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RARE CONTINUITY REVISITED VIA δb-OPEN SETS

Uğur Şengül[1]✉, Hatice Bilge Soğukoğlu[2]

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Abstract

In this paper we introduce and study the concept of rare δb-continuity in topological spaces as a generalization of rare continuity and weak δb-continuity. We investigate several properties of rarely δb-continuous functions. Rare δb-continuity implied by rare δs-continuity and implies rare b-continuity. We also introduce the concept of $I.\delta b$-continuity, which is weaker than δb-continuity and stronger than rare δb-continuity.

Key words: δb-open sets, rare δb-continuity, $I.\delta b$-continuity
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1. Introduction and preliminaries. One of the research lines of general topology is to study various continuity types and their weak or strong forms of them using different generalized open sets. We will focus on a special form of rare continuity using a variant of b-open sets. The rarely continuous function concept was introduced and studied by Popa [1] in 1979, as a generalization of weak continuity. After that Long and Herrington [2] studied some special properties of rare continuity and of rare sets. These two papers were followed by studies in which different authors generalized rare continuity using different generalized open set notions [3–7]. The first author is introduced and studied rarely b-continuous functions [8] as a generalization of weak b-continuity [9]. Kaymakci [10] introduced and investigated weak δb-continuity as a generalization of weak continuity. The purpose of the present paper is to introduce concept of rare δb-continuity in topological spaces as a generalization of rare continuity and weak

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δb-continuity. We investigate several properties of rarely δb -continuous functions. Rare δb-continuity implied by rare δs-continuity and implies rare b-continuity. We also introduced the notion of $I.\delta b$-continuity which is weaker than δb-continuity and stronger than rare δb-continuity. It is shown that if Y is a regular space, then the function $f : X \to Y$ is $I.\delta b$-continuous on X if and only if f is rarely δb-continuous on X.

Throughout this paper, X and Y are topological spaces. We denote interior and closure of a subset S of X by $\mathring{S}$ and $\overline{S}$, respectively. Recall that a rare set R is a set R such that $\mathring{R} = \varnothing$. A subset S of a space (X, τ) is called regular open [11] (resp. regular closed [11]) if $S = \mathring{\overline{S}}$ (resp. $S = \overline{\mathring{S}}$). The family of all open (resp. regular open) sets of X denoted by $O.(X)$ (resp. $R.O.(X)$) and the family of all regular open sets of X containing a point $x \in X$ is denoted by $O.(X; x)$ (resp. $R.O.(X; x)$). A point $x \in X$ is called a δ-cluster point of S [12] if for all $U \in R.O.(X; x)$ we have $S \cap U \neq \varnothing$. The set of all δ-cluster points of S is called the δ-closure of S and is denoted by $\delta\text{-}cl(S)$. If $S = \delta\text{-}cl(S)$, then S is said to be δ-closed. The complement of a δ-closed set is said to be δ-open. The set $\{x \in X : \exists V \in R.O.(X; x) \text{ such that } V \subset S\}$ is called the δ-interior of S and is denoted by $\delta\text{-}int(S)$. A subset S of a space (X, τ) is called semi-open [13] (resp. b-open [14] or γ-open [15], δ-semi open [16], δb-open [17] (= z-open [18])) if $S \subset \overline{\mathring{S}}$ (resp. $S \subset \mathring{\overline{S}} \cup \overline{\mathring{S}}$, $S \subset \overline{\delta\text{-}int(S)}$, $S \subset \mathring{\overline{S}} \cup \overline{\delta\text{-}int(S)}$). The complement of a semiopen (resp. b-open, δ-semi-open, δb-open) set is said to be semi-closed (resp. b-closed, δ-semi-closed, δb-closed ([17])). If S is a subset of a space (X, τ), then the δb-closure of S, denoted by $\delta b\text{-}cl(S)$, is the intersection of the δb-closed sets containing S ([17]). If S is a subset of a space (X, τ), then the δb-interior of S, denoted by $\delta b\text{-}int(S)$, is the union of δb-open sets contained in S. The family of all b-closed, δb-open, δb-closed, δ-semi-open (shortly δs-open), semi open and b-open sets of a space (X, τ) will be denoted by $B.C.(X)$, $\delta\text{-}B.O.(X)$, $\delta B.C.(X)$, $\delta\text{-}S.O.(X)$, $S.O.(X)$ and $B.O.(X)$, respectively. The family of all δb-open (resp. b-open) sets of X containing a point $x \in X$ is denoted by $\delta\text{-}B.O.(X; x)$ (resp. $B.O.(X; x)$).

