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FUNDAMENTAL THEOREMS FOR TIMELIKE SURFACES IN THE MINKOWSKI 4-SPACE

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Abstract

In the present paper, we study timelike surfaces free of minimal points in the four-dimensional Minkowski space. For each such surface we introduce a geometrically determined pseudo-orthonormal frame field and writing the derivative formulas with respect to this moving frame field and using the integrability conditions, we obtain a system of six functions satisfying some natural conditions. In the general case, we prove a Fundamental Bonnet-type theorem (existence and uniqueness theorem) stating that these six functions, satisfying the natural conditions, determine the surface up to a motion. In some particular cases, we reduce the number of functions and give the fundamental theorems.

Key words: timelike surfaces, fundamental theorems

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1. Introduction. In the local theory of surfaces in Euclidean and pseudo-Euclidean spaces, one of the basic problems is to determine the surface by a system of some functions satisfying some differential equations. This is the fundamental Bonnet-type theorem giving the natural conditions under which the surface is determined up to a motion. In the Euclidean space $\mathbb{R}^4$, the general fundamental theorem states that each surface free of minimal points is determined up to a motion in $\mathbb{R}^4$ by eight invariant functions satisfying some natural conditions

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(differential equations) [1]. For the class of minimal surfaces in $\mathbb{R}^4$, the number of the invariant functions and the number of the differential equations determining the surfaces are reduced to two [2]. The surfaces with parallel normalized mean curvature vector field are determined uniquely up to a motion by three invariant functions satisfying a system of three partial differential equations [3].

Similar results hold for spacelike surfaces in the Minkowski 4-space $\mathbb{R}^4_1$. The local theory of spacelike surfaces in $\mathbb{R}^4_1$ whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector, is developed in [4]. This class of surfaces is determined up to a motion in $\mathbb{R}^4_1$ by eight invariant functions satisfying some natural conditions. Spacelike surfaces in $\mathbb{R}^4_1$ whose mean curvature vector at any point is a lightlike vector are called marginally trapped surfaces. These surfaces were defined by Penrose [5] in order to study the global properties of spacetime and play an important role in the theory of cosmic black holes. Recently, marginally trapped surfaces satisfying some extra conditions have been studied intensively from a mathematical viewpoint [6–8], etc. The invariant theory of marginally trapped surfaces is developed in [9], where it is proved that the marginally trapped surfaces in $\mathbb{R}^4_1$ are determined up to a motion by seven invariant functions. Maximal spacelike and minimal timelike surfaces in $\mathbb{R}^4_1$ are studied in [10] and [11], respectively, and it is proved that their geometry is determined by two invariant functions, satisfying a system of two partial differential equations.

In the present paper, we study timelike surfaces free of minimal points in the Minkowski 4-space $\mathbb{R}^4_1$. For each such surface we introduce a pseudo-orthonormal frame field $\{x, y, n_1, n_2\}$, which is geometrically determined by the two lightlike directions in the tangent space of the surface and the mean curvature vector field H. We call this pseudo-orthonormal frame field a *geometric frame field* of the surface. Writing the derivative formulas with respect to the geometric frame field and using the integrability conditions, we obtain a system of six functions satisfying some differential equations (natural conditions). We prove a Fundamental Bonnet-type theorem stating that, in the general case, these six functions, satisfying the natural conditions, determine the surface up to a motion in $\mathbb{R}^4_1$ (Theorem 3.2). Then we give the fundamental theorems in some particular cases (Theorem 3.3 and Theorem 3.4).

2. Preliminaries. Let $\mathbb{R}^4_1$ be the four-dimensional Minkowski space endowed with the standard flat metric $\langle\cdot,\cdot\rangle$ of signature $(3,1)$ given in local coordinates by $dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$. The considerations in this paper are local and all functions are supposed to be of class C^∞.

Let $\mathcal{M}^2 = (\mathcal{D}, z)$ be a surface in $\mathbb{R}^4_1$, where $\mathcal{D} \subset \mathbb{R}^2$ and $z\colon \mathcal{D} \to \mathbb{R}^4_1$ is an immersion, i.e. $\mathcal{M}^2$ is locally parameterized by $\mathcal{M}^2\colon z = z(u,v)$, $(u,v) \in \mathcal{D}$.

A surface $\mathcal{M}^2$ in $\mathbb{R}^4_1$ is said to be:

- *timelike*, if the restriction of $\langle\cdot,\cdot\rangle$ to each tangent space of $\mathcal{M}^2$ is indefinite;
- *spacelike*, if the restriction of $\langle\cdot,\cdot\rangle$ to each tangent space of $\mathcal{M}^2$ is positive definite;

- *lightlike*, if the restriction of $\langle\cdot,\cdot\rangle$ to each tangent space of $\mathcal{M}^2$ is degenerate.

