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SIMULATION ON A GENERALIZED OSCILLATOR MODEL: WEB-BASED APPLICATION

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Abstract

In this paper, we propose new generalized oscillator model. Considerations in the light of the Melnikov's approach are also given. We focus on some interesting simulations based on the proposed new model. This is an integral part of a planned much more general Web-based application for scientific computing.

Key words: generalized oscillator, Melnikov's approach, diagram factor, web-based application

1. Introduction. A number of authors devote their research to the classical differential model:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\sin x + \epsilon F(x,y)y, \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$. The publications on this topic are significant and varied. The reader can find detailed information in the magnificent studies of Tricomi [1], Stoker [2], Levi, Hoppensteadt and Miranker [3], Guckenheimer and

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Holmes [4], and Perko [5]. In [6] Gavrilov and Iliev compute the cyclicity of open period annuli of the following generalized Rayleigh–Lienard planar system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -ax - bx^3 + \epsilon F(x,y)y \end{cases} \tag{2}$$

with

$$F(x,y) = \lambda_1 + \lambda_2 x^2 + \lambda_3 y^2 + \lambda_4 x^4 + \lambda_5 y^4 + \lambda_6 x^6, \tag{3}$$

where $0 \leq \epsilon < 1$ and λ_i are real parameters. For other hypothetical oscillator models see [7]. Obviously, these studies can be successfully continued. In this article, we offer a general summary of model (1) using "correction" of the type (3). Considerations in the light of Melnikov [8] are also given. Program modules have been generated that can be included in our envisioned upgrade to a Web-based application (for details see [9–13]).

The plan of the paper is as follows. We state our model in Section 2. Our results in the light of the Melnikov's approach can be found in Section 3. Some simulations are presented in Section 4. We conclude by Section 5.

2. The model. We consider the following new generalized model

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\sin x + \epsilon \left(a_1 + \displaystyle\sum_{i=0}^{\frac{n}{2}-1} a_{n-2i} x^{n-2i} + \sum_{i=0}^{\frac{n}{2}-2} b_{n-2-2i} y^{n-2-2i} \right) y, \end{cases} \tag{4}$$

where $0 \leq \epsilon < 1$ and n is even. The total energy of this system ($\epsilon = 0$) is $H(x,y) = \frac{1}{2}y^2 + 1 - \cos x$. Some details can be found in monograph [4]. The orbit is given by (see Fig. 1): $x_0(t) = \pm 2 \arctan(\sinh t)$; $y_0(t) = \pm 2 \operatorname{sech} t$.

3. Results in the light of the Melnikov's approach. The Melnikov's method gives us an analytic tool for establishing the existence of transfer homoclinic points of the Poincare map for a periodic orbit of a perturbed dynamical system.

3.1. The case n = 6. The model has the form

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\sin x + \epsilon \left(a_1 + a_6 x^6 + a_4 x^4 + a_2 x^2 + b_4 y^4 + b_2 y^2 \right) y, \end{cases} \tag{5}$$

where $0 \leq \epsilon < 1$. We can fix without community restriction $a_2 = a_4 = a_6 = b_2 = 1$. On the upper orbit, a time-independent Melnikov function which depends on the

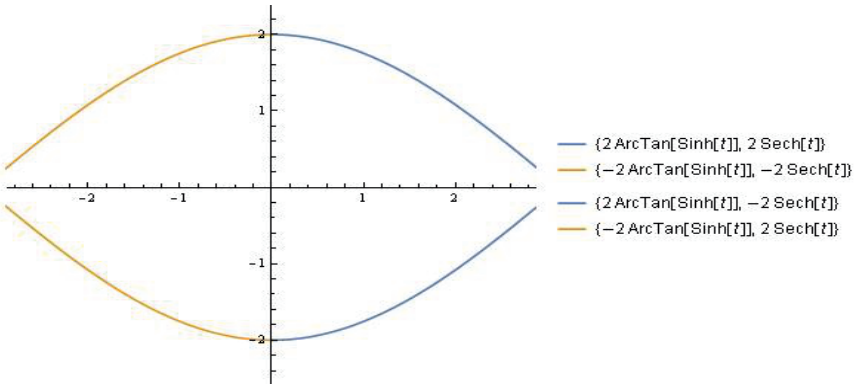


Fig. 1. The orbits $(x_0(t), y_0(t))$

two parameters: a_1 and b_4 is given by

$$M^+(a_1, b_4) = \int_{-\infty}^{\infty} y_0(t) \left(a_1 + x_0^6(t) + x_0^4(t) + x_0^2(t) + b_4 y_0^4(t) + y_0^2(t)\right) y_0(t)\, dt.$$

