

LIE-ADMISSIBLE, ASSOCIATIVE-ADMISSIBLE OPERADS
AND NON-SYMMETRIC VERSIONS OF SYMMETRIC
OPERADS

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Abstract

A non-symmetric version of a symmetric operad is defined as the operad of algebras, where all consequences of the identity of degree 2 are valid in degree higher than 2 and not valid in degree 2. The main identity that should be included for such an operad is the reverse-associative identity $a(bc) = (cb)a$. An algebra is $\pm$ -Jacobi admissible and $\pm$ -associative-admissible if it satisfies the Jacobi identity and the associative identity under $\pm$ -commutators, respectively. In our paper we consider operads related to Lie-admissible, associative-admissible, and associative-Lie-admissible algebras. In particular we find the dimension sequences of their multilinear parts.

Key words: operads, Koszul duality, Lie-admissible algebras, associative-admissible algebras, polynomial identities

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1. Non-symmetric versions of symmetric operads. An algebra is a vector space with multiplication. We can imagine the multiplication as a binary tree with two inputs and one output. Moreover, one can consider n -ary trees, i.e., trees with n inputs and one output as n -ary operations. We can compose these kinds of trees: just use the output of the first tree as one of the inputs of the

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second tree. In general, there are n ways to compose an m -ary operation with an n -ary operation and the result is always an $(n + m - 1)$ -ary operation. An operad is the collection of all such operations, together with the compositions. These compositions satisfy three natural axioms that use actions of permutation groups.

A variety of algebras is the class of algebras satisfying a set of polynomial identities. Operads of algebras and their varieties have deep connections. If $F(n)$ is the multilinear part of the free algebra generated by $x_1, \dots, x_n$, then $F(n)$ can be considered a module over permutation group S_n . Let

$$f(t) = \sum_{i \geq 1} (-1)^i \frac{d_i t^i}{i!}$$

be the skew-symmetric generating function for $d_n = \dim F(n)$. One of the important notions which appeared in operads theory is the notion of Koszul duality. Koszul duals of algebraic operads are defined for quadratic operads. In terms of identities quadratic algebras are defined as algebras with identity of degree 3. The operad of algebras is called symmetric if its algebras satisfy an identity of degree 2 like commutativity or anti-commutativity. The operad is non-symmetric if it has no identity of degree 2. The operad $\mathcal{A}$ and its Koszul dual $\mathcal{A}^!$ have the following connection: $\mathcal{A} \otimes \mathcal{A}^!$ is Lie-admissible. A necessary condition for an operad to be Koszul is $f(f^!(t)) = t$. A detailed information about operads one can find in [1].

An algebra A with identity *revas* = 0, where

$$\text{revas}(t_1, t_2, t_3) = t_1(t_2 t_3) - (t_3 t_2)t_1$$

is called *reverse associative*. Such algebras appear in considering non-symmetric versions of symmetric operads. Let the operad $\mathcal{A} = \langle f_1, f_2, \dots \rangle$ be symmetric, i.e., an operad generated by multilinear polynomial identities

$$f_1 \equiv 0, f_2 \equiv 0, \dots,$$

where $f_1 = t_1 t_2 + \varepsilon t_2 t_1$ with $\varepsilon^2 = 1$, and $\deg f_i > 2$ for $i > 1$. Denote by $\mathcal{A}^b$ the new operad generated by the polynomial identities of $\mathcal{A}$, where the identity of degree 2 is replaced by the following identities of degree 3: $f_{1,r}$ and *revas*, where $f_{1,r}^r(t_1, t_2, t_3) = f(t_1, t_2) t_3$. Here instead of $f_{1,r}$ one can also use the polynomial of degree three $f_{1,l}(t_1, t_2, t_3) = t_3 f_1(t_1, t_2)$.