This paper is devoted to the study of timelike surfaces in $\mathbb{R}^4_1$, so $\langle\cdot,\cdot\rangle$ induces a Lorentzian metric g on $\mathcal{M}^2$.

We denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections on $\mathbb{R}^4_1$ and $\mathcal{M}^2$, respectively. So, we have the following formulas of Gauss and Weingarten

$$\widetilde{\nabla}_x y = \nabla_x y + \sigma(x,y),$$
$$\widetilde{\nabla}_x \xi = -A_\xi x + D_x \xi,$$

where x and y are vector fields tangent to $\mathcal{M}^2$ and ξ is a normal vector field. These formulas determine the second fundamental tensor σ, the normal connection D and the shape operator A_ξ with respect to ξ. In general, A_ξ is not diagonalizable.

The mean curvature vector field H of $\mathcal{M}^2$ is defined as

$$H = \frac{1}{2}\operatorname{tr}\sigma.$$

If the mean curvature vector vanishes identically, i.e. $H = 0$, the surface is called *minimal.*

It is well known that for a timelike surface $\mathcal{M}^2$ in $\mathbb{R}^4_1$, locally there exists a coordinate system (u,v) such that the metric tensor g of $\mathcal{M}^2$ has the following form [12]:

$$g = -f^2(u,v)(du \otimes dv + dv \otimes du)$$

for some positive function $f(u,v)$. We suppose that $z = z(u,v)$, $(u,v) \in \mathcal{D}$ is such a local parametrization on $\mathcal{M}^2$. Then, the coefficients of the first fundamental form are

$$E = \langle z_u, z_u\rangle = 0, \quad F = \langle z_u, z_v\rangle = -f^2(u,v), \quad G = \langle z_v, z_v\rangle = 0,$$

where z_u and z_v denote the derivatives of the vector function $z(u,v)$, i.e. $z_u = \dfrac{\partial z}{\partial u}$, $z_v = \dfrac{\partial z}{\partial v}$. Since $\langle z_u, z_u\rangle = 0$ and $\langle z_v, z_v\rangle = 0$, the parameters (u,v) are called isotropic parameters of the surface.

Let us consider the pseudo-orthonormal tangent frame field of $\mathcal{M}^2$ defined by $x = z_u/f$, $y = z_v/f$. Obviously, $\langle x,x\rangle = 0$, $\langle x,y\rangle = -1$, $\langle y,y\rangle = 0$. Hence, the mean curvature vector field H is given by

$$H = -\sigma(x,y).$$

Since we consider surfaces free of minimal points, i.e. $H \neq 0$ at all points, we can choose a unit normal vector field n_1 which is collinear with H, i.e. $H = \nu n_1$ for a smooth function $\nu = \|H\|$. Then, $\sigma(x,y) = -\nu n_1$. We consider the unit normal vector field n_2 such that $\{n_1, n_2\}$ is an orthonormal frame field of the normal bundle (n_2 is determined up to orientation). So, we can write the following formulas for the second fundamental tensor σ:

$$\begin{aligned} \sigma(x,x) &= \lambda_1 n_1 + \mu_1 n_2, \\ \sigma(x,y) &= -\nu n_1, \\ \sigma(y,y) &= \lambda_2 n_1 + \mu_2 n_2, \end{aligned} \tag{2.1}$$

where λ_1, μ_1, λ_2, μ_2 are smooth functions defined by:

$$\lambda_1 = \langle \widetilde{\nabla}_x x, n_1 \rangle, \quad \mu_1 = \langle \widetilde{\nabla}_x x, n_2 \rangle, \quad \lambda_2 = \langle \widetilde{\nabla}_y y, n_1 \rangle, \quad \mu_2 = \langle \widetilde{\nabla}_y y, n_2 \rangle.$$

Having in mind that $x = z_u/f$, $y = z_v/f$ and using $\langle z_u, z_u \rangle = 0$, $\langle z_u, z_v \rangle = -f^2(u,v)$, $\langle z_v, z_v \rangle = 0$, after differentiation we obtain:

$$\begin{aligned} \nabla_x x &= \frac{f_u}{f^2}\, x, \\ \nabla_x y &= \qquad\quad -\frac{f_u}{f^2}\, y, \\ \nabla_y x &= -\frac{f_v}{f^2}\, x, \\ \nabla_y y &= \qquad\quad \frac{f_v}{f^2}\, y. \end{aligned} \tag{2.2}$$

