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OPTICAL SOLITON PERTURBATION WITH THE CONCATENATION MODEL: SEMI-INVERSE VARIATION

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Abstract

This paper retrieves bright 1-soliton solution to the perturbed concatenation model by the application of the semi-inverse variational principle. The perturbation terms are from inter-modal dispersion and self-steepening effect that is considered with maximum allowable intensity. The parameter constraints that naturally emerged from the integration scheme are enumerated.

Key words: concatenation, self-steepening, self-frequency shift

OCIS codes: 060.2310, 060.4510, 060.5530, 190.3270, 190.4370

1. Introduction. One of the most innovative models to describe soliton transmission dynamics, across intercontinental distances, is a conjunction of three previously known equations. They are the nonlinear Schrödinger's equation (NLSE), Lakshmanan–Porsezian–Daniel (LPD) equation and the Sasa–Satsuma equation (SSE). These three separate models are conjoined to create a new one that first appeared in 2014 [1, 2]. It is referred to as the concatenation model. Later, it gained popularity and a lot of research work was conducted. A full spectrum of soliton solutions was secured, the conservation laws were recovered, the Painleve analysis was performed and many more [3–5].

The current paper considers the perturbed version of the concatenation model. The two perturbation terms are from inter-modal dispersion and self-steepening effect which is considered with maximum allowable intensity. It is this perturbed

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version of the concatenation model that will be addressed in the paper. The focus is on its integrability aspect so that a bright 1-soliton solution can be recovered for the model. The integration algorithm that is adopted is the semi-inverse variational principle (SVP) that leads to an analytical soliton solution to the model [3, 6–10]. The details are enumerated in the rest of the paper after a succinct introduction and acclimatization with the equation.

1.1. Governing model. The dimensionless version of the concatenation model in presence of perturbation terms is written as:

$$\begin{aligned} & iq_t + aq_{xx} + b\left|q\right|^2 q \\ & + c_1\left[\sigma_1 q_{xxxx} + \sigma_2\left(q_x\right)^2 q^* + \sigma_3\left|q_x\right|^2 q + \sigma_4\left|q\right|^2 q_{xx} + \sigma_5 q^2 q_{xx}^* + \sigma_6\left|q\right|^4 q\right] \\ & + ic_2\left[\sigma_7 q_{xxx} + \sigma_8\left|q\right|^2 q_x + \sigma_9 q^2 q_x^*\right] \\ & = i\left[\lambda\left(\left|q\right|^{2m} q\right)_x + \theta_1\left(\left|q\right|^{2m}\right)_x q + \theta_2\left|q\right|^{2m} q_x\right]. \end{aligned} \tag{1}$$

Here in (1), $q(x,t)$ is the complex-valued dependent variable and represents the wave amplitude. The independent variables x and t are from spatial and temporal components, respectively. The perturbed concatenation model given by (1) is as meaningful as the name implies. The perturbation terms are on the right hand side that come from self-steepening effect and self-frequency shift whose coefficients are λ and θ_j, for $j = 1, 2$, respectively. The power-law parameter m accounts for maximum allowable intensity and has its threshold count. It also represents full nonlinearity. With $\lambda = \theta_j = 0$, (1) reduces to the unperturbed concatenation model.

For the unperturbed model when $c_1 = 0$, (1) reduces to the familiar SSE, while for $c_2 = 0$, equation (1) reduces to LPD model. However, if $c_1 = c_2 = 0$, (1) with $\lambda = \theta_j = 0$ condenses to the familiar NLSE. The unperturbed concatenation is thus an extended version of the familiar NLSE, SSE and LPD models. The concatenated version therefore models the the propagation of solitons through an optical fibre in a far more precise manner. The current paper will reveal the bright 1-soliton solution to the perturbed concatenation model, given by (1) by the aid of SVP.