One can extract the following diagram from [18]:

$$\delta\text{-semi-open}(=\delta s\text{-open}) \Longrightarrow \delta b\text{-open} \Rightarrow b\text{-open}$$

Our next definition contains some types of functions used throughout this paper.

Definition 1. A function $f : X \to Y$ is called:

(a) Weakly continuous [19] (resp. weakly b-continuous [9], weakly δb-continuous [10]) if for each $x \in X$ and each open set G containing $f(x)$, there exists an $U \in O.(X; x)$ (resp. $U \in B.O.(X; x)$, $U \in \delta\text{-}B.O.(X; x)$) such that $f(U) \subset \overline{G}$;

(b) b-continuous [15] if $f^{-1}(V)$ is b-open in X for every $V \in O.(Y)$;

(c) Rarely continuous [1] if for each point $x \in X$ and each open set $W \subset Y$ containing $f(x)$, there exists a rare set R_W with $W \cap \overline{R_W} = \varnothing$ and an $U \in O.(X; x)$ such that $f(U) \subset W \cup R_W$;

(d) δb-continuous [10] if $f^{-1}(V)$ is δb-open in X for every $V \in O.(Y)$;

(e) Rarely b-continuous [8] (resp. rarely δs-continuous [6]) if for each point $x \in X$ and each open set $W \subset Y$ containing $f(x)$, there exists a rare set R_W with $W \cap \overline{R_W} = \varnothing$ and a b-open (resp. δs-open) set U containing x such that $f(U) \subset W \cup R_W$.

2. Rarely δb-continuous functions.

Definition 2. A function $f : X \to Y$ is called rarely δb-continuous at $x \in X$ if for each open set $W \subset Y$ containing $f(x)$, there exist a rare set R_W with $W \cap \overline{R_W} = \varnothing$ and $U \in \delta\text{-}B.O.(X; x)$ such that $f(U) \subset W \cup R_W$. If f has this property at each point $x \in X$, then we say that f is rarely δb-continuous on X.

Theorem 1. *The following statements are equivalent for a function $f : X \to Y$:*

(a) *The function is rarely δb-continuous at $x \in X$.*

(b) *For each $W \in O.(Y; f(x))$, there exists a rare set R_W with $W \cap \overline{R_W} = \varnothing$ such that $x \in \delta b\text{-}int(f^{-1}(W \cup R_W))$.*

(c) *For each $W \in O.(Y; f(x))$, there exists a rare set R_W with $\overline{W} \cap R_W = \varnothing$ such that $x \in \delta b\text{-}int(f^{-1}(\overline{W} \cup R_W))$.*

(d) *For each $W \in R.O.(Y; f(x))$, there exists a rare set R_W with $W \cap \overline{R_W} = \varnothing$ such that $x \in \delta b\text{-}int(f^{-1}(W \cup R_W))$.*

(e) *For each $W \in O.(Y; f(x))$, there exists an $U \in \delta\text{-}B.O.(X; x)$ such that $(f(U) \cap (Y \setminus W))^\circ = \varnothing$.*

(f) *For each $W \in O.(Y; f(x))$, there exists an $U \in \delta\text{-}B.O.(X; x)$ such that $(f(U))^\circ \subset \overline{W}$.*

Proof. $(a) \Rightarrow (b)$: Let $x \in X$ and $W \in O.(Y; f(x))$. Then there exist a rare set R_W with $W \cap \overline{R_W} = \varnothing$ and an $U \in \delta\text{-}B.O.(X; x)$ such that $f(U) \subset W \cup R_W$. It follows that $x \in U \subset f^{-1}(W \cup R_W)$, then we have $x \in \delta b\text{-}int(f^{-1}(W \cup R_W))$.