Denoting $\gamma_1 = f_u/f^2 = x(\ln f)$ and $\gamma_2 = f_v/f^2 = y(\ln f)$, from (2.1) and (2.2) we obtain the following derivative formulas:

$$\begin{aligned} \widetilde{\nabla}_x x &= \gamma_1 x \qquad\quad + \lambda_1 n_1 + \mu_1 n_2, \\ \widetilde{\nabla}_x y &= \qquad -\gamma_1 y - \nu n_1, \\ \widetilde{\nabla}_y x &= -\gamma_2 x \qquad - \nu n_1, \\ \widetilde{\nabla}_y y &= \qquad\quad \gamma_2 y + \lambda_2 n_1 + \mu_2 n_2. \end{aligned} \tag{2.3}$$

For the normal frame field $\{n_1, n_2\}$ by use of (2.3) we can derive the formulas:

$$\begin{aligned} \widetilde{\nabla}_x n_1 &= -\nu x + \lambda_1 y \qquad + \beta_1 n_2, \\ \widetilde{\nabla}_y n_1 &= \lambda_2 x - \nu y \qquad + \beta_2 n_2, \\ \widetilde{\nabla}_x n_2 &= \qquad + \mu_1 y - \beta_1 n_1, \\ \widetilde{\nabla}_y n_2 &= \mu_2 x \qquad - \beta_2 n_1, \end{aligned} \tag{2.4}$$

where $\beta_1 = \langle \widetilde{\nabla}_x n_1, n_2 \rangle$ and $\beta_2 = \langle \widetilde{\nabla}_y n_1, n_2 \rangle$.

Remark 2.1. The pseudo-orthonormal frame field $\{x, y, n_1, n_2\}$ is geometrically determined: x, y are the two lightlike directions in the tangent space (determined up to notation); n_1 is the unit normal vector field collinear with the mean curvature vector field H; n_2 is determined by the condition that $\{n_1, n_2\}$ is an orthonormal frame field of the normal bundle (n_2 is determined up to a sign). We call this pseudo-orthonormal frame field $\{x, y, n_1, n_2\}$ a *geometric frame field* of the surface [13].

Formulas (2.3) and (2.4) are the derivative formulas of the surface with respect to the geometric frame field $\{x, y, n_1, n_2\}$.

The functions β_1 and β_2 characterize the class of surfaces with parallel mean curvature vector field. In [13], we proved that:

- The timelike surface $\mathcal{M}^2$ has parallel mean curvature vector field if and only if $\beta_1 = \beta_2 = 0$ and $\nu = \text{const}$.

- The timelike surface $\mathcal{M}^2$ has parallel normalized mean curvature vector field if and only if $\beta_1 = \beta_2 = 0$ and $\nu \neq \text{const}$.

In the present paper we study timelike surfaces in $\mathbb{R}^4_1$ with $\beta_1^2 + \beta_2^2 \neq 0$.

3. Fundamental theorems. Let $\mathcal{M}^2 \colon z = z(u, v)$, $(u, v) \in \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$ be a local parametrization with respect to isotropic parameters on a timelike surface $\mathcal{M}^2$ in $\mathbb{R}^4_1$ and $\{x, y, n_1, n_2\}$ be the geometric pseudo-orthonormal frame field introduced above.

Since the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{R}^4_1$ is flat, we have

$$\widetilde{R}(x, y, x) = 0, \quad \widetilde{R}(x, y, y) = 0, \quad \widetilde{R}(x, y, n_1) = 0, \quad \widetilde{R}(x, y, n_2) = 0, \tag{3.1}$$

where

$$\widetilde{R}(x, y, z) = \widetilde{\nabla}_x \widetilde{\nabla}_y z - \widetilde{\nabla}_y \widetilde{\nabla}_x z - \widetilde{\nabla}_{[x,y]} z$$

for arbitrary vector fields x, y, z. Using (3.1) and the derivative formulas (2.3) and (2.4), we obtain the following integrability conditions:

$$x(\lambda_2) + y(\nu) + 2\gamma_1 \lambda_2 - \mu_2 \beta_1 = 0, \tag{3.2}$$

$$x(\nu) + y(\lambda_1) + 2\gamma_2 \lambda_1 - \mu_1 \beta_2 = 0, \tag{3.3}$$

$$x(\mu_2) + 2\gamma_1 \mu_2 + \nu \beta_2 + \lambda_2 \beta_1 = 0, \tag{3.4}$$

$$y(\mu_1) + 2\gamma_2 \mu_1 + \nu \beta_1 + \lambda_1 \beta_2 = 0 \tag{3.5}$$

$$x(\gamma_2) + y(\gamma_1) + 2\gamma_1 \gamma_2 - \nu^2 + \lambda_1 \lambda_2 + \mu_1 \mu_2 = 0, \tag{3.6}$$

$$x(\beta_2) - y(\beta_1) + \mu_1 \lambda_2 - \lambda_1 \mu_2 + \gamma_1 \beta_2 - \gamma_2 \beta_1 = 0. \tag{3.7}$$

Remark 3.1. If we assume that both μ_1 and μ_2 are zero functions, i.e. $\mu_1(u, v) = 0$ and $\mu_2(u, v) = 0$ for all $(u, v) \in \mathcal{D}$, then the surface consists of inflection points, i.e. at each point $p \in \mathcal{M}^2$ the first normal space $\operatorname{Im} \sigma_p = \operatorname{span}\{\sigma(x, y) \colon x, y \in T_p\mathcal{M}^2\}$ is one-dimensional. In such case, the surface is developable or lies in a 3-dimensional space [14].

