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ON THE COINCIDENCE THEOREM

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Abstract

We are proving Coincidence theorem due to Walsh for the case when the total degree of a polynomial is less than the number of arguments. Also, the following result has been proven: if $p(z)$ is a complex polynomial of degree n, then closed disk D that contains at least $n-1$ of its zeros (counting multiplicity) contains at least $\left[\dfrac{n-2k+1}{2}\right]$ zeros of its k-th derivative, provided that the arithmetical mean of these zeros is also centre of D. We also prove a variation of the classical composition theorem due to Szegö.

Key words: Coincidence theorem, zeros of polynomial, critical points of a polynomial, apolar polynomials

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Let $a(z) = \sum_{k=0}^{n} a_k z^k$ and $b(z) = \sum_{k=0}^{n} b_k z^k$ be two complex polynomials of degree n. For them we can define linear operator $A(a,b) = \sum_{k=0}^{n} (-1)^k \dfrac{a_k b_{n-k}}{\binom{n}{k}}$.

If $A(a,b) = 0$, then a and b are said to be apolar polynomials. For apolar polynomials holds the classical theorem due to Grace:

Theorem of Grace. *If all zeros of a polynomial $a(z)$ are contained in some circular region S, then at least one zero of $b(z)$ is contained in S, provided that $a(z)$ and $b(z)$ are apolar.*

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Circular region is (open or closed) disk or half-plane or their exterior. Some generalizations of the theorem of Grace can be found in [1] and [2].

One consequence of this theorem is a classical Coincidence theorem due to Walsh:

Coincidence theorem. *Let $p(z_1, z_2, \ldots, z_n)$ be a symmetric complex polynomial of total degree n, and of degree 1 in each z_i. Suppose that $w_1, w_2, \ldots, w_n$ are complex numbers that are contained in some circular region S. Then there exists $z \in S$ such that $p(w_1, w_2, \ldots, w_n) = p(z, z, \ldots, z)$.*

In his papers [3], [4] and [5] Aziz showed that the above theorem holds if total degree is less then n, provided that the mentioned circular region is convex. In general, Coincidence theorem is not true is S is exterior of a disk and total degree is less then n. For example, if we set $p(z_1, z_2) = z_1 + z_2$, and S is closed exterior of the unit disk, then the equation $0 = p(-1, 1) = 2z$ has no solution in S.

We are now going to improve the above result of Aziz and also extend it to the exterior of a disk.

Theorem 1. *Let $p(z_1, z_2, \ldots, z_n)$ be a symmetric complex polynomial of total degree $m \leq n$, and of degree 1 in all z_i. Suppose now that $w_1, w_2, \ldots, w_n$ are complex numbers, such that zeros of $q(z)^{(n-m)}$ are contained in some circular region S, where $q(z) = \prod_{i=1}^{n}(z - w_i)$. Then equation $p(w_1, w_2, \ldots, w_n) = p(z, z, \ldots, z)$ has a solution in S. We shall assume that $m < n$, since for $n = m$ the theorem is reduced to the original theorem of Walsh.*

Proof. We can assume that $p(w_1, w_2, \ldots, w_n) = 0$. Otherwise we can replace $p(z_1, z_2, \ldots, z_n)$ with $p(z_1, z_2, \ldots, z_n) - p(w_1, w_2, \ldots, w_n)$. By a well-known representation theorem, polynomial p can be represented as a linear combination of e_k, where e_k is an elementary symmetric polynomial in $z_1, z_2, \ldots, z_n$ of degree k ($e_0 = 1$). Hence, $p(z_1, z_2, \ldots, z_n) = \sum_{k=0}^{m} E_k e_k$, for some complex constants E_k. We can also write $q(z) = \prod_{k=1}^{n}(z - w_k) = \sum_{k=0}^{n}(-1)^{n-k} e_{n-k} z^k$, and so $q^{(i)}(z) = \sum_{k=0}^{n-i}(i + k) \cdots (1 + k)(-1)^{n-k-i} e_{n-k-i} z^k$. It can be easily checked that

$$p(z_1, z_2, \ldots, z_n) = \sum_{k=0}^{m} E_k e_k = \frac{1}{n(n-1)\cdots(m+1)} \sum_{k=0}^{m} \frac{E_k \binom{n}{k}(n-k)\cdots(m-k+1)e_k}{\binom{m}{k}}.$$

Then,

$$p(w_1, w_2, \ldots, w_n) = \frac{1}{n(n-1)\cdots(m+1)} \sum_{k=0}^{m} \frac{E_k \binom{n}{k}(n-k)\cdots(m-k+1)e_k}{\binom{m}{k}} = 0$$

is equivalent to $A(r, q^{(n-m)}) = 0$, where $r(z) = \sum_{k=0}^{m} E_k \binom{n}{k} z^k = p(z, z, \ldots, z)$. That means that $q^{(n-m)}$ and $p(z, z, \ldots, z)$ are apolar. Hence, equation $p(z, z, \ldots, z) = 0$ has a solution in S, and the theorem is proved. □

Let us note that in case when S is convex, Theorem 1 is stronger (in general) than the mentioned results of Aziz, since any convex set that contains zeros of some polynomial contains also its critical points due to Gauss–Lucas theorem.

Our next result is an application of Theorem 1. It generalizes the result given in [6].