$(b) \Rightarrow (c)$: Suppose that $W \in O.(Y; f(x))$. Then there exists a rare set R_W with $W \cap \overline{R_W} = \varnothing$ such that $x \in \delta b\text{-}int(f^{-1}(W \cup R_W))$. $W \cap \overline{R_W} = \varnothing$ implies

$$R_W = R_W \cap \big((Y \setminus \overline{W}) \cup (\overline{W} \setminus W)\big) = \big(R_W \cap (Y \setminus \overline{W})\big) \cup \big(R_W \cap (\overline{W} \setminus W)\big)$$

$$\subset \big(R_W \cap (Y \setminus \overline{W})\big) \cup (\overline{W} \setminus W).$$

So that, we have $R_W \subset \big(R_W \cap (Y \setminus \overline{W})\big) \cup (\overline{W} \setminus W)$. Set $R^* = R_W \cap (Y \setminus \overline{W})$. It follows that R^* is a rare set where $\overline{W} \cap R^* = \varnothing$. Then we have

$$W \cup R_W \subset W \cup (\overline{W} \setminus W) \cup \big(R_W \cap (Y \setminus \overline{W})\big) = \overline{W} \cup R^*$$
$$\Rightarrow \delta b\text{-}int(f^{-1}(W \cup R_W)) \subset \delta b\text{-}int(f^{-1}(\overline{W} \cup R^*)).$$

Therefore $x \in \delta b\text{-}int(f^{-1}(W \cup R_W)) \subset \delta b\text{-}int(f^{-1}(\overline{W} \cup R^*))$.

$(c) \Rightarrow (d)$: Assume that $x \in X$ and $W \in R.O.(Y; f(x))$. Then there exists a rare set R_W with $\overline{W} \cap R_W = \varnothing$ such that $x \in \delta b\text{-}int(f^{-1}(\overline{W} \cup R_W))$. Set $R^* = R_W \cup (\overline{W} \setminus W)$. It follows that R^* is a rare set and $W \cap \overline{R^*} = \varnothing$. Hence

$$x \in \delta b\text{-}int(f^{-1}(\overline{W} \cup R_W)) = \delta b\text{-}int\left[f^{-1}(W \cup (\overline{W} \setminus W) \cup R_W)\right]$$
$$= \delta b\text{-}int\left[f^{-1}(W \cup R^*)\right].$$

$(d) \Rightarrow (e)$: Let $W \in O.(Y; f(x))$. Then using $f(x) \in W \subset \overset{\circ}{\overline{W}}$ and the fact that $\overset{\circ}{\overline{W}} \in R.O.(Y; f(x))$, there exists a rare set $R_{\overset{\circ}{\overline{W}}}$ with $\overset{\circ}{\overline{W}} \cap \overline{R_{\overset{\circ}{\overline{W}}}} = \varnothing$ such that $x \in \delta b\text{-}int(f^{-1}(\overset{\circ}{\overline{W}} \cup R_{\overset{\circ}{\overline{W}}}))$. Suppose $U = \delta b\text{-}int(f^{-1}(\overset{\circ}{\overline{W}} \cup R_{\overset{\circ}{\overline{W}}}))$ then $U \in \delta$-$B.O.(X; x)$ and therefore $f(U) \subset \overset{\circ}{\overline{W}} \cup R_{\overset{\circ}{\overline{W}}}$. Then we have

$$(f(U) \cap (Y \setminus W))^\circ = (f(U))^\circ \cap (Y \setminus W)^\circ \subset \left(\overset{\circ}{\overline{W}} \cup R_{\overset{\circ}{\overline{W}}}\right)^\circ \cap (Y \setminus \overline{W})$$
$$\subset \left(\overline{W} \cup \left(R_{\overset{\circ}{\overline{W}}}\right)^\circ\right) \cap (Y \setminus \overline{W}) = \varnothing.$$