Theorem 1.1. *The operad $\mathcal{A}^b$ is non-symmetric, but it is almost symmetric in the following sense: it satisfies all consequences of the identity $f_1 \equiv 0$ for degrees higher than 2.*

Example. Let

$$\mathcal{L}ie = \langle f_1, f_2 \rangle$$

be the operad of Lie algebras:

$$f_1(t_1, t_2) = t_1 t_2 + t_2 t_1, f_2(t_1, t_2, t_3) = (t_1 t_2) t_3 + (t_2 t_3) t_1 + (t_3 t_1) t_2.$$

Then $\mathcal{Lie}^b = \langle f_{1,r}, revas, f_2 \rangle$ coincides with the two-sided Leibniz operad. Note that another version of the Leibniz operad generated by the polynomial identities $\{f_{1,r}, f_{1,l}\}$ does not satisfy some consequences of degree 3 of the skew-symmetric identity $f_1 \equiv 0$. For example, $revas \equiv 0$ is not an identity.

Let

$$arevas(t_1, t_2, t_3) = t_1(t_2 t_3) + (t_3 t_2)t_1$$

be anti-reverse associative polynomial.

2. Reverse-associative and anti-reverse associative operads. A tree is *non-planar* if the branches of each node can be permuted. In our paper by trees we understand equivalence classes of non-planar trees up to permuting of the branches. A *series-reduced tree* is defined as a non-planar tree without node of degree 1. We endow a tree by orientation taking the edge (u, v) with endpoints $u, v \in V$ *positive directed* if a path from the root to an endpoint passes first u and then v . For a node $u \in V$ an edge (u, v) is called *out-edge*, if (u, v) is positive directed. The degree of a node $v \in V$ is the number of its out-edges. A node of degree 0 is called a *leaf*. We colour nodes by two colours w (white) and b (black). A coloured tree is called *plus-coloured* if the colours of all nodes coincide with the colour of the root. A coloured tree is *minus-coloured* if the colours of the end nodes of any edge are different. Note that the colours of the nodes of $\pm$ -coloured trees are uniquely defined by colour of the root.

A base of the free anti-reverse-associative algebra $G(X)$ can be constructed by coloured binary rooted trees with alternative coloured nodes. Let

$$G(X)^2 = \{uv \mid u, v \in G(X)\}$$

be its subalgebra generated by all products and let

$$G(X)^{\bullet 2} = \{u \bullet v \mid u, v \in G(X)\},$$

$$G(X)^{\circ 2} = \{u \circ v \mid u, v \in G(X)\},$$

be the subalgebras generated by the black and white products, respectively.

For $u \in G(X)^2$ say that u is a black element if the root of the tree corresponding to u is black. Similarly, $u \in G(X)^2$ is called white, if the root of the tree corresponding to u is white. For $u \in G(X)^2$ denote by u_+ and u_- the projection of u in $G(X)^{\bullet 2}$ and $G(x)^{\circ 2}$, respectively.

A similar base can be constructed for the free reverse-associative algebras. Let us pay attention to the differences between the notations u_+ and u_- in the reverse-associative and anti-reverse-associative cases. In the reverse-associative case by u_+ we understand an element generated by rooted binary trees whose nodes are black, and in the anti-reverse-associative case the colours of nodes are alternating and the colour of the root is black. Similarly, in the reverse-associative case the notation u_- denotes an element generated by rooted binary trees whose

nodes are white, and in the anti-reverse-associative case the colours of the nodes are alternating and the colour of the root is white.

Theorem 2.1. *The reverse-associative and anti-reverse associative operads have the following properties:*

