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# A STUDY ON HARMONIC FUNCTIONS

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**Abstract**

The class of functions that have bounded boundary rotation and bounded radius rotation are the generalization of the convex and the starlike functions, respectively. The concept of such functions was introduced by Löwner [1]. But he did not use the present terminology. It was Paatero [2,3] who systematically developed their properties and made an exhaustive study of the class of functions that have bounded boundary rotation. We will examine in this paper, the subclass of $\mathcal{S}_H$ that is related to the class of functions that have bounded radius rotation.

**Key words:** harmonic function, bounded radius rotation
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**1. Introduction.** $\Omega$ is the class of functions $\mu(z)$ which are analytic in the open unit disc $\mathbb{D} = \{z| \, |z| < 1\}$ and normalized $\mu(0) = 0$, $|\mu(z)| < 1$ for every $z \in \mathbb{D}$. $\mathcal{P}$ is family of regular $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ functions in $\mathbb{D}$.

$$p(z) \in \mathcal{P} \Leftrightarrow p(z) = \frac{1+\mu(z)}{1-\mu(z)}, \tag{1.1}$$

where $\mu(z) \in \Omega$ for all $z \in \mathbb{D}$ [4].

Also, $\mathcal{A}$ is the class of functions that are given with these properties $f(0) = 0, f'(0) = 1$ in the open unit disc $\mathbb{D}$ . If $f(\mathbb{D})$ is starlike domain with respect to

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the origin, then we say that $f(z) \in A$ and starlike in $\mathbb{D}$, and we represent the class of all such functions by $S^*$. It is known that

$$f(z) \in \mathcal{A} \text{ and starlike} \Leftrightarrow \text{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0 \quad z \in \mathbb{D}. \tag{1.2}$$

If $f(\mathbb{D})$ is a convex domain, we say that the function $f(z) \in \mathcal{A}$ is convex. The set of convex functions is represented by $\mathcal{C}$. If $f(z) \in \mathcal{C}$

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0 \quad z \in \mathbb{D} \tag{1.3}$$

is realized and there exist $p_1(z), p_2(z) \in \mathcal{P}$ [5], where

$$p(z) \in \mathcal{P}(k) \Leftrightarrow p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), k \geq 4. \tag{1.4}$$

Now, let us define the following classes:

$$\mathcal{R}_k = \left\{ f(z) \in \mathcal{A} \middle| z\frac{f'(z)}{f(z)} \in \mathcal{P}(k) \right\}, \quad z \in \mathbb{D} \tag{1.5}$$

$$\mathcal{V}_k = \left\{ f(z) \in \mathcal{A} \middle| \left(1 + z\frac{f'(z)}{f(z)}\right) \in \mathcal{P}(k) \right\}, \quad z \in \mathbb{D}. \tag{1.6}$$

Here we write the relation

$$f(z) \in \mathcal{V}_k \Leftrightarrow zf'(z) \in \mathcal{R}_k. \tag{1.7}$$

The classes $\mathcal{R}_k$ and $\mathcal{V}_k$ are generalized of starlike and convex functions, respectively [5–7].

Finally, a planar harmonic function defined in the unit disc $\mathbb{D}$ is a complex-valued function. This function maps $\mathbb{D}$ onto the same planar domain. Since $\mathbb{D}$ is simply connected domain, the mapping $f$ has a canonical decomposition $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the series expansion as follows:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n,$$

where $a_n, b_n \in \mathbb{C}$, $n = 1, 2, 3, \ldots$. We call $h(z)$ an analytic part of $f$, and $g(z)$ a co-analytic part of $f$.

Theory of harmonic mappings is examined in detail in Duren's monograph [8]. The harmonic mapping $f$ is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2$ is non-zero in $\mathbb{D}$ [9]. Locally univalent harmonic mapping in the open unit disc is sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$, and sense reserving if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$. We will consider sense-preserving harmonic mapping in this paper. $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ is non-zero in $\mathbb{D}$, and second dilatation $w(z) = \left(g'(z)/f'(z)\right)$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. The class of sense-preserving harmonic mappings in the open unit disc $\mathbb{D}$ which have the properties $a_0 = b_0 = 0, a_1 = 1$ is denoted by $S_H$. If $f \in S_H$ with the properties $g'(0) = 0$, $b_1 = 0$, then it is denoted by $S_H^0$. Thus, we get the relation $S \subset S_H^0 \subset S_H$.