$(e) \Rightarrow (f)$: Since

$$(f(U) \cap (Y \setminus W))^\circ = (f(U))^\circ \cap (Y \setminus \overline{W}) = \varnothing$$

we have

$$[f(U)]^\circ \subset \overline{W}.$$

$(f) \Rightarrow (a)$: Assume that $x \in X$ and $W \in O.(Y; f(x))$. Then by (f), there exists an $U \in \delta$-$B.O.(X; x)$ such that $[f(U)]^\circ \subset \overline{W}$. Then

$$f(U) = [f(U) \setminus (f(U))^\circ] \cup (f(U))^\circ$$
$$\subset [f(U) \setminus (f(U))^\circ] \cup \overline{W} = [f(U) \setminus (f(U))^\circ] \cup W \cup (\overline{W} \setminus W).$$

Set $R^* = [f(U) \setminus (f(U))^\circ] \cap (Y \setminus W)$ (notice that $Y \setminus (f(U))^\circ \subset Y \setminus W$) and $R^{**} = (\overline{W} \setminus W)$. Then R^* and R^{**} are rare sets. Moreover $R_W = R^* \cup R^{**}$ is a rare set, $\overline{R_W} \cap W = \varnothing$ and $f(U) \subset W \cup R_W$. $\square$

Theorem 2. *A function $f : X \to Y$ is rarely δb-continuous if and only if $f^{-1}(W) \subset \delta b\text{-}int(f^{-1}(W \cup R_W))$ for every open set W in Y, where R_W is a rare set with $W \cap \overline{R_W} = \varnothing$.*

Proof. Clear from Theorem 1. □

Remark 1. Rare δb-continuity implied by rare δs-continuity and implies rare b-continuity, but the converse implications are not true in general as the following examples show.

Example 1. Let (X, τ_1) and (Y, σ_1) be topological spaces such that $X = Y = \{a, b, c, d\}$, $\tau_1 = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\sigma_1 = \{\varnothing, Y, \{b\}, \{c\}, \{b, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$. Then $B.O.(X) = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$, $\delta\text{-}B.O.(X) = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$ and the identity function $f : (X, \tau_1) \to (Y, \sigma_1)$ is rarely b-continuous but it is not rarely δb-continuous (there is no $U \in \delta\text{-}B.O.(X)$ satisfying $f(U) \subset \{a, d\} \cup R_{\{a,d\}}$ for $R_{\{a,d\}} = \varnothing$).

Example 2. Let (X, τ_2) and (Y, σ_2) be topological spaces such that $X = Y = \{a, b, c, d\}$, $\tau_2 = \{\varnothing, X, \{a, b\}, \{a, b, c\}\}$ and $\sigma_2 = \{\varnothing, Y, \{a, b\}, \{c, d\}\}$. Then $\delta\text{-}B.O.(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c\}, \{a, c, d\}\}$, $\delta\text{-}S.O.(X) = \{\varnothing, X\}$ and the identity function $f : (X, \tau_2) \to (Y, \sigma_2)$ is rarely δb-continuous, but it is not rarely δs-continuous (there is no $U \in \delta\text{-}S.O.(X)$ satisfying $f(U) \subset \{a, b\} \cup R_{\{a,b\}}$ for $R_{\{a,b\}} = \{c\}$).

