})$-diaphony.

Definition 5. For an arbitrary integer $N \geq 1$ the weighted anchored $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})$-diaphony of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is defined as

$$F_N(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};\xi) = \left(\frac{1}{C(\gamma;\mathbf{w};\mathcal{B}_s)} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k}) \left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}},$$

where the coefficient $\rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k})$ and the constant $C(\gamma;\mathbf{w};\mathcal{B}_s)$ are defined by equalities (6) and (7), respectively.

In Theorem 3, we will show that the weighted anchored $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})$-diaphony is a quantitative measure for the irregularity of the distribution of sequences.

Theorem 3. *The sequence ξ is uniformly distributed in $[0,1)^s$ if and only if for each choice of the vectors γ and $\mathbf{w}$ the limit equality $\lim\limits_{N\to\infty} F_N(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};\xi) = 0$ holds.*

Theorem 4 will give us the relationship that exists between the mean square worst-case error of the integration in the space $H_{\mathrm{Sob},s,\gamma,\mathbf{w}}$ and the weighted anchored $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})$-diaphony of the net of the nodes of the integration.

Theorem 4. *Let P_N be an arbitrary net of N points in $[0,1)^s$. Then, the mean square worst-case error $\tilde{e}(H_{\mathrm{Sob},s,\gamma,\mathbf{w}};P_N)$ of the integration in the space $H_{\mathrm{Sob},s,\gamma;\mathbf{w}}$, by using the net P_N, and the weighted anchored diaphony $F(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};P_N)$ of this net are related with the equality*

$$\tilde{e}(H_{\mathrm{Sob},s,\gamma,\mathbf{w}};P_N) = \sqrt{C(\gamma;\mathbf{w};\mathcal{B}_s)}\cdot F(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};P_N),$$

where the constant $C(\gamma;\mathbf{w};\mathcal{B}_s)$ is defined by equality (7).

Proof. We will use the exact formula for $\tilde{e}^2(H_{Sob,s,\gamma,\mathbf{w}};P_N)$ presented in Theorem 2, the Fourier expansion of the kernel $K_{ds,s,\gamma,\mathbf{w}}(\mathbf{x},\mathbf{y})$ shown in Lemma 2, and the denotation (6) to obtain that

$$\tilde{e}^2\left(H_{\mathrm{Sob},s,\gamma,\mathbf{w}};P_N\right) = -\prod_{j=1}^{s}\left[1+\gamma_j\left(w_j^2 - w_j + \tfrac{1}{3}\right)\right] + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} K_{ds,s,\gamma,\mathbf{w}}(\mathbf{x}_n,\mathbf{x}_m)$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k})\,{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)\,{}_{\mathcal{B}_s}\overline{\gamma}_{\mathbf{k}}(\mathbf{x}_m)$$

$$= \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k})\left[\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)\right]\cdot\left[\frac{1}{N}\sum_{m=0}^{N-1} {}_{\mathcal{B}_s}\overline{\gamma}_{\mathbf{k}}(\mathbf{x}_m)\right]$$

$$= \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k})\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)\right|^2 = C(\gamma;\mathbf{w};\mathcal{B}_s)\cdot F^2(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};P_N).$$

Thus, we obtained the equality

$$\tilde{e}\left(H_{\mathrm{Sob},s,\gamma,\mathbf{w}};P_N\right)=\sqrt{C(\gamma;\mathbf{w};\mathcal{B}_s)}\cdot F(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};P_N)$$

and Theorem 4 is finally proved. □

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