- (a) *Operads $Revas$ and $Arevas$ are Koszul.*
- (b) *Any anti-reverse-associative algebra is associative-admissible and Lie-admissible.*
- (c) *$Revas^! = Arevas$.*
- (d) *$Revas = \langle \{t_1, [t_2, t_3]\}, [t_1, \{t_2, t_3\}] \rangle$, where $\{t_1, t_2\} = t_1t_2 + t_2t_1$ and $[t_1, t_2] = t_1t_2 - t_2t_1$.*
- (e) *$Arevas = \langle [t_1, [t_2, t_3]], \{t_1, \{t_2, t_3\}\} \rangle$.*
- (f) *Plus-coloured trees generate a base of free reverse-associative operad. In particular, $Revas(n) = Com^+(n) \oplus Com^-(n)$.*
- (g) *The minus-coloured trees generate a base of the free anti-reverse-commutative operad. In particular, $Arevas(n) = Com^\pm(n) \oplus Com^\mp(n), n > 1$.*
- (h) *$\dim Arevas(n) = \dim Revas(n) = 2(2n - 1)!!$, $n > 1$, $\dim Arevas(1) = \dim Revas(1) = 1$.*
- (i) *$Arevas = ComNil_2 * AcomNil_2$, where $ComNil_2 = \langle t_1t_2 - t_2t_1, (t_1t_2)t_3 \rangle$, $AcomNil_2 = \langle t_1t_2 + t_2t_1, (t_1t_2)t_3 \rangle$.*
- (j) *The multiplication table in the free reverse associative algebra $F_{revas}(X)$ generated by the set of elements $X = \{x_1, x_2, \dots, x_n\}$ can be defined by*

$$x_i x_j = x_i \bullet x_j + x_i \circ x_j, \quad x_i u = x_i \bullet u_+ + x_i \circ u_-,$$

$$u x_j = u_+ \bullet x_j + u_- \circ x_j, \quad uv = u_+ \circ v_+ + u_- \circ v_-,$$

where $u, v \in F_{revas}(X)^2, 1 \leq i, j \leq n$.

- (k) *The multiplication table in the free anti-reverse associative algebra $F_{arevas}(X)$ generated by the elements of $X = \{x_1, x_2, \dots, x_n\}$ can be defined by*

$$x_i x_j = x_i \bullet x_j + x_i \circ x_j, \quad x_i u = x_i \bullet u_- + x_i \circ u_+,$$

$$u x_j = u_- \bullet x_j + u_+ \circ x_j, \quad uv = u_- \circ v_- + u_+ \circ v_+,$$

where $u, v \in F_{arevas}(X)^2, 1 \leq i, j \leq n$.

Here and below we will denote by $\mathcal{A} * \mathcal{B}$ the free product of the operads $\mathcal{A}$ and $\mathcal{B}$.

3. Associative-admissible operads. The left-Leibniz and right-Leibniz operads are defined by the identities $llei = 0$, and $rlei = 0$, respectively, where

$$llei = (t_1 t_2) t_3 - t_1 (t_2 t_3) + t_2 (t_1 t_3), \quad rlei = t_1 (t_2 t_3) - (t_1 t_2) t_3 + (t_1 t_3) t_2.$$

The Leibniz operad $\mathcal{L}ei = \mathcal{L}ie^b$ can be generated by the left- and right-Leibniz identities.

For an algebra A denote by A^+ and A^- its plus- and minus-algebras, i.e., the algebras with the vector space A with multiplication the Jordan and Lie commutator, respectively. An algebra A is called plus-associative-admissible, if its plus-algebra A^+ is associative. Similarly, A is minus-associative-admissible, if its minus-algebra A^- is associative. Let $\mathcal{A}com$ and $\mathcal{C}om$ be the anti-commutative and commutative operads, i.e., operads generated by the anti-commutative and commutative identity, respectively.

Theorem 3.1. *The plus-associative-admissible operad $\mathcal{A}sAdm^+$ has the following properties:*

- (a) $\mathcal{L}ie^b = \mathcal{L}ei$.
- (b) $\dim \mathcal{L}ie^b(n) = (n-1)!$, if $n \neq 2$, and $= 2$, if $n = 2$.
- (c) The operads $\mathcal{A}sAdm^+$ and $\mathcal{L}ie^b$ are Koszul.
- (d) $\mathcal{A}sAdm^{+!} = \mathcal{L}ie^b$.
- (e) $\mathcal{A}sAdm^+ = \mathcal{A}sCom \star \mathcal{A}com$.
- (f) The dimensions of the multilinear parts of plus-associative-admissible operad $d_n = \dim \mathcal{A}sAdm^+(n)$ can be found by the following recurrence relation:

$$d_n = \sum_{k=1}^{n-1} k! F_{k+2} B_{n-1,k}(d_1, d_2, \dots, d_{n-k}), \quad n > 1, \quad d_1 = 1,$$

where F_n are the Fibonacci numbers and $B_{n,k}(x_1, \dots, x_{n-k+1})$ are the Bell polynomials.