Definition 3. A function $f : X \to Y$ is called $I.\delta b$-continuous at $x \in X$ if for each open set $W \subset Y$ containing $f(x)$, there exists a δb-open set U containing x such that $[f(U)]^\circ \subset W$. If f has this property at each point $x \in X$, then we say that f is $I.\delta b$-continuous on X.

Remark 2. It is clear that $I.\delta b$-continuity is weaker than δb-continuity and stronger than rare δb-continuity.

Theorem 3. *Let Y be a regular space. Then the function $f : X \to Y$ is $I.\delta b$-continuous on X if and only if f is rarely δb-continuous on X.*

Proof. Necessity is clear.

Sufficiency. Let f be rarely δb-continuous on X. Suppose that $f(x) \in W$, where W is an open set in Y and $x \in X$. By the regularity of Y, there exists an open set W_1 in Y such that $f(x) \in W_1$ and $\overline{W_1} \subset W$. Since f is rarely δb-continuous, then there exists $U \in \delta\text{-}B.O.(X; x)$ such that $[f(U)]^\circ \subset \overline{W_1}$. This implies $[f(U)]^\circ \subset W$ which means that f is $I.\delta b$-continuous on X. □

Example 3. Let (X, τ_1) and (Y, σ_2) be topological spaces such that $X = Y = \{a, b, c, d\}$, $\tau_1 = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\sigma_2 = \{\varnothing, Y, \{a, b\}, \{c, d\}\}$. Then the identity function $f : (X, \tau_1) \to (Y, \sigma_2)$ is rarely δb-continuous, but not $I.\delta b$-continuous (there is no $U \in \delta\text{-}B.O.(X)$ satisfying $[f(U)]^\circ \subset \{c, d\}$).

Example 4. Example 12 of [6] provides an example of a function which is $I.\delta b$-continuous but not δb-continuous.

Definition 4. A space X is called r-separate [3] if for every pair of distinct points x and y in X, there exist rare sets R_{U_x} and R_{U_y}, and open sets U_x and U_y with $U_x \cap \overline{R_{U_x}} = \varnothing$ and $U_y \cap \overline{R_{U_y}} = \varnothing$ such that $(U_x \cup R_{U_x}) \cap (U_y \cup R_{U_y}) = \varnothing$.

Definition 5. A space X is said to be b-T_2 [20] (resp. δb-T_2 [18]) if for each pair of distinct points x and y in X, there exist b-open (resp. δb-open) sets U and V of X containing x and y, respectively, such that $U \cap V = \varnothing$.

Theorem 4. *If $f : X \to Y$ is rarely δb-continuous injection and Y is r-separate, then X is b-T_2.*

Proof. Since the function f is injective, $f(x) \neq f(y)$ for any distinct points x and y in X. By hypothesis, Y is r-separate, so there exist open sets W_x and W_y in Y such that $f(x) \in W_x$ and $f(y) \in W_y$, respectively, and rare sets R_{W_x} and R_{W_y} where $W_x \cap \overline{R_{W_x}} = \varnothing$ and $W_y \cap \overline{R_{W_y}} = \varnothing$ such that $(W_x \cup R_{W_x}) \cap (W_y \cup R_{W_y}) = \varnothing$. Therefore

$$\delta b\text{-}int(f^{-1}(W_x \cup R_{W_x})) \cap \delta b\text{-}int(f^{-1}((W_y \cup R_{W_y})) = \varnothing.$$

By Theorem 2 we have

$$x \in f^{-1}(W_x) \subset \delta b\text{-}int(f^{-1}(W_x \cup R_{W_x}))$$

and

$$y \in f^{-1}(W_y) \subset \delta b\text{-}int(f^{-1}(W_y \cup R_{W_y})).$$

Then

$$\delta b\text{-}int(f^{-1}(W_x \cup R_{W_x})),\ \delta b\text{-}int(f^{-1}(W_y \cup R_{W_y})) \in \delta\text{-}B.O.(X)$$

and by Theorem 8 of [17], X is b-T_2. □

Definition 6. A function $f : X \to Y$ is called strongly δb-open if for every $U \in \delta\text{-}B.O.(X)$, $f(U)$ is open.