2. Mathematical preliminaries. To construct soliton solutions for the system (1) we assume it supports solutions of the form

$$q(x,t) = g(x - vt)e^{i\phi(x,t)} \tag{2}$$

from which the phase factor is:

$$\phi(x,t) = -\kappa x + \omega t + \theta_0, \tag{3}$$

where $g(s)$ represents the bright soliton while κ represents the soliton wave number, and ω and θ_0 give the soliton frequency and phase constant, respectively. Also in (2), v gives the speed of the bright soliton. By substituting (2) into (1) and splitting into real and imaginary parts, one obtains for the real part

$$
\begin{aligned}
&(c_1\sigma_1\kappa^4 - c_2\sigma_7\kappa^3 - a\kappa^2 - \omega)g + \{b - c_1(\sigma_2 - \sigma_3 + \sigma_4 + \sigma_5)\kappa^2 \\
&+ c_2\left(\sigma_8 - \sigma + \left(a - 6c_1\sigma_1\kappa^2 + 3c_2\sigma_7\kappa\right) g_9''\right)\kappa\}g^3 + c_1\sigma_6 g^5 \\
&+ c_1\sigma_1 g^{(iv)} + c_1(\sigma_4 + \sigma_5) g^2 g'' + c_1(\sigma_2 + \sigma_3) g \left(g'\right)^2 \\
&= \kappa(\lambda + \theta_2) g^{2m+1}, \qquad (4)
\end{aligned}
$$

while the imaginary counterpart turns out to be,

$$
\begin{aligned}
&\left(v + 2a\kappa + 3c_2\sigma_7\kappa^2 - 4c_1\sigma_1\kappa^3\right) g' \\
&- \{c_2(\sigma_8 + \sigma_9) - 2\kappa c_1(\sigma_2 + \sigma_4 - \sigma_5)\} g^2 g' - (c_2\sigma_7 - 4\kappa c_1\sigma_1) g''' \\
&= -\{(2m+1)\lambda + 2\theta_1 m + \theta_2\} g^{2m} g'. \qquad (5)
\end{aligned}
$$

Here, the notations $g' = dg/dx$, $g'' = d^2g/dx^2$ and so on are adopted. From the imaginary part the soliton speed comes out to be

$$v = -\kappa\left(2a + 3c_2\sigma_7\kappa - 4c_1\sigma_1\kappa^2\right), \qquad (6)$$

provided

$$c_2(\sigma_8 + \sigma_9) = 2\kappa c_1(\sigma_2 + \sigma_4 - \sigma_5), \qquad (7)$$

and

$$c_2\sigma_7 = 4\kappa c_1\sigma_1 \text{ and } (2m+1)\lambda + 2\theta_1 m + \theta_2 = 0 \qquad (8)$$

hold. Next, from the real part equation given by (4), upon setting the coefficients of g^2g'' and $g\left(g'\right)^2$ to zero, for integrability purposes, leads to

$$\sigma_4 + \sigma_5 = 0, \qquad (9)$$

and

$$\sigma_2 + \sigma_3 = 0. \qquad (10)$$

Thus equation (4) reduces to:

$$P_1 g - P_2 g^3 - P_3 g^5 - P_4 g'' - P_5 g^{(iv)} - \kappa(\lambda + \theta_2) g^{2m+1} = 0, \qquad (11)$$

where

$$P_1 = -\omega - a\kappa^2 - 3c_1\sigma_1\kappa^4, \qquad (12)$$

$$P_2 = -b + c_1(\sigma_2 - \sigma_3 + \sigma_4 + \sigma_5)\kappa^2 - c_2(\sigma_8 - \sigma_9)\kappa, \qquad (13)$$

$$P_3 = -c_1\sigma_6, \qquad (14)$$

$$P_4 = -a - 6c_1\sigma_1\kappa^2, \qquad (15)$$

and

$$P_5 = -c_1\sigma_1. \qquad (16)$$

Equation (11) will be now further analyzed, by SVP, to retrieve the bright 1-soliton solution to the governing model given by (1). The details are narrated in the subsequent section.