In this paper, we will produce and study the following subclass of the $S_H$:

$$S_H(k) = \left\{ f = h(z) + \overline{g(z)} \,\middle|\, w(z) = \frac{f'(z)}{g'(z)} \prec b_1 p(z), p(z) \in \mathcal{P}(z) \right\}.$$

**2. Main results.**

**Lemma 2.1.** *If $h(z) \in \mathcal{R}_k$, then the boundary value of* $\left(z\dfrac{h'(z)}{h(z)}\right)$ *is*

$$\left(\frac{1 + kre^{i\theta} + r^2}{1 - r^2}\right).$$

**Proof.** Robertson [10] showed that

$$s(z) \in \mathcal{V}(k) \Rightarrow \left| z\frac{s''(z)}{s'(z)} - \frac{2r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2}.$$

Therefore we have

$$\left| \left(1 + z\frac{s''(z)}{s'(z)}\right) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2}. \tag{2.1}$$

By means of the definition of $\mathcal{V}(k)$, inequality (2.1) can be expressed as

$$\left| p(k) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2}.$$

Moreover, using the definition of the class of $\mathcal{R}(k)$ we obtain

$$\left| z\frac{h'(z)}{h(z)} - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2}.$$

□

**Theorem 2.2.** *If $f(z) \in S_H(k)$, then*

$$\frac{g(z)}{h(z)} \prec b_1 p_k(z).$$

**Proof.** By means of the definition of $S_H(k)$, we write the following inequality:

$$\left|\frac{g'(z)}{h'(z)} - b_1\frac{1+r^2}{1-r^2}\right| \leq \frac{|b_1|\, kr}{1-r^2}, \quad |b_1| < 1.$$

Thus

$$w(\mathbb{D}_r) = \left\{\frac{g'(z)}{h'(z)} \,\middle|\, \left|\frac{g'(z)}{h'(z)} - b_1\frac{1+r^2}{1-r^2}\right| \leq \frac{|b_1|\, kr}{1-r^2}, \quad |b_1| < 1\right\}. \tag{2.2}$$

We can give $\mu(z)$ as

$$\frac{g(z)}{h(z)} = b_1\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+\mu(z)}{1-\mu(z)} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-\mu(z)}{1+\mu(z)}\right]. \tag{2.3}$$

We observe that $\mu(z)$ is analytic and $\mu(0) = 0$. It is clear from the definition of $S_H(k)$ and the subordination $\frac{g'(z)}{h'(z)} \prec b_1 p_k(z)$ that $|\mu(z)| < 1$ $(z \in \mathbb{D}_r)$. Indeed, we give the counterexample that there exists $z_0 \in \mathbb{D}_r$ such that $|\mu(z_0)| = 1$. Using JACK's lemma [11] $z_0\mu'(z_0) = m\mu(z_0)$, $m \geq 1$, we get

$$\begin{aligned}\frac{g'(z_0)}{h'(z_0)} = b_1 \Bigg[&\frac{A\,(1+\mu(z_0))}{1-\mu(z_0)} - \frac{B\,(1-\mu(z_0))}{1+\mu(z_0)} \\ &+ \left(\frac{2Am\mu(z_0)}{(1-\mu(z_0))^2} + \frac{2Bm\mu(z_0)}{(1+\mu(z_0))^2}\right)\frac{h(z_0)}{z_0 h'(z_0)}\Bigg],\end{aligned} \tag{2.4}$$

where $A = \left(\frac{k}{4}+\frac{1}{2}\right)$, and $B = \left(\frac{k}{4}-\frac{1}{2}\right)$. In this step, using lemma (2.1), the equality (2.4) can be written in the following form:

$$\begin{aligned}w(z_0) = \frac{g'(z_0)}{h'(z_0)} = b_1 \Bigg[&\frac{A\,(1+\mu(z_0))}{1-\mu(z_0)} - \frac{B\,(1-\mu(z_0))}{1+\mu(z_0)} \\ &+ \left(\frac{2Am\mu(z_0)}{(1-\mu(z_0))^2} + \frac{2Bm\mu(z_0)}{(1+\mu(z_0))^2}\right)\frac{1+kre^{i\theta}+r^2}{1-r^2}\Bigg]\end{aligned}$$

that is not in $w(\mathbb{D}_r)$, so it contradicts with (2.2). Indeed our hypothesis is wrong, so $|\mu(z)| < 1$ $(z \in \mathbb{D}_r)$. □

**Theorem 2.3.** *If $f(z) \in S_H(k)$, then the inequalities*

$$\frac{|b_1|\, r\left(1-kr+r^2\right)}{(1-r)^{2-k/2}(1+r)^{2+k/2}} \leq |g(z)| \leq \frac{|b_1|\, r\left(1+kr+r^2\right)}{(1-r)^{2+k/2}(1+r)^{2-k/2}} \tag{2.5}$$

$$\frac{|b_1|\left(1-kr+r^2\right)^2}{(1-r)^{3-k/2}(1+r)^{3+k/2}} \leq \left|g'(z)\right| \leq \frac{|b_1|\left(1+kr+r^2\right)^2}{(1-r)^{3+k/2}(1+r)^{3-k/2}} \tag{2.6}$$

*are realized.*

**Proof.** Using Theorem 2.2 we can get

$$|b_1|\,|h(z)|\,\frac{1-kr+r^2}{1-r^2} \le |g(z)| \le |b_1|\,|h(z)|\,\frac{1+kr+r^2}{1-r^2} \tag{2.7}$$

and

$$|b_1|\,\left|h'(z)\right|\frac{1-kr+r^2}{1-r^2} \le \left|g'(z)\right| \le |b_1|\,\left|h'(z)\right|\frac{1+kr+r^2}{1-r^2}. \tag{2.8}$$

On the other hand, KAHRAMANER et al. [12] proved the boundaries of $|h(z)|$ and $\left|h'(z)\right|$. Considering (2.7), (2.8) and boundaries of $|h(z)|$ and $\left|h'(z)\right|$ together, we obtain (2.5) and (2.6). □

By means of the extremal function

$$\frac{z\,(1-z)^{k/2-1}}{(1+z)^{k/2-1}}$$

we conclude that last inequalities are sharp.

**Corollary 2.4.** *Using Theorem* 2.2 *and Theorem* 2.3 *with the following equalities, we get approximate value of the quantity* $J_{f(z)}$ *and* $|f|$ *as*

$$J_{f(z)} = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 = \left|h'(z)\right|^2\left(1-|w(z)|^2\right),$$

$$\left(\left|h'(z)\right| - \left|g'(z)\right|\right)|dz| \le |df| \le \left(\left|h'(z)\right| + \left|g'(z)\right|\right)|dz|$$

*or*

$$\left|h'(z)\right|(1-|w(z)|)\,|dz| \le |df| \le \left|h'(z)\right|(1+|w(z)|)\,|dz|\,.$$

**3. Conclusion.** In this paper mainly, we analyse the properties of $f = h(z) + \overline{g(z)}$ function that is a solution mapping of the equation $w(z) = \dfrac{\overline{f_{\bar{z}}}}{f_z} = \dfrac{g'(z)}{h(z)}$. It is provided that second dilatation $w(z) = \dfrac{g'(z)}{h(z)}$ of harmonic mapping $f = h(z) + \overline{g(z)}$ is subordinate to the function

$$b_1\mathcal{P}_k(z) = b_1\left[\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right].$$

The method of solution is completely built on the Subordination Principle, also it can be considered a generalization of the Shear Construction.

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