Theorem 2. *Let $p(z)$ be a complex polynomial of degree n. Suppose that some closed disk D contains $n-1$ zeros of $p(z)$, such that the centre of D is also the arithmetical mean of these zeros. Then disk D contains at least $\left[\frac{n-2k+1}{2}\right]$ zeros of the k-th derivative of $p(z)$, where $[\ ]$ denotes integer part.*

Proof. Let $z_1, z_2, \ldots, z_n$ be zeros of $p(z)$ such that $z_1, z_2, \ldots, z_n$ are contained in a disk D, and let c be the centre of D, $c = \frac{1}{n-1}\sum_{k=1}^{n-1} z_k$. We can assume that z_n is outside D, otherwise our theorem follows immediately from Gauss–Lucas theorem. Due to suitable rotation and translation, we can also assume that $z_n = 0$ and c is real and positive.

If we find k-th derivative of $p(z) = \prod_{k=1}^{n}(z - z_k)$ it will be a sum of products of the form $(z - z_{l_1})\cdots(z - z_{l_{n-k}})$. We will group in one sum those products in which $z_n = 0$ occurs, and the rest of them in the other sum, i.e.:

$$p^{(k)}(z) = z\Sigma_1 + \Sigma_2.$$

Polynomial $p^{(k)}(z)$ can be viewed as a polynomial in $z_1, z_2, \ldots, z_{n-1}$ of total degree $n-k$ that satisfies conditions of Theorem 1. If we set $p^{(k)}(z) = z\sum_1 + \sum_2 = 0$, then by Theorem 1 exists $y \in D$, such that

$$\begin{aligned} & p^{(k)}(z, z_1, z_2, \cdots, z_{n-1}) \\ & = p^{(k)}(z, y, y, \cdots, y) \\ & = (n-k)!\binom{n-1}{n-k-1} z(z-y)^{n-k-1} + (n-k)!\binom{n-1}{n-k}(z-y)^{n-k} = 0 \end{aligned}$$

and this is equivalent to $(z-y)^{n-k-1}(z-\frac{k}{n}y) = 0$. Hence, it follows that $p^{(k)}(z) = 0$ implies that either $z = y$(i.e. $z \in D$) or $z = \frac{k}{n}y$ for some $y \in D$. Now, let $w_1, w_2, \ldots, w_{n-k}$ be all zeros of $p^{(k)}(z)$. We can arrange these points such that first m of them are in D, while all other points are outside D. So, all w_i for $i > m$, are of the form $w_i = \frac{k}{n}y_i$ for some $y_i \in D$. Arithmetical means of zeros of $p(z)$ and $p^{(k)}(z)$ are equal, so $\frac{1}{n}\sum_{i=0}^{n} z_i = \frac{1}{n-k}\sum_{i=0}^{n-k} w_i$, i.e.

$$\frac{(n-1)c}{n} = \frac{1}{n-k}\sum_{i=0}^{n-k} w_i = \frac{1}{n-k}\left(\sum_{i=0}^{m} w_i + \frac{k}{n}\sum_{i=m+1}^{n-k} y_i\right),$$

for some $y_i \in D$. All w_i, $i \leq m$, and y_i in the above equation lie in D. So their real parts are less then $2c$. Therefore, if we take real parts of both sides in the above equation, we obtain the following inequality:

$$\frac{(n-1)c}{n} < \frac{2c}{n-k}(m + \frac{k}{n}(n-k-m))$$

and this is equivalent to $\frac{n-2k-1}{2} < m$ i.e. $\left[\frac{n-2k+1}{2}\right] \leq m$, and the proof is completed. $\square$

Now we want to prove another consequence of Theorem 1. It is a variation of the classical composition theorem due to Szegö. The original form of that theorem can be found in [7].

Theorem 3. *Let* $A(z) = \sum_{k=0}^{n} a_z z^k$, $B(z) = \sum_{k=0}^{m} b_k z^k$ *be two monic complex polynomials of degree* n *and* m*, respectively,* $m \leq n$*. Suppose that zeros of* $A^{(n-m)}(z)$ *lie in some circular region* S *that do not contain zero. Then any zero* c *of the polynomial*

$$C(z) = \sum_{k=0}^{m} \frac{a_{n-m+k}b_k}{\binom{n}{m-k}} z^k = \frac{a_{n-m}b_0}{\binom{n}{m}} + \frac{a_{n-m+1}b_1 z}{\binom{n}{m-1}} + \cdots + \frac{a_n b_m z^m}{\binom{n}{0}}$$

is of the form $c = -ab$*, where* $a \in S$ *and* b *is a zero of* $B(z)$*, provided* $a_{n-m}b_0 \neq 0$*.*

Proof. Let c be a zero of $C(z)$, i.e. $C(c) = \sum_{k=0}^{m} \frac{a_{n-m+k}b_k}{\binom{n}{m-k}} c^k = 0$. Coefficients $a_n, a_{n-1}, \ldots, a_{n-m}$ are symmetric functions on zeros of polynomial $A(z)$. Since they are of degree 1 in respect to every of these zeros, condition $a_{n-m}b_0 \neq 0$

implies that $C(c)$ is of total order m. Therefore we can apply Theorem 1 to $C(c)$. We obtain that

$$C(c) = \sum_{k=0}^{m} b_k (-1)^k a^{m-k} c^k = 0$$

for some $a \in S, a \neq 0$. So we can write $C(c) = a^m \sum_{k=0}^{m} b_k \left(-\frac{c}{a}\right)^k = 0$. That means that $-\frac{c}{a} = b$ is a zero of $B(z)$, i.e. $c = -ab$ and the theorem is proved. $\square$

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