3. Semi-inverse variation. Multiplying (11) by g' and integrating gives

(17) $$6P_1g^2 + 3P_2g^4 + 2P_3g^6 + 6P_4\left(g'\right)^2 + 18P_5\left(g''\right)^2 - \frac{6\left(\lambda+\theta_2\right)\kappa g^{2m+2}}{m+1} = K,$$

where K is the integration constant. The stationary integral is defined as

(18) $$J = \int_{-\infty}^{\infty}\left[6P_1g^2+3P_2g^4+2P_3g^6+6P_4\left(g'\right)^2+18P_5\left(g''\right)^2-\frac{6\left(\lambda+\theta_2\right)\kappa}{m+1}g^{2m+2}\right]dx.$$

SVP states that the solution of the perturbed concatenation model given by (1) will be the same as that of its unperturbed version, namely with $\lambda = \theta_j = 0$. However, the amplitude (A) and the inverse width (B) of the perturbed soliton will change and their variations can be recovered from the solution of the coupled system [3]:

(19) $$\frac{\partial J}{\partial A} = 0,$$

and

(20) $$\frac{\partial J}{\partial B} = 0.$$

Now the solution of the unperturbed concatenation model is given by [4]

(21) $$g(x-vt) = A\,\mathrm{sech}[B(x-vt)].$$

Substituting (21) into (18) leads to

(22) $$J = 3P_1\frac{A^2}{B} + P_2\frac{A^4}{B} + \frac{4P_3}{15}\frac{A^6}{B} + 2P_4A^2B - \frac{171P_5}{5}A^2B^3 - \frac{QA^{2m+2}}{B},$$

where

(23) $$Q = \frac{3m\left(\lambda+\theta_2\right)\kappa}{(m+1)(2m+1)}\frac{\Gamma\left(m\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}.$$

From J given by (22), equations (19) and (20) reduce to

(24) $$3P_1 - P_2A^2 - \frac{4P_3}{15}A^4 + 2P_4B^2 - \frac{171P_5}{5}B^4 - (m+1)QA^{2m} = 0,$$

and

(25) $$-3P_1 - P_2A^2 - \frac{8P_3}{15}A^4 + 2P_4B^2 - \frac{513P_5}{5}B^4 + QA^{2m} = 0,$$

respectively. Uncoupling (23) and (24) gives the biquadratic equation for B in terms of A as

(26) $$2052P_5B^4 - 60P_4B^2 - 8P_3A^4 - 15P_2A^2 + 15mQA^{2m} = 0,$$

which expresses the width B of the soliton in terms of the amplitude A as:

(27) $$B = \left[\frac{15P_4 + 3\sqrt{25P_4^2 + 57P_5\left(15P_2A^2 + 8P_3A^4 - 15mQA^{2m}\right)}}{513P_5}\right]^{\frac{1}{2}}.$$

The solution given by (27) is valid provided

$$25P_4^2 + 57P_5\left(15P_2A^2 + 8P_3A^4 - 15mQA^{2m}\right) > 0, \tag{28}$$

and

$$P_5\left[15P_4 + 3\sqrt{25P_4^2 + 57P_5\left(15P_2A^2 + 8P_3A^4 - 15mQA^{2m}\right)}\right] > 0. \tag{29}$$

Finally, the bright 1-soliton solution to the perturbed concatenation model (1) is given by

$$q(x,t) = A\,\mathrm{sech}\left[B(x - vt)\right] e^{i(-\kappa x + \omega t + \theta_0)}, \tag{30}$$

where the relation between the amplitude A and its inverse width B is given by (26) and is subjected to the several parameter constraints listed throughout.

Figures 1, 2 and 3 show the profile of a bright 1-soliton solution for the concatenation model with the parameter values chosen as: $A = 0.9$, $a = 1$, $b = 1$, $\kappa = 1$, $\lambda = 1$, $m = 0.5$, $c_1 = 1$, $\sigma_1 = 1$, $\sigma_2 = 1$, $\sigma_3 = 1$, $\sigma_4 = 1$, $\sigma_5 = 1$, $\sigma_6 = 1$, $\sigma_8 = 1$, $\sigma_9 = 1$, and $\theta_2 = 1$. Figure 1 focuses on the effect of changing the soliton