Example.

k	1	2	3	4	5	6	7	8	9	10
F_k	1	1	2	3	5	8	13	21	34	55
d_k	1	2	10	86	1036	16052	304060	6807656	175881016	5150163272

Theorem 3.2. *The minus-Associative-admissible operad $\mathcal{A}sAdm^-$ has the following properties:*

- (a) $\mathcal{A}sAdm^{-!} = \mathcal{C}om^b$.

- (b) $\dim \text{Com}^b(n) = (2n - 3)!!$, if $n \neq 2$ and $\dim \text{Com}^b(2) = 2$.
- (c) The operads $\mathcal{AsAdm}^-$ and Com^b are Koszul.
- (d) $\mathcal{AsAdm}^-$ is a free product of the operads Com and $\mathcal{AcomNil}_2$.
- (e) The Dimensions of the multilinear parts of the minus-associative-admissible operad $d_n = \dim \mathcal{AsAdm}^-(n)$ can be found by the following recurrence relations:

$$d_n = \sum_{k=2}^n (-1)^k \mu_k B_{n,k}(d_1, d_2, \dots, d_{n-k}), \quad n > 1, \quad d_1 = 1,$$

where $\mu_k = \delta_{k,2} + (2k - 3)!!$.

Example.

k	1	2	3	4	5	6	7	8	9	10
μ_k	1	2	3	15	105	945	10395	135135	2027025	34459425
d_n	1	2	9	75	855	12600	226800	4826115	118503000	3297969675

Theorem 3.3. Let p be prime. The dimensions $d_n = \dim \mathcal{AsAdm}(n)$ of the multilinear parts of the plus-associative-admissible operad satisfy the following congruences

$$d_{n+p-1} \equiv d_n \pmod{p}$$

for any prime $p > 2$. If $p = 2$, then all d_n are even for $n > 1$. Moreover,

$$d_{p-1} \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \neq 3, \\ -1 \pmod{p}, & \text{if } p = 3, \end{cases}$$

$$d_p \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \neq 2, \\ 0 \pmod{p}, & \text{if } p = 2, \end{cases}$$

$$d_{p+1} \equiv 2 \pmod{p}, \quad d_{p+2} \equiv 10 \pmod{p}.$$

4. $\pm$ -Jacobi-admissible operads. Let

$$\text{jac}(a, b, c) = (ab)c + (bc)a + (ca)$$

be Jacobi polynomial. An algebra A is called *minus-Jacobi-admissible* if its minus-algebra A^- satisfies the identity

$$\text{jac}^-(a, b, c) = [[a, b], c] + [[b, c], a] + [[c, a], b] \equiv 0.$$

Similarly, an algebra A is called *plus-Jacobi-admissible* if its plus-algebra A^+ satisfies the identity

$$\text{jac}^+(a, b, c) = \{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} \equiv 0.$$

Usually, minus-Jacobi-admissible algebras are called Lie-admissible.

Theorem 4.1. *If $\mathcal{LieAdm}$ is a Lie-admissible operad, then*

- (a) $\mathcal{LieAdm}^! = (\mathcal{Com} \cap \mathcal{As})^b$.
- (b) $\dim(\mathcal{Com} \cap \mathcal{As})^b(n) = 1$, if $n \neq 2$ and $= 2$ if $n = 2$.
- (c) *The operads $\mathcal{LieAdm}$ and $(\mathcal{Com} \cap \mathcal{As})^b$ are Koszul.*
- (d) *The operad $\mathcal{LieAdm}$ is a free product of the Lie operad and the commutative operad.*
- (e) *The dimensions of the multilinear parts of the Lie-admissible operad $d_n = \dim \mathcal{LieAdm}^-(n)$ can be found by the following recurrence relations:*

$$d_n = \sum_{k=2}^n (-1)^k \mu_k B_{n,k}(d_1, d_2, \dots, d_{n-k+1}), \quad n > 1, \quad d_1 = 1,$$

where $\mu_k = 1 + \delta_{k,2}$.