So, further we consider timelike surfaces in $\mathbb{R}^4_1$ that are free of inflection points, i.e. we assume that $\mu_1^2+\mu_2^2 \neq 0$ at least in a sub-domain $\mathcal{D}_0$ of $\mathcal{D}$. Without loss of generality we may assume that $\mu_1 \neq 0$. We will consider separately the two cases:

I. $\mu_1 \neq 0$ and $\mu_2 \neq 0$ in a sub-domain;

II. $\mu_1 \neq 0$ and $\mu_2 = 0$ in a sub-domain.

Case I. $\mu_1 \neq 0$ and $\mu_2 \neq 0$ in a sub-domain.

In this case we call the surface a timelike surface of *first type*.

From (3.2) and (3.3) we express the functions β_1 and β_2 as follows:

$$\beta_1 = \frac{1}{\mu_2}\Big(x(\lambda_2) + y(\nu) + 2\gamma_1\lambda_2\Big), \quad \beta_2 = \frac{1}{\mu_1}\Big(x(\nu) + y(\lambda_1) + 2\gamma_2\lambda_1\Big). \tag{3.8}$$

Having in mind that $x = \dfrac{1}{f}\dfrac{\partial}{\partial u}$, $y = \dfrac{1}{f}\dfrac{\partial}{\partial v}$ and $\gamma_1 = \dfrac{f_u}{f^2}$, $\gamma_2 = \dfrac{f_v}{f^2}$, from (3.8) we obtain

$$\beta_1 = \frac{1}{f\mu_2}\Big((\lambda_2)_u + \nu_v + \lambda_2(\ln f^2)_u\Big), \quad \beta_2 = \frac{1}{f\mu_1}\Big(\nu_u + (\lambda_1)_v + \lambda_1(\ln f^2)_v\Big).$$

Hence, the functions β_1 and β_2 are expressed by f, ν, λ_1, λ_2, μ_1, μ_2. So, we have six functions f, ν, λ_1, λ_2, μ_1, μ_2 satisfying the following four equations, which are derived from (3.4), (3.5), (3.6), and (3.7), respectively:

$$\begin{gathered}
(\lambda_2^2+\mu_2^2)_u + (\ln f^4)_u(\lambda_2^2+\mu_2^2) + 2\lambda_2\nu_v + \frac{2\nu\mu_2}{\mu_1}\Big(\nu_u + (\lambda_1)_v + \lambda_1(\ln f^2)_v\Big) = 0,\\
(\lambda_1^2+\mu_1^2)_v + (\ln f^4)_v(\lambda_1^2+\mu_1^2) + 2\lambda_1\nu_u + \frac{2\nu\mu_1}{\mu_2}\Big((\lambda_2)_u + \nu_v + \lambda_2(\ln f^2)_u\Big) = 0,\\
\frac{2ff_{uv} - 2f_uf_v}{f^4} + \lambda_1\lambda_2 + \mu_1\mu_2 - \nu^2 = 0,\\
(\mu_1\lambda_2 - \mu_2\lambda_1)\left(f^2\mu_1\mu_2 - (\ln f^2)_{uv}\right) + \nu_{uu}\mu_2 - \nu_{vv}\mu_1 + (\lambda_1)_{uv}\mu_2 - (\lambda_2)_{uv}\mu_1\\
+ \mu_2(\lambda_1)_u(\ln f^2)_v - \mu_1(\lambda_2)_v(\ln f^2)_u - \frac{\mu_2(\mu_1)_u}{\mu_1}\left(\nu_u + (\lambda_1)_v + \lambda_1(\ln f^2)_v\right)\\
+ \frac{\mu_1(\mu_2)_v}{\mu_2}\left((\lambda_2)_u + \nu_v + \lambda_2(\ln f^2)_u\right) = 0.
\end{gathered}$$

Now we can give the fundamental theorem:

Theorem 3.2. *Let $f(u,v) > 0$, $\nu(u,v)$, $\lambda_1(u,v)$, $\mu_1(u,v)$, $\lambda_2(u,v)$, $\mu_2(u,v)$, $\mu_1\mu_2 \neq 0$ be six smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying* *the conditions*

(3.9)