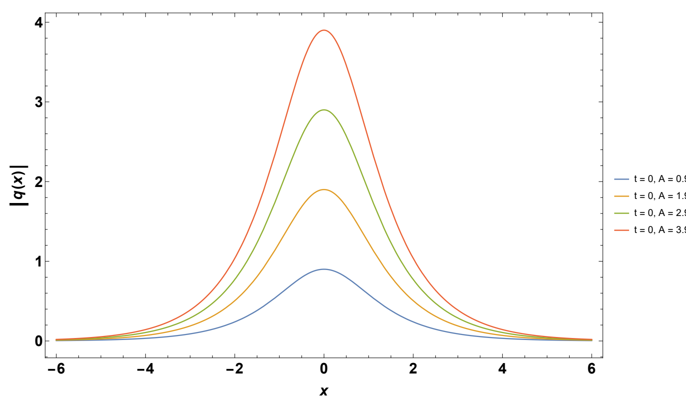

Fig. 1. Effect of the soliton amplitude (A)

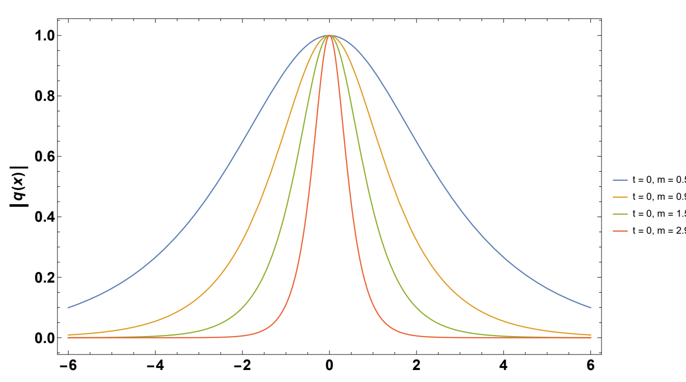

Fig. 2. Effect of the soliton inverse width (B) along with the power-law parameter (m)

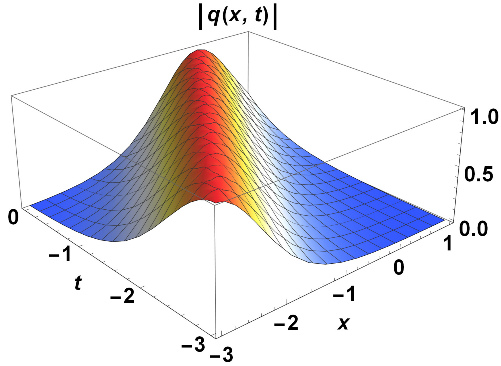

Fig. 3. Profile of a bright soliton solution

amplitude (A) on the bright 1-soliton solution. Figure 1 contains a graphical representation showing how the soliton profile changes as A varies. Figure 2, on the other hand, explores the relationship between two parameters: the soliton's inverse width (B) and the power-law parameter (m). Figure 2 presents graphical results that show how changes in m affect the soliton inverse width (B), and vice versa, within the context of the concatenation model. Also, Figure 3 exhibits the surface plot of a bright soliton solution. As a result, Figures 1, 2 and 3 are used to visually represent the behaviour of a bright 1-soliton solution in the concatenation model under different parameter settings. We have chosen specific values for these parameters to investigate their impact on the soliton's profile. These figures provides valuable insights into how changes in these parameters influence the soliton solution in the context of the model being studied.

4. Conclusions. The paper recovered a bright 1-soliton solution to the perturbed concatenation model where the perturbation term was considered with maximum allowable intensity. The full nonlinearity parameter (m) is considered to be arbitrary at this stage. However, the threshold value of the parameter m can be determined with the usage of the modulation instability analysis and that is a future project. It must be noted that the analysis for the concatenation model is being reported here for the first time. However, for the individual equations, namely the NLSE and LPD model, the SVP has already been studied and those results have been already reported [9, 10]. This includes cubic-quartic optical solitons for the LPD model [5]. It must be noted that the soliton solution is not exact since it is recovered by the aid of SVP.

Later, the model will be extended to address it with power-law nonlinearity and use these approaches including Kudryashov's method to handle it [4, 5]. In future, this study can be extended further along to fibres with differential group delay and to dispersion-flattened fibres. Another avenue to look at is to obtain the quasi-monochromatic dynamics of such solitons and thus recover the optical soliton cooling effect for the concatenation model [6–10].

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