Example.

k	1	2	3	4	5	6	7	8	9	10
μ_k	1	2	1	1	1	1	1	1	1	1
d_n	1	2	11	101	1299	21484	434314	10376729	286071990	8938291341

Remark. All results of Theorem 4.1 except the dimension formula are known (see [2]).

Theorem 4.2. *If $\mathcal{JacAdm}$ is the plus-Jacobi-admissible operad, then*

- (a) $\mathcal{JacAdm}^! = (\mathcal{Acom} \cap \mathcal{Aas})^b$, where $\mathcal{Acom} = \langle ab + ba \equiv 0 \rangle$ and $\mathcal{Aas} = \langle a(bc) + (ab)c \equiv 0 \rangle$.
- (b) $\dim \mathcal{Acom}^b \cap \mathcal{Aas}(n) = \begin{cases} 1, & \text{if } n = 1 \text{ or } n = 3, \\ 2, & \text{if } n = 2, \\ 0, & \text{if } n > 3. \end{cases}$
- (c) $\mathcal{JacAdm}$ and $(\mathcal{Acom} \cap \mathcal{Aas})^b$ are not Koszul.

Proof of (c). The dimension sequences for the operad $\mathcal{JacAdm}$ and for its dual $\mathcal{JacAdm}^!$ are

$$d_n = 1, 2, 11, 100, 1275, 20910, \dots, \quad d_n^! = 1, 2, 1, 0, 0, \dots$$

Therefore the skew-generating functions are

$$f(x) = -x + \frac{2x^2}{2!} - \frac{11x^3}{3!} + \frac{100x^4}{4!} - \frac{1275x^5}{5!} + \dots,$$

$$f^!(x) = -x + \frac{2x^2}{2!} - \frac{x^3}{3!},$$

and

$$f(f^!(x)) = x + \frac{x^5}{24} + O(x^6) \neq x.$$

So, by Koszulity criterium ([1], [3]) the operad $\mathcal{AsAdm}^-$ is not Koszul. $\square$

4.1. Dimensions of the Lie-admissible operad in characteristic p .

Theorem 4.3. *Let $p = 2$. Then the dimensions of multilinear parts of the Lie-admissible operad satisfy the following recurrence relations:*

$$d_n \equiv d_{n-2} + d_{n-3} \pmod{2}, \quad n > 3,$$

$$d_1 \equiv 1 \pmod{2}, \quad d_2 \equiv 0 \pmod{2}, \quad d_3 \equiv 1 \pmod{2}.$$

The ordinary generating function for the dimensions in characteristic $p = 2$ is

$$\begin{aligned} \sum_{i \geq 1} d_i x^i &= x + x^3 + x^4 + x^5 + x^8 + x^{10} + x^{11} + x^{12} + x^{15} + x^{17} + x^{18} + x^{19} + \dots \\ &= \frac{x}{1 + x^2 + x^3}. \end{aligned}$$

In particular, the dimensions of the multilinear parts of the Lie-admissible operad satisfy the following congruences: If $k > 0$, then

$$d_i \equiv 0 \pmod{2}, \quad \text{for } i = 7k - 5, 7k - 1, 7k.$$

If $k \geq 0$, then

$$d_i \equiv 1 \pmod{2}, \quad \text{for } i = 7k + 1, 7k + 3, 7k + 4, 7k + 5.$$

Theorem 4.4. *Let $p > 2$ be prime. Then the dimensions of the multilinear parts of the Lie-admissible operad satisfy the following congruences:*

$$d_{p-2} \equiv \begin{cases} \frac{p+1}{2} \pmod{p}, & \text{if } p > 5, \\ 1 \pmod{p}, & \text{if } p = 3, 5, \end{cases}$$

$$d_{p-1} \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \neq 3, \\ -1 \pmod{p}, & \text{if } p = 3, \end{cases}$$

$$d_p \equiv -1 \pmod{p}, \quad d_{p+1} \equiv -1 \pmod{p}, \quad d_{p+2} \equiv -6 \pmod{p},$$

$$d_{p+3} \equiv -56 \pmod{p}, \quad d_{p+4} \equiv -725 \pmod{p}.$$

Another formulation of these results. The sequence of the dimensions d_n for $p = 2$ has period 7. It generates shift-register sequences [4], [5] and [6] (see OEIS A011656). So, the dimension d_n is even with probability ≈ 0.43 and odd with probability ≈ 0.57 . In particular, the number of even dimensions is infinite, and similarly, the number of odd dimensions is also infinite. This observation is false if we consider the distribution of the dimension congruences $d_n \pmod{p}$ for $p > 2$. For example, dimension d_n ends by 0 or by 5 beginning with $n = 24$. We see that