Studying the dynamics of model 5 is also related to the task of finding the zeros of the function $M(a_1, b_4)$. The following proposition stands.

Proposition 1. *Let $n = 6$. Then the Melnikov function $M^+(a_1, b_4)$ has a zero when the parameters a_1 and b_4 satisfy the equation*

$$-\frac{137\,104}{3} + a_1 + \frac{128}{15} b_4 + \pi^2(5713 - 119\pi^2 + \pi^4) = 0.$$

Proof. For the Melnikov function $M^+(a_1, b_4)$ we have

$$\begin{aligned} M^+(a_1, b_4) = & \int_{-\infty}^{\infty} (2\operatorname{sech} t)^2 \left(a_1 + (2\arctan(\sinh t))^2 + (2\arctan(\sinh t))^4\right. \\ & \left. + (2\arctan(\sinh t))^6 + (2\operatorname{sech} t)^2 + b_4 (2\operatorname{sech} t)^4\right)\, dt \\ = & -\tfrac{1\,096\,832}{3} + 8a_1 + \tfrac{1024}{15} b_4 + 45704\pi^2 - 952\pi^4 + 8\pi^6 \\ = & \tfrac{1}{8}\left(-\tfrac{137\,104}{3} + a_1 + \tfrac{128}{15} b_4 + \pi^2(5713 - 119\pi^2 + \pi^4)\right). \end{aligned}$$

This completes the proof of Proposition 1. □

Remark 1. The function $M^-(a_1, b_4)$ is calculated in an analogous way and will be omitted here.

3.2. The case n = 8. The model has the form

(6)
$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\sin x + \epsilon \left(a_1 + a_8 x^8 + a_6 x^6 + a_4 x^4 + a_2 x^2 + b_6 y^6 + b_4 y^4 + b_2 y^2\right) y, \end{cases}$$

where $0 \leq \epsilon < 1$. We can fix without community restriction $a_8 = a_6 = a_4 = a_2 = b_2 = b_4 = 1$. The following proposition stands.

Proposition 2. *Let $n = 8$. Then the Melnikov function $M^+(a_1, b_6)$ has a zero when the parameters a_1 and b_6 satisfy the equation*

$$\frac{51\,381\,136}{5} + a_1 + \frac{1024}{35}b_6 - \pi^2(1\,284\,527 - 26\,761\pi^2 + 223\pi^4 - \pi^6) = 0.$$

Proof. For the Melnikov function $M^+(a_1, b_6)$ we have

$$\begin{aligned} M^+(a_1, b_6) = & \int_{-\infty}^{\infty} (2\operatorname{sech} t)^2 \left(a_1 + (2\arctan(\sinh t))^2 + (2\arctan(\sinh t))^4\right. \\ & +(2\arctan(\sinh t))^6 + (2\arctan(\sinh t))^8 + (2\operatorname{sech} t)^2 \\ & \left. +(2\operatorname{sech} t)^4 + b_6(2\operatorname{sech} t)^6\right)\, dt \\ = & \tfrac{411\,049\,088}{5} + 8a_1 + \tfrac{8192}{35}b_6 - 10\,276\,216\pi^2 + 214\,088\pi^4 - 1784\pi^6 + 8\pi^8 \\ = & \tfrac{1}{8}\left(\tfrac{51\,381\,136}{5} + a_1 + \tfrac{1024}{35}b_6 - \pi^2(1\,284\,527 - 26\,761\pi^2 + 223\pi^4 - \pi^6)\right). \end{aligned}$$

This completes the proof of Proposition 2. □

4. Some simulations. Here we will focus on some interesting simulations.

Example 1. For given $n = 6$, $a_1 = 35$, $b_4 = 15$, $\epsilon = 0.0001$, $a_2 = a_4 = a_6 = b_2 = 1$ the simulations on the system (5) for $x_0 = 0.99$, $y_0 = 0.1$ are depicted in Fig. 2a.