(i) $(\lambda_2^2+\mu_2^2)_u+(\ln f^4)_u(\lambda_2^2+\mu_2^2)+2\lambda_2\nu_v+\dfrac{2\nu\mu_2}{\mu_1}\big(\nu_u+(\lambda_1)_v+\lambda_1(\ln f^2)_v\big)=0,$

(ii) $(\lambda_1^2+\mu_1^2)_v+(\ln f^4)_v(\lambda_1^2+\mu_1^2)+2\lambda_1\nu_u+\dfrac{2\nu\mu_1}{\mu_2}\big((\lambda_2)_u+\nu_v+\lambda_2(\ln f^2)_u\big)=0,$

(iii) $\dfrac{2ff_{uv}-2f_uf_v}{f^4}+\lambda_1\lambda_2+\mu_1\mu_2-\nu^2=0,$

(iv) $(\mu_1\lambda_2-\mu_2\lambda_1)\big(f^2\mu_1\mu_2-(\ln f^2)_{uv}\big)+\nu_{uu}\mu_2-\nu_{vv}\mu_1+(\lambda_1)_{uv}\mu_2-(\lambda_2)_{uv}\mu_1$
$+\mu_2(\lambda_1)_u(\ln f^2)_v-\mu_1(\lambda_2)_v(\ln f^2)_u-\dfrac{\mu_2(\mu_1)_u}{\mu_1}\big(\nu_u+(\lambda_1)_v+\lambda_1(\ln f^2)_v\big)$
$+\dfrac{\mu_1(\mu_2)_v}{\mu_2}\big((\lambda_2)_u+\nu_v+\lambda_2(\ln f^2)_u\big)=0.$

If $\{x_0,y_0,(n_1)_0,(n_2)_0\}$ *is a pseudo-orthonormal frame at a point* $p_0\in\mathbb{R}^4_1$*, then there exists a subdomain* $\mathcal{D}_0\subset\mathcal{D}$ *and a unique timelike surface of first type* $\mathcal{M}^2\colon z=z(u,v)$*,* $(u,v)\in\mathcal{D}_0$*, parameterized by isotropic parameters, such that* $\mathcal{M}^2$ *passes through* p_0*, and* $\{x_0,y_0,(n_1)_0,(n_2)_0\}$ *is the geometric frame of* $\mathcal{M}^2$ *at* p_0*.*

Proof. Using the given functions we define $\gamma_1=f_u/f^2$, $\gamma_2=f_v/f^2$, $\beta_1=\dfrac{(\lambda_2)_u+\nu_v+\lambda_2(\ln f^2)_u}{f\mu_2}$, $\beta_2=\dfrac{\nu_u+(\lambda_1)_v+\lambda_1(\ln f^2)_v}{f\mu_1}$ and consider the following system of partial differential equations for the unknown vector functions $x=x(u,v)$, $y=y(u,v)$, $n_1=n_1(u,v)$, $n_2=n_2(u,v)$ in $\mathbb{R}^4_1$:

$$\begin{array}{ll} x_u=f(\gamma_1x+\lambda_1n_1+\mu_1n_2) & x_v=f(-\gamma_2x-\nu n_1)\\ y_u=f(-\gamma_1y-\nu n_1) & y_v=f(\gamma_2y+\lambda_2n_1+\mu_2n_2)\\ (n_1)_u=f(-\nu x+\lambda_1y+\beta_1n_2) & (n_1)_v=f(\lambda_2x-\nu y+\beta_2n_2)\\ (n_2)_u=f(\mu_1y-\beta_1n_1) & (n_2)_v=f(\mu_2x-\beta_2n_1). \end{array}\tag{3.10}$$

For convenience, we denote

$$\mathcal{W}=\begin{pmatrix}x\\y\\n_1\\n_2\end{pmatrix},\quad \mathcal{A}=f\begin{pmatrix}\gamma_1&0&\lambda_1&\mu_1\\0&-\gamma_1&-\nu&0\\-\nu&\lambda_1&0&\beta_1\\0&\mu_1&-\beta_1&0\end{pmatrix},\quad \mathcal{B}=f\begin{pmatrix}-\gamma_2&0&-\nu&0\\0&\gamma_2&\lambda_2&\mu_2\\\lambda_2&-\nu&0&\beta_2\\\mu_2&0&-\beta_2&0\end{pmatrix}.$$

Then, system (3.10) can be written in the following matrix form:

$$\begin{array}{l}\mathcal{W}_u=\mathcal{A}\,\mathcal{W},\\ \mathcal{W}_v=\mathcal{B}\,\mathcal{W}.\end{array}\tag{3.11}$$

The integrability conditions of system (3.11) are $\mathcal{W}_{uv}=\mathcal{W}_{vu}$, i.e.