Theorem 5. *If a function $f : X \to Y$ is strongly δb-open and rarely δb-continuous, then f is weakly δb-continuous.*

Proof. Suppose that $x \in X$ and W is any open set of Y containing $f(x)$. Since f is rarely δb-continuous, there exist a rare set R_W with $W \cap \overline{R_W} = \varnothing$ and $U \in \delta\text{-}B.O.(X; x)$ such that $f(U) \subset W \cup R_W$. Then $f(U) \cap (Y \setminus \overline{W}) \subset R_W$. Since f is strongly δb-open $f(U) \cap (Y \setminus \overline{W})$ is open. But the rare set R_W has no interior point. Then $f(U) \cap (Y \setminus \overline{W}) = \varnothing$. This implies that $f(U) \subset \overline{W}$. Hence f is weakly δb-continuous. □

Lemma 1 ([18]). *The intersection of an open and a δb-open set is a b-open set.*

Theorem 6. *If a function $f : X \to Y$ is rarely δb-continuous, then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every $x \in X$, is rarely b-continuous.*

Proof. Suppose that $x \in X$ and W is any open set containing $g(x)$. Then there exist open sets U and V in X and Y, respectively, such that $(x, f(x)) \in$

$U \times V \subset W$. Since f is rarely δb-continuous, there exists $G \in \delta\text{-}B.O.(X;x)$ such that $[f(G)]^\circ \subset \overline{V}$. Let $O = U \cap G$. By Lemma 1, $O \in B.O.(X;x)$ and we have $[g(O)]^\circ \subset [U \times f(G)]^\circ \subset U \times \overline{V} \subset \overline{W}$. Therefore, g is rarely b-continuous. $\square$

Definition 7. A topological space (X, τ) is said to be δb-compact [10] if every δb-open cover of X has a finite subcover.

Definition 8. Let $\mathcal{U} = \{W_\ell\}$ be a class of subsets of X. If for each $W_{\ell'} \in \{W_\ell\}$ there exists a rare set $R_{W_{\ell'}}$ such that $W_{\ell'} \cap \overline{R_{W_{\ell'}}} = \varnothing$, then the family $\{W_\ell \cup R_{W_\ell}\}$ is called rarely union sets [3].

Definition 9. A topological space (X, τ) is called rarely almost compact [3] if each open cover of X has a finite subfamily whose rarely union sets cover the space.

Theorem 7. *Let $f : X \to Y$ be rarely δb-continuous and S be a δb-compact subset in X. Then $f(S)$ is a rarely almost compact subset of Y.*

Proof. Suppose that $\mathcal{U}$ is an open cover of $f(S)$. Set $\mathcal{U}^* = \{V \in \mathcal{U} : V \cap f(S) \neq \varnothing\}$. Then $\mathcal{U}^*$ is an open cover of $f(S)$. Hence for each $x \in S$, there is some $V_x \in \mathcal{U}^*$ such that $f(x) \in V_x$. Since f is rarely δb-continuous there exist a rare set R_{V_x} with $V_x \cap \overline{R_{V_x}} = \varnothing$ and a δb-open set U_x containing x such that $f(U_x) \subset V_x \cup R_{V_x}$. Hence there is a subfamily $\{U_{x_i}\}_{x_i \in S_0}$ which covers S, where S_0 is a finite subset of S. The subfamily $\{V_{x_i} \cup R_{V_{x_i}}\}_{x_i \in S_0}$ also covers $f(S)$. $\square$

Lemma 2 ([2])**.** *If $g : Y \to Z$ is continuous and one-to-one, then g preserves rare sets.*

Theorem 8. *If $f : X \to Y$ is rarely δb-continuous surjection and $g : Y \to Z$ is continuous and one-to-one, then $gof : X \to Z$ is rarely δb-continuous.*

