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Number theory

MERSENNE NUMBERS IN GENERALIZED LUCAS SEQUENCES

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Abstract

Let $k \geq 2$ be an integer and let $(L_n^{(k)})_{n \geq 2-k}$ be the k-generalized Lucas sequence with certain initial k terms and each term afterward is the sum of the k preceding terms. Mersenne numbers are the numbers of the form $2^a - 1$, where a is any positive integer. The aim of this paper is to determine all Mersenne numbers which lie inside k-Lucas sequences.

Key words: k-generalized Lucas sequences, k-Lucas numbers, Mersenne numbers, linear forms in logarithms, Lucas numbers

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1. Introduction. Let $k \geq 2$ be an integer. The k-generalized Lucas sequence is defined by

$$L_n^{(k)} = L_{n-1}^{(k)} + \cdots + L_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2$$

with the initial conditions $L_0^{(k)} = 2$, $L_1^{(k)} = 1$ for all $k \geq 2$ and $L_{2-k}^{(k)} = \cdots = L_{-1}^{(k)} = 0$ for all $k \geq 3$. For each value of k, $L_n^{(k)}$ produces a different sequence. For example, the first few non zero terms of this sequence for some small values of k are as follows

$$\begin{aligned}
k = 2 &: 2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322,\ \ldots \\
k = 3 &: 2,\ 1,\ 3,\ 6,\ 10,\ 19,\ 35,\ 64,\ 118,\ 217,\ 399,\ 734,\ 1350,\ \ldots \\
k = 4 &: 2,\ 1,\ 3,\ 6,\ 12,\ 22,\ 43,\ 83,\ 160,\ 308,\ 594,\ 1145,\ 2207,\ \ldots \\
k = 5 &: 2,\ 1,\ 3,\ 6,\ 12,\ 24,\ 46,\ 91,\ 179,\ 352,\ 692,\ 1360,\ 2674,\ \ldots.
\end{aligned}$$

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Especially for $k=2$, $L_n^{(2)}$ is the classical Lucas sequence and in this case we denote it just by L_n. For simplicity, we call the sequence $L_n^{(k)}$ a k-Lucas sequence.

In recent years, all k-Lucas numbers possessing some special forms have been investigated in many studies. For example all repdigits, that are the numbers consisting of only one distinct digit and concatenations of two repdigits which lie inside k-Lucas sequences have been determined in [1, 2], respectively. Also, Fermat numbers which belong to k-Lucas sequences have been investigated in [3]. In this paper, we continue to search k-Lucas sequences in this line and determine all k-Lucas numbers which are Mersenne numbers. Recall that Mersenne numbers are positive integers of the form 2^a-1. Recently, all Mersenne numbers belonging to some interesting sequences have been determined by a number of authors, see for example [4–7]. In particular, in [8], the authors found all Mersenne numbers which are k-Fibonacci numbers. Thus, this last mentioned study motivated us to search all Mersenne numbers that lie inside generalized Lucas sequences. We state our main result as the following theorem.

Theorem 1. *The only positive integer solutions of the Diophantine equation*

$$L_n^{(k)} = 2^a - 1 \tag{1.1}$$

are $L_1^{(k)} = 2^1-1=1$, $L_2^{(k)} = 2^2-1=3$ *for all* $k\geq 2$ *and* $L_4 = 2^3-1=7$.

For the proof of Theorem 1, we use some properties of the logarithmic height of an algebraic number, which is a concept that elegantly quantifies the size of an algebraic number relative to its degree and its conjugates. Matveev's Theorem, (Theorem 2), will be in the centre of our investigation. This theorem is a powerful result that offers a threshold beyond which algebraic numbers must reside in order for certain expressions to remain non-zero. As we delve deeper, we encounter an essential lemma, which is a variant of a result established by Dujella and Pethő [9]. This lemma bestows upon us a method of reduction, that allow us to reduce potential bounds on variables with precision and efficiency. We give the details of these tools in the next section.