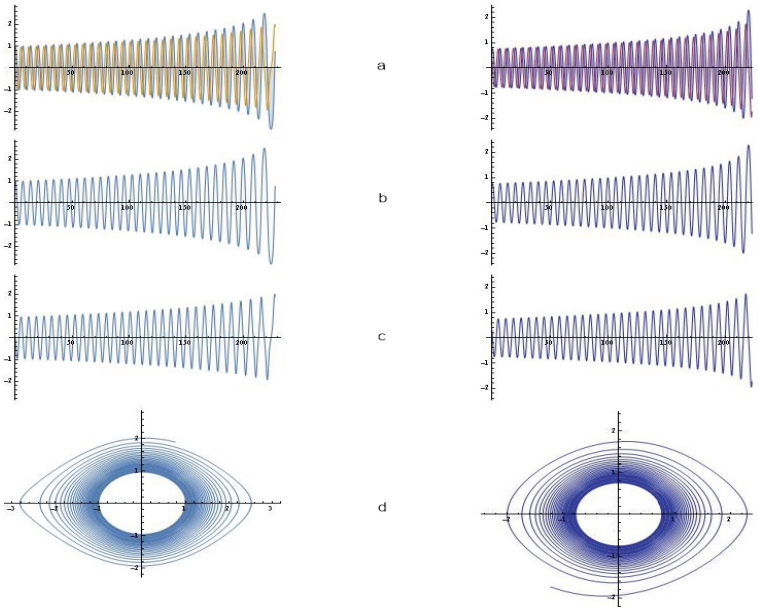


Fig. 2. Example 1 (left) and Example 2 (right): a) solutions of the system (5); b) x-time series; c) y-time series; d) phase space

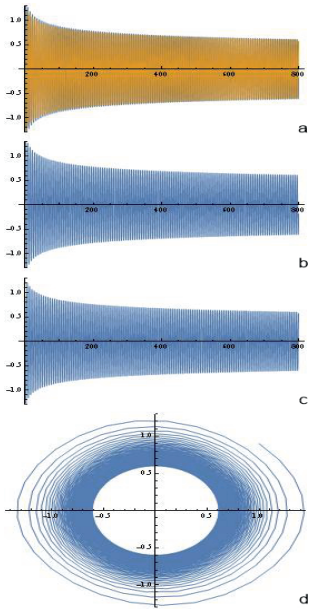


Fig. 3. Example 3: a) solutions of the system (6);
b) x-time series; c) y-time series; d) phase space

Example 2. For given $n = 8$, $a_1 = 45$, $b_6 = 25$, $\epsilon = 0.0001$, $a_8 = a_6 = a_4 = a_2 = b_2 = b_4 = 1$ the simulations on the system (6) for $x_0 = 0.7$ and $y_0 = 0.3$ are depicted in Fig. 2b.

Example 3. For given $n = 8$, $a_1 = 0.5$, $a_2 = -11.5$, $\epsilon = 0.00005$, $a_4 = 0.8$, $a_6 = -20.2$, $a_8 = 1.7$, $b_2 = 0.01$, $b_4 = -0.5$, $b_6 = -300$ the simulations on the system (6) for $x_0 = 1$ and $y_0 = 0.9$ are depicted in Fig. 3.