Proof. Suppose that $x \in X$ and $g(f(x)) \in V$, where V is an open subset in Z. By hypothesis, g is continuous, therefore there exists an open set $W \subset Y$ containing $f(x)$ such that $g(W) \subset V$. Since f is rarely δb-continuous, there exist a rare set R_W with $W \cap \overline{R_W} = \varnothing$ and a δb-open set U containing x such that $f(U) \subset W \cup R_W$. It follows from Lemma 2 that $g(R_W)$ is a rare set in Z. Since R_G is a subset of $Y \setminus W$ and g is injective, we have $\overline{g(R_W)} \cap V = \varnothing$. This implies that $g(f(U)) \subset V \cup g(R_W)$. Hence the result follows. $\square$

Definition 10. A function $f : X \to Y$ is called pre δb-open if for every $U \in \delta\text{-}B.O.(X)$ we have $f(U) \in \delta\text{-}B.O.(Y)$.

Theorem 9. *If $f : X \to Y$ is pre δb-open and $g : Y \to Z$ a function such that $gof : X \to Z$ is rarely δb-continuous. Then g is rarely δb-continuous.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let G be an open set containing $g(f(x))$. Then there exist a rare set R_G with $G \cap \overline{R_G} = \varnothing$ and a δb-open set U containing x such that $g(f(U)) \subset G \cup R_G$. But $f(U)$ is a δb-open set containing $f(x) = y$ such that $g(f(U)) = (gof)(U) \subset G \cup R_G$. This shows that g is rarely δb-continuous at y. $\square$

Definition 11. A function $f : X \to Y$ is called relatively rare δb-continuous if for each point $x \in X$ and each open set $W \subset Y$ containing $f(x)$, there exists a rare set R_W with $W \cap \overline{R_W} = \varnothing$ such that the set $f^{-1}(W)$ is δb-open in the subspace $f^{-1}(W \cup R_W)$.

Theorem 10. *If $f : (X, \tau) \to (Y, \sigma)$ is rarely δb-continuous, then for each $W \in \sigma$ there exists an $U \in \delta$-$B.O.(X)$ such that $f^{-1}(W) \subset f^{-1}(W \cup R_W) \cap U$.*

Proof. Take $U = \delta b\text{-}int(f^{-1}(W \cup R_W))$. □

Theorem 11. *A function $f : X \to Y$ is δb-continuous if and only if it is rarely δb-continuous and relatively rare δb-continuous.*

Proof. Necessity is clear.

Sufficiency. By relatively rare δb-continuity, we have $f^{-1}(W) = f^{-1}(W \cup R_W) \cap W_1$ where W is open in Y, R_W is a rare set and $W_1 \in \delta$-$B.O.(X)$. We will prove $f^{-1}(W)$ is δb-open in X. Suppose that $x \in f^{-1}(W)$. This means that $f(x) \in W$ and $x \in W_1$. Since f is also rarely δb-continuous, there exist a rare set R_W with $W \cap \overline{R_W} = \varnothing$ and a subset $U \in \delta$-$B.O.(X; x)$ such that $f(U) \subset W \cup R_W$. Then we have $U \subset f^{-1}(W \cup R_W)$. Since $\delta b\text{-}int(f^{-1}(W \cup R_W))(\subset f^{-1}(W \cup R_W))$ is also δb-open we may assume $U \subset W_1$. Hence we have $x \in U \subset f^{-1}(W \cup R_W) \cap W_1 = f^{-1}(W)$. This shows that f is δb-continuous. □

Definition 12. A function $f : X \to Y$ satisfies interiority rare δb condition if $\delta b\text{-}int(f^{-1}(W \cup R_W)) \subset f^{-1}(W)$ for each open set W in Y, where R_W is a rare set with $W \cap \overline{R_W} = \varnothing$.