$$\frac{\partial a_i^k}{\partial v}-\frac{\partial b_i^k}{\partial u}+\sum_{j=1}^{4}(a_i^j\,b_j^k-b_i^j\,a_j^k)=0,\quad i,k=1,\dots,4,\tag{3.12}$$

where by a_i^j and b_i^j $(i,j = 1,\dots,4)$ we denote the elements of the matrices $\mathcal{A}$ and $\mathcal{B}$, respectively. Taking into consideration the conditions given in (3.9), one can check that equalities (3.12) are fulfilled. Hence, there exists a subdomain $\mathcal{D}_1 \subset \mathcal{D}$ and unique vector functions $x = x(u,v)$, $y = y(u,v)$, $n_1 = n_1(u,v)$, $n_2 = n_2(u,v)$, $(u,v) \in \mathcal{D}_1$, which satisfy system (3.10) and the initial conditions

$$x(u_0,v_0) = x_0, \quad y(u_0,v_0) = y_0, \quad n_1(u_0,v_0) = (n_1)_0, \quad n_2(u_0,v_0) = (n_2)_0.$$

In order to prove that the vector functions $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ form a pseudo-orthonormal frame in $\mathbb{R}^4_1$ for each $(u,v) \in \mathcal{D}_1$, we consider the functions:

$$\begin{array}{lllll} h_1 = \langle x,x\rangle, & h_3 = \langle n_1,n_1\rangle - 1, & h_5 = \langle x,y\rangle + 1, & h_7 = \langle x,n_2\rangle, & h_9 = \langle y,n_2\rangle, \\ h_2 = \langle y,y\rangle, & h_4 = \langle n_2,n_2\rangle - 1, & h_6 = \langle x,n_1\rangle, & h_8 = \langle y,n_1\rangle, & h_{10} = \langle n_1,n_2\rangle, \end{array}$$

defined for $(u,v) \in \mathcal{D}_1$. Having in mind that $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ satisfy (3.10), we obtain the system

$$\frac{\partial h_i}{\partial u} = k_i^j\, h_j, \quad \frac{\partial h_i}{\partial v} = m_i^j\, h_j; \qquad i = 1,\dots,10, \tag{3.13}$$

where k_i^j, m_i^j, $i,j = 1,\dots,10$ are functions of $(u,v) \in \mathcal{D}_1$. System (3.13) is a linear system of partial differential equations for the functions $h_i(u,v)$, satisfying the conditions $h_i(u_0,v_0) = 0$ for all $i = 1,\dots,10$, since $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ is a pseudo-orthonormal frame. Therefore, $h_i(u,v) = 0$, $i = 1,\dots,10$ for each $(u,v) \in \mathcal{D}_1$. Hence, the vector functions $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ form a pseudo-orthonormal frame in $\mathbb{E}^4_1$ for each $(u,v) \in \mathcal{D}_1$.

Now, we consider the following system of PDEs for the vector function $z(u,v)$:

$$\begin{aligned} z_u &= f\,x \\ z_v &= f\,y. \end{aligned} \tag{3.14}$$

Using (3.9) and (3.10), one can check that the integrability conditions $z_{uv} = z_{vu}$ of system (3.14) are fulfilled. Consequently, there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}_1$ and a unique vector function $z = z(u,v)$, $(u,v) \in \mathcal{D}_0$, satisfying $z(u_0,v_0) = p_0$.

Finally, we consider the surface $\mathcal{M}^2\colon z = z(u,v)$, $(u,v) \in \mathcal{D}_0$. Obviously, $\mathcal{M}^2$ is a timelike surface in $\mathbb{R}^4_1$ parameterized by isotropic parameters (u,v), since $\langle z_u, z_u\rangle = 0$, $\langle z_v, z_v\rangle = 0$, $\langle z_u, z_v\rangle = -f^2(u,v)$. □

Case II. $\mu_1 \neq 0$ and $\mu_2 = 0$ in a sub-domain.

In this case, we have the following subcases:

(a) $\lambda_2 \neq 0$;

(b) $\lambda_2 = 0$.

Subcase II (a). $\lambda_2 \neq 0$.

In this subcase we call the surface a timelike surface of *second type*.

Now, using the integrability conditions under the assumptions $\mu_2 = 0$ and $\lambda_2 \neq 0$, we obtain the following expressions for the functions β_1 and β_2:

$$\begin{aligned}
\beta_2 &= \frac{1}{\mu_1}(x(\nu) + y(\lambda_1) + 2\gamma_2\lambda_1) = \frac{1}{f\mu_1}(\nu_u + (\lambda_1)_v + (\ln f^2)_v\lambda_1),\\
\beta_1 &= -\frac{\nu\beta_2}{\lambda_2} = -\frac{\nu}{f\mu_1\lambda_2}(\nu_u + (\lambda_1)_v + (\ln f^2)_v\lambda_1)
\end{aligned}$$

and the next four equations:

$$\begin{gathered}
(\lambda_2)_u + \nu_v + (\ln f^2)_u\lambda_2 = 0,\\
\frac{2ff_{uv} - 2f_uf_v}{f^4} - (\nu^2 - \lambda_1\lambda_2) = 0,\\
(\mu_1)_v + (\ln f^2)_v\mu_1 - \frac{\nu^2 - \lambda_1\lambda_2}{\lambda_2\mu_1}(\nu_u + (\lambda_1)_v + (\ln f^2)_v\lambda_1) = 0,\\
\lambda_2\left(\nu_{uu} + (\lambda_1)_{uv} + (\lambda_1)_u(\ln f^2)_v + \lambda_1(\ln f^2)_{uv}\right)\\
+ \nu\left(\nu_{uv} + (\lambda_1)_{vv} + (\lambda_1)_v(\ln f^2)_v + \lambda_1(\ln f^2)_{vv}\right) + f^2\mu_1^2\lambda_2^2\\
- \left(\nu_u + (\lambda_1)_v + \lambda_1(\ln f^2)_v\right)\left(\lambda_2(\ln|\mu_1|)_u + \nu_v - \nu(\ln|\mu_1|)_v - \nu(\ln|\lambda_2|)_v\right) = 0.
\end{gathered}$$

Finally, we can formulate the fundamental theorem in this subcase:

Theorem 3.3. *Let $f(u,v) > 0$, $\nu(u,v)$, $\lambda_1(u,v)$, $\mu_1(u,v)$, $\lambda_2(u,v) \neq 0$, be five smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

(i) $(\lambda_2)_u + \nu_v + (\ln f^2)_u\lambda_2 = 0$;

(ii) $(\mu_1)_v + (\ln f^2)_v\mu_1 - \dfrac{\nu^2 - \lambda_1\lambda_2}{\lambda_2\mu_1}(\nu_u + (\lambda_1)_v + (\ln f^2)_v\lambda_1) = 0$;

(iii) $\dfrac{2ff_{uv} - 2f_uf_v}{f^4} - (\nu^2 - \lambda_1\lambda_2) = 0$;

(iv) $\lambda_2\left(\nu_{uu} + (\lambda_1)_{uv} + (\lambda_1)_u(\ln f^2)_v + \lambda_1(\ln f^2)_{uv}\right)$
$+\nu\left(\nu_{uv} + (\lambda_1)_{vv} + (\lambda_1)_v(\ln f^2)_v + \lambda_1(\ln f^2)_{vv}\right) + f^2\mu_1^2\lambda_2^2$
$-\left(\nu_u + (\lambda_1)_v + \lambda_1(\ln f^2)_v\right)\left(\lambda_2(\ln|\mu_1|)_u + \nu_v - \nu(\ln|\mu_1|)_v - \nu(\ln|\lambda_2|)_v\right) = 0.$

If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a pseudo-orthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface of second type $\mathcal{M}^2\colon z = z(u,v)$, $(u,v) \in \mathcal{D}_0$, parameterized by isotropic parameters, such that $\mathcal{M}^2$ passes through p_0, and $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of $\mathcal{M}^2$ at p_0.

Proof. Let us denote $\beta_1 = -\dfrac{\nu}{f\mu_1\lambda_2}(\nu_u + (\lambda_1)_v + (\ln f^2)_v\lambda_1)$, $\beta_2 = \dfrac{1}{f\mu_1}(\nu_u + (\lambda_1)_v + (\ln f^2)_v\lambda_1)$, $\gamma_1 = f_u/f^2$, $\gamma_2 = f_v/f^2$, and consider the following system of partial differential equations for the unknown vector functions $x = x(u,v)$, $y = y(u,v)$, $n_1 = n_1(u,v)$, $n_2 = n_2(u,v)$ in $\mathbb{R}^4_1$:

$$
\begin{array}{ll}
x_u = f\left(\gamma_1 x + \lambda_1 n_1 + \mu_1 n_2\right) & x_v = f\left(-\gamma_2 x - \nu n_1\right) \\
y_u = f\left(-\gamma_1 y - \nu n_1\right) & y_v = f\left(\gamma_2 y + \lambda_2 n_1\right) \\
(n_1)_u = f\left(-\nu x + \lambda_1 y + \beta_1 n_2\right) & (n_1)_v = f\left(\lambda_2 x - \nu y + \beta_2 n_2\right) \\
(n_2)_u = f\left(\mu_1 y - \beta_1 n_1\right) & (n_2)_v = f\left(-\beta_2 n_1\right).
\end{array}
\tag{3.15}
$$