5. Concluding remarks. From a numerical point of view, the task of finding the roots of $M(a_1, b_4)$ (the case $n = 6$) and $M(a_1, b_6)$ (the case $n = 8$) is more interesting given that the parameters appearing in the proposed differential model are subject to a number of restrictions (of a physical and practical nature). If the user is interested for example, in the root of the equation $M^+(a_1, a_2, a_4, a_6, b_2, b_4) = 0$ (for fixed $n = 6$) then using the indicated technique we obtain

$$M^+(a_1, a_2, a_4, a_6, b_2, b_4) = 8a_1 + \frac{64}{3}b_2 + \frac{1024}{15}b_4 + 8a_2(-8 + \pi^2) + 8a_4(384 - 48\pi^2 + \pi^4) + 8a_6(-46\,080 + 5760\pi^2 - 120\pi^4 + \pi^6).$$

The reader can consider the corresponding approximation problem for arbitrarily chosen n. Some specialized modules for investigating the dynamics of the new oscillator model have been demonstrated. They will be an integral part of the mentioned above Web-based application for scientific computing. Let us note that the

theoretical apparatus for studying the circuit implementation (design, fabricating, etc.) of the considered differential model is extremely complex and requires a serious investigation before being adapted for its possible inclusion in our planned Web platform. Define the normalized diagram factor as $|y(b\cos\theta + c)|/N$. We note that the use of the solution component $y(\theta)$ of the corresponding planar differential system as a diagram factor is very complicated. With fixed parameters from Example 3, the diagram factors in the interval $[0, 2\pi]$ for

a) $a = 0.4$, $b = 0.12$, $x_0 = 1$, $y_0 = 0.9$;

b) $a = 0.4$, $b = 0.12$, $x_0 = 0.8$, $y_0 = 0.33$;

c) $a = 0.4$, $b = 0.12$, $x_0 = 0.6$, $y_0 = 0.1$;

d) $a = 0.4$, $b = 0.12$, $x_0 = 0.01$, $y_0 = 0.01$

are depicted in Fig. 4.

Some of our previous research on this issue encouraged us to begin developing specialized modules as part of a much more general Web-based application, with the ability to upgrade and include additional modules that existing platforms (with paid and free access) do not provide to users (for example, modules for automatic generation of theorems for the number and type of limit cycles (in the light of Melnikov's considerations), generation of radiation diagrams, etc.). Where

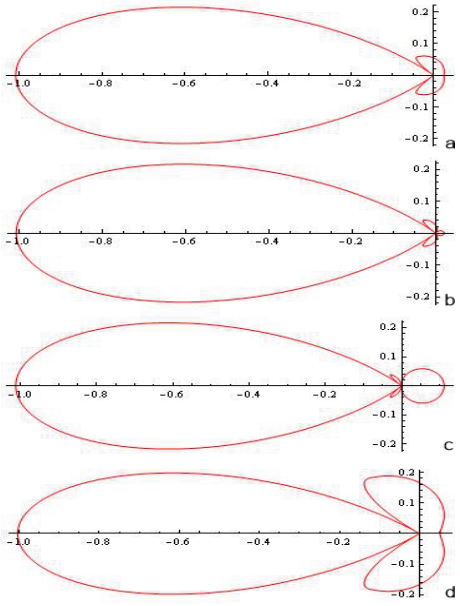


Fig. 4. The diagram factor $F(\theta)$

possible, we employ various optimization techniques for high performance calculations, including multi-processor and multi-threading calculations, and hardware intrinsics [14, 15]. Some of the algorithms used in this article:

- algorithm for generating Melnikov's functions;
- specialized algorithm for detailed Hamiltonian study of Rayleigh–Lienard systems and visualization of "level curves" (assuming implementation of software tools in a user-selected computer algebraic system for scientific calculations);
- algorithm for matching the initial approximations when solving the differential system, given its interesting specificity and behaviour of the solution in confidential time intervals;
- algorithm for control and visualization of the "diagram factor".

The research can continue for e-learning and test theory: evolutionary development from CBT to e-learning [16], learning environment [18], DeLC educational portal [17], Virtual Educational Space [20], generation of test questions [19].

The results of Propositions 1 and 2 reveal the possibility of formulating the classical Melnikov's criterion for occurring chaos in the considered differential model. The obtained theoretical results are of great importance for studying the circuit implementation (design, fabricating, etc.) and are intended for specialists working in the field of engineering sciences.

Specialists working in this scientific direction have the floor.

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