Theorem 12. *If $f : X \to Y$ is rarely δb-continuous and satisfies interiority rare δb condition, then f is δb-continuous.*

Proof. By the hypothesis of the theorem, we have $f^{-1}(W) \subset \delta b\text{-}int(f^{-1}(W \cup R_W))$, where W is an open set in Y and R_W is a rare set with $W \cap \overline{R_W} = \varnothing$. On the other hand, by the interiority rare δb condition we have $\delta b\text{-}int(f^{-1}(W \cup R_W)) \subset f^{-1}(W)$. Therefore $f^{-1}(W)$ is δb-open in X and consequently f is δb-continuous. □

REFERENCES

[1] Popa V. (1979) Sur certaine décomposition de la continuité dans les espaces topologiques, Glasnik Mat. Ser. III, **14(34)**(2), 359–362.

[2] Long P. E., L. L. Herrington (1982) Properties of rarely continuous functions, Glasnik Mat., Ser. III. **17(37)**(1), 147–153.

[3] Jafari S. (1995) A note on rarely continuous functions, Univ. Bacău. Stud. Cerc. Şt. Ser. Mat. (Romania), **5**, 29–34.

[4] Jafari S. (2000) On rarely β-continuous functions, J. Inst. of Math. Comput. Sci. Math. Ser., **13**(2), 247–251.

[5] Jafari S. (2000) On rarely precontinuous functions, Far East J. Math. Sci. (FJMS) Spec. Vol., 305–314.

[6] Caldas M., S. Jafari, S. P. Moshokoa, T. Noiri (2008) On rarely δs-continuous functions, Note Mat., **28**(2), 107–115.

[7] Popa V., T. Noiri (1991) Some properties of rarely quasicontinuous functions, An. Univ. Timişoara, Ştiinţ. Mat., **29**(1), 65–71.

[8] JAFARI S., U. ŞENGÜL (2013) Rarely b-continuous functions, Ital. J. Pure Appl. Math., **30**, 109–116.
[9] ŞENGÜL U. (2009) Weakly b-continuous functions, Chaos Solitons Fractals, **41**(3), 1070–1077.
[10] KESKIN KAYMAKCI A. (2015) Weakly δ-b-continuous functions, Gen. Math. Notes, **27**(1), 24–39.
[11] STONE M. H. (1937) Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., **41**(3), 375–381.
[12] VELIČKO N. V. (1968) H-closed topological spaces, American Mathematical Society Translations, **78**, 103–118.
[13] LEVINE N. (1963) Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, **70**, 36–41.
[14] ANDRIJEVIĆ D. (1996) On b-open sets, Mat. Vesnik, **48**(1–2), 59–64.
[15] EL-ATIK A. A. (1997) A study on some types of mappings on topological spaces, M.Sc. Thesis, Tanta University, Egypt.
[16] PARK J. H., B. Y. LEE, M. J. SON (1997) On δ-semiopen sets in topological spaces, J. Indian Acad. Math, **19**(1), 59–67.
[17] KESKIN KAYMAKCI A. (2019) A New Class of Generalized b-open Sets. In: IV. International Natural and Engineering Sciences Congress, 2019.
[18] EL-MAGHARABI A. I., A. M. MUBARKI (2011) z-open sets and z-continuity in topological spaces, Int. J. Math. Arch., **2**(10), 1819–1827.
[19] LEVINE N. (1961) A decomposition of continuity in topological spaces, Amer. Math. Monthly, **68**, 44–46.
[20] CALDAS M., S. JAFARI (2007) On some applications of b-open sets in topological spaces, Kochi J. Math, **2**, 11–19.

[1]*Department of Mathematics, Faculty of Science, Marmara University,*
34722 Göztepe-İstanbul, Turkey
e-mail: usengul@marmara.edu.tr

[2]*Institute of Pure and Applied Sciences, Marmara University,*
34722 Göztepe-İstanbul, Turkey
e-mail: hbsogukoglu@gmail.com