We denote

$$
\mathcal{W} = \begin{pmatrix} x \\ y \\ n_1 \\ n_2 \end{pmatrix}, \quad \mathcal{A} = f \begin{pmatrix} \gamma_1 & 0 & \lambda_1 & \mu_1 \\ 0 & -\gamma_1 & -\nu & 0 \\ -\nu & \lambda_1 & 0 & \beta_1 \\ 0 & \mu_1 & -\beta_1 & 0 \end{pmatrix}, \quad \mathcal{B} = f \begin{pmatrix} -\gamma_2 & 0 & -\nu & 0 \\ 0 & \gamma_2 & \lambda_2 & 0 \\ \lambda_2 & -\nu & 0 & \beta_2 \\ 0 & 0 & -\beta_2 & 0 \end{pmatrix}
$$

and rewrite system (3.15) in the matrix form:

$$
\begin{array}{l}
\mathcal{W}_u = \mathcal{A}\,\mathcal{W}, \\
\mathcal{W}_v = \mathcal{B}\,\mathcal{W}.
\end{array}
$$

Further, the proof of the theorem follows the steps in the proof of Theorem 3.2. □

Subcase II (b). $\lambda_2 = 0$.

We call the surfaces satisfying $\mu_2 = 0$ and $\lambda_2 = 0$ timelike surfaces of *third type*.

In this subcase, under the assumption $\lambda_2 = \mu_2 = 0$ from the integrability conditions we obtain that $\nu = \nu(u)$, $\beta_2 = 0$, $\beta_1 = -\left((\mu_1)_v + \mu_1(\ln f^2)_v\right)/\nu f$ and the following three equations:

$$
\begin{array}{c}
\nu_u + (\lambda_1)_v + \lambda_1(\ln f^2)_v = 0, \\
(\mu_1)_{vv} + (\mu_1)_v(\ln f^2)_v + \mu_1(\ln f^2)_{vv} = 0, \\
\dfrac{2ff_{uv} - 2f_u f_v}{f^4} = \nu^2.
\end{array}
\tag{3.16}
$$

The last equality implies that the function ν is expressed by the function f and its derivatives as follows:

$$
\nu^2 = f^{-2}\left(\ln f^2\right)_{uv}. \tag{3.17}
$$

Hence, the function $f^{-2}\left(\ln f^2\right)_{uv}$ depends only on the parameter u, since $\nu = \nu(u)$. Using (3.17) we can write equality (3.16) in the form:

$$
(\lambda_1)_v + \lambda_1(\ln f^2)_v + \left(f^{-1}\sqrt{(\ln f^2)_{uv}}\right)_u = 0. \tag{3.18}
$$

Now, we can give the fundamental theorem in this subcase.

Theorem 3.4. *Let $f(u,v) > 0$, $\lambda_1(u,v)$ and $\mu_1(u,v)$ be three smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{aligned}
&\text{(i)}\ \left(f^{-2}\left(\ln f^2\right)_{uv}\right)_v = 0,\\
&\text{(ii)}\ (\lambda_1)_v + \lambda_1(\ln f^2)_v + \left(f^{-1}\sqrt{(\ln f^2)_{uv}}\right)_u = 0,\\
&\text{(iii)}\ (\mu_1)_{vv} + (\mu_1)_v(\ln f^2)_v + \mu_1(\ln f^2)_{vv} = 0.
\end{aligned}$$

If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a pseudo-orthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface of third type $\mathcal{M}^2\colon z = z(u,v)$, $(u,v) \in \mathcal{D}_0$, parameterized by isotropic parameters, such that $\mathcal{M}^2$ passes through p_0, and $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of $\mathcal{M}^2$ at the point p_0.

Proof. Let us denote

$$\nu = f^{-1}\sqrt{(\ln f^2)_{uv}},\ \beta_1 = -\left((\ln f^2)_{uv}\right)^{-\frac{1}{2}}\left((\mu_1)_v + \mu_1(\ln f^2)_v\right),\ \gamma_1 = \frac{f_u}{f^2},\ \gamma_2 = \frac{f_v}{f^2},$$

and consider the following system of partial differential equations for the unknown vector functions $x = x(u,v)$, $y = y(u,v)$, $n_1 = n_1(u,v)$, $n_2 = n_2(u,v)$ in $\mathbb{R}^4_1$:

$$\begin{array}{ll}
x_u = f\left(\gamma_1 x + \lambda_1 n_1 + \mu_1 n_2\right) & x_v = f\left(-\gamma_2 x - \nu n_1\right)\\
y_u = f\left(-\gamma_1 y - \nu n_1\right) & y_v = f\gamma_2 y\\
(n_1)_u = f\left(-\nu x + \lambda_1 y + \beta_1 n_2\right) & (n_1)_v = -f\nu y\\
(n_2)_u = f\left(\mu_1 y - \beta_1 n_1\right) & (n_2)_v = 0.
\end{array}$$

Further, the proof of the theorem follows the steps in the proof of Theorem 3.2. □

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