- $d_n \equiv 0 \pmod{3}$, if $n \geq 9$,
- $d_n \equiv 0 \pmod{5}$, if $n \geq 24$,
- $d_n \equiv 0 \pmod{7}$, if $n \geq 47$.

Probably $d_n \equiv 0 \pmod{p}$, if $p > 2$ and $n > p^2$.

4.2. How to construct a base? We endow the set of series-reduced trees by order. Let t be a series-reduced tree with number of leaves n , a root degree k and branches $t_1, \dots, t_k$ at the root. Similarly, let t' be a series-reduced tree with number of leaves n' , a root degree k' and branches $t'_1, \dots, t'_{k'}$ at the root. For two series-reduced trees t and t' we say that $t \preceq t'$

- if $n < n'$
- $t_1 = t'_1, \dots, t_{s-1} = t'_{s-1}$, but $t_s \preceq t'_s$ for some $s \leq \min(k, k')$, if $n = n'$.

Let G be the set of sequences of elements $a_1, a_2, \dots$. We endow it by lexicographic order, say from left to right. Now we label leaves by elements of G and extend the order $\preceq$ for leaf labelled trees. If $t \prec t'$ as leaf unlabelled trees, then $t \prec t'$ as leaf labelled tree. If $t = t'$ as leaf unlabelled tree, then $t \preceq t'$ if the sequence of leaf labels of the tree t is less than the sequence of leaf labels of the tree t' .

Now we give an algorithm how to construct a multilinear base of the free product of an operad $\mathcal{W}$ with white multiplication and an operad $\mathcal{B}$ with black multiplication by its multilinear bases. We show how to construct a base of $(\mathcal{W} * \mathcal{B})(n)$ generated by n elements $a_1, \dots, a_n$.

1. Construct all series-reduced trees with n leaves. Below we show the construction of base elements for a given tree.

2. Colour all nodes of the tree by two colours w and b , such that the ends of each edge have different colours. It can be done in two ways and the colouring is uniquely defined by the colour of the root.

3. Label the leaves by generators by some order, say in the increasing order from left to right.

4. Prolong this labelling from the leaves to the inner nodes in the usual way: if the labels of the sons of a node v with colour $r = w, b$, are $l_1, l_2, \dots, l_s$, then the label of v is $r(l_1, l_2, \dots, l_s)$.

5. Construct by the label of the root set of the base elements generated by the tree and the generators. Follow by colours that will refer to bases of corresponding operads.

6. Repeat this procedure for all series-reduced trees and obtain a base of $(\mathcal{W} * \mathcal{B})(n)$.

Example. The list of the 11 multilinear base elements of the free Lie-admissible algebra generated by the elements a, b, c in degree 3 is the following:

$$\begin{array}{cccc}
 \begin{array}{c} b \\ \swarrow \quad \searrow \\ a \quad c \\ \swarrow \quad \searrow \\ \circ \end{array} & \begin{array}{c} a \\ \swarrow \quad \searrow \\ b \quad c \\ \swarrow \quad \searrow \\ \circ \end{array} & \begin{array}{c} a \\ \swarrow \quad \searrow \\ c \quad b \\ \swarrow \quad \searrow \\ \circ \end{array} & \begin{array}{c} b \\ \swarrow \quad \searrow \\ a \quad c \\ \swarrow \quad \searrow \\ \circ \end{array} \\
 a \circ (b \bullet c) & b \circ (a \bullet c) & c \circ (a \bullet b) & \{a \circ (b \circ c), \quad b \circ (a \circ c)\} \\
 \\
 \begin{array}{c} b \\ \swarrow \quad \searrow \\ a \quad c \\ \swarrow \quad \searrow \\ \bullet \end{array} & \begin{array}{c} a \\ \swarrow \quad \searrow \\ b \quad c \\ \swarrow \quad \searrow \\ \bullet \end{array} & \begin{array}{c} a \\ \swarrow \quad \searrow \\ c \quad b \\ \swarrow \quad \searrow \\ \bullet \end{array} & \begin{array}{c} b \\ \swarrow \quad \searrow \\ a \quad c \\ \swarrow \quad \searrow \\ \bullet \end{array} \\
 a \bullet (b \circ c) & b \bullet (a \circ c) & c \bullet (a \circ b) & \{a \bullet (b \bullet c), \quad b \bullet (a \bullet c), \quad c \bullet (a \bullet b)\}
 \end{array}$$

5. Associative-Lie-admissible operad. Let $\mathcal{AsLieAdm}$ be the operad representing the associative-admissible and Lie-admissible algebras, i.e. the algebras with the following identities

$$\{t_1, \{t_2, t_3\}\} = \{\{t_1, t_2\}, t_3\}, \quad [[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_1, t_2], t_3] = 0.$$

Let $\mathcal{AsCom}$ be the operad for the associative and commutative algebras and let

$$aslia = aslia(t_1, t_2, t_3) = [t_1, [t_2, t_3]] + 2(t_2(t_3t_1) - t_3(t_2t_1) - (t_1t_2)t_3 + (t_1t_3)t_2).$$

Let $\mathcal{Pois}$ and $\mathcal{TPois}$ be the operads of the Poisson and transposed Poisson algebras. Recall that the Poisson algebras are associative-Lie-admissible and satisfy the Leibniz identity

$$[a, \{b, c\}] = \{[a, b], c\} + \{b, [a, c]\}.$$

The transposed Poisson algebras are also associative-Lie-admissible and satisfy the transposed Leibniz identity [7]

$$2\{a, [b, c]\} = [\{a, b\}, c] + [b, \{a, c\}].$$

Theorem 5.1. *The operad $\mathcal{AsLieAdm}$ has the following properties:*

- (a) *The operad $\mathcal{AsLieAdm}$ is Koszul.*
- (b) *$\mathcal{AsLieAdm} = \langle aslia \rangle$, if $p \neq 3$.*

(c) $\mathcal{A} sLieAdm = \mathcal{A} sCom \star \mathcal{L} ie.$

(d) $\mathcal{A} sLieAdm^! = \mathcal{P} ois \cap \mathcal{T} P ois.$

(e) $d_n^! = \dim \mathcal{A} sLieAdm^!(n) = (n-1)! + 1.$

(f) *The Poincaré series is* $f_{\mathcal{A} sLieAdm}^!(x) = \sum_{i \geq 1} d_i^! \frac{x^i}{i!} = -1 + e^x - x - \ln(1-x).$

(g) $d_n = \dim \mathcal{A} sLieAdm(n) = \sum_{k=1}^{n-1} (-1)^k \lambda_k B_{n-1,k}(d_1, d_2, \dots, d_{n-k}),$ *where*

$$\lambda_k = \sum_{s=1}^k (-1)^s s! B_{k,s}(1! + 1, 2! + 1, \dots, i! + 1, \dots, (k-s+1)! + 1).$$

Example.

k	1	2	3	4	5	6	7	8	9	10
λ_k	-2	5	-19	93	-581	4303	-37451	370905	-4143241	51345531
d_k	1	2	9	67	695	9256	150570	2893717	64155404	1611799961

Theorem 5.2. *Let p be prime and let d_n be the dimension of multilinear part of free associative-admissible and Lie-admissible algebra generated by n elements. Then*

$$d_{p-1} \equiv 2 \pmod{p}, \quad p > 2,$$

$$d_p \equiv 0 \pmod{p}, \quad d_{p+1} \equiv 1 \pmod{p}, \quad d_{p+2} \equiv 5 \pmod{p}.$$

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