2. The tools. Let θ denote an algebraic number, and consider its minimal polynomial over $\mathbb{Z}$ of degree d given by

$$c_0x^d + c_1x^{d-1} + \cdots + c_d = c_0\prod_{i=1}^{d}(x-\theta^{(i)}),$$

where c_i are mutually prime integers with $c_0>0$, and $\theta^{(i)}$ represents the conjugates of θ. The logarithmic height of θ is defined as follows:

$$h(\theta) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d}\log\left(\max\{|\theta^{(i)}|,1\}\right)\right).$$

The subsequent properties proved to be highly valuable when calculating the logarithmic height:

- $h(\theta_1 \pm \theta_2) \leq h(\theta_1) + h(\theta_2) + \log 2$.
- $h(\theta_1 \theta_2^{\pm 1}) \leq h(\theta_1) + h(\theta_2)$.
- $h(\theta^s) = |s| h(\theta)$, $s \in \mathbb{Z}$.

The following theorem is a special case of Matveev's result for $t = 3$.

Theorem 2 (MATVEEV's Theorem, [10]). *Assume that $\alpha_1, \alpha_2, \alpha_3$ are positive real algebraic numbers in a real algebraic number field $\mathbb{F}$ of degree $d_{\mathbb{F}}$ and let b_1, b_2, b_3 be rational integers, such that*

$$\Lambda := \alpha_1^{b_1} \alpha_2^{b_2} \alpha_3^{b_3} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.44 \cdot 10^{11} d_{\mathbb{F}}^2 (1 + \log d_{\mathbb{F}})(1 + \log B) A_1 A_2 A_3\right),$$

where

$$A_i \geq \max\{d_{\mathbb{F}} h(\alpha_i), |\log \alpha_i|, 0.16\}, \ i = 1, 2, 3 \quad \textit{and} \quad B \geq \max\{|b_1|, |b_2|, |b_3|\}.$$

For a real number θ, we put $||\theta|| = \min\{|\theta - n| : n \in \mathbb{Z}\}$, which represents the distance from θ to the nearest integer. Now, we cite the following lemma which is a variation of the result due to Dujella and Pethő [9].

Lemma 1 ([11], Lemma 1). *Let M be a positive integer, and let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$. Let A and μ be some real numbers with $A > 0$ and $B > 1$. If $\epsilon := ||\mu q|| - M||\tau q|| > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

Lemma 2 ([12], Lemma 2.2). *Let $r, \theta \in \mathbb{R}$ and $0 < r < 1$. If $|\theta| < r$, then*

$$|\log(1 + \theta)| < \frac{-\log(1 - r)}{r} |\theta|.$$

3. Properties of k-Lucas numbers. The characteristic polynomial of the k-generalized Lucas sequences is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

which is an irreducible polynomial over $\mathbb{Q}[x]$. Let $\alpha := \alpha_1, \cdots, \alpha_k$ be the roots of $\Psi_k(x)$. Among all these roots of $\Psi_k(x)$, there is exactly one real distinguished root $\alpha := \alpha_1$, outside the unit circle [13–15]. In fact α is located in the interval

$$2(1-2^{-k}) < \alpha < 2 \quad \text{for all} \quad k \geq 2,$$

whereas all other roots are strictly inside the unit circle [14].

Let

$$f_k(x) = \frac{x-1}{2+(k+1)(x-2)}.$$

We summarize the fundamental properties of k-Lucas sequences in the following lemma.

Lemma 3. *Let α and $f_k(x)$ be as above.*

(i) $1/2 < f_k(\alpha) < 3/4$ *and* $|f_k(\alpha_i)| < 1$ *for* $2 \leq i \leq k$.

(ii) $h(f_k(\alpha)) < 3\log k$ *holds for all* $k \geq 2$.

(iii) $L_n^{(k)} = \sum_{i=1}^{k}(2\alpha_i - 1)f_k(\alpha_i)(\alpha_i)^{n-1}$ *and* $\left|L_n^{(k)} - f_k(\alpha)(2\alpha-1)\alpha^{n-1}\right| < 3/2$ *for all* $k \geq 2$.

(iv) $\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n$ *for all* $n \geq 1$ *and* $k \geq 2$.

(v) $L_i^{(k)} = 3 \cdot 2^{i-2}$ *for all* $2 \leq i \leq k$.

Proof. (i) and (ii) are from [11], Lemma 2, for others see [1]. □

4. Proof of theorem. If $n \leq 2$, then all the solutions of (1.1) are $L_1^{(k)} = 2-1 = 1$ and $L_2^{(k)} = 2^2 - 1 = 3$ for all $k \geq 2$. If $3 \leq n \leq 10$, then it is easy to see that $L_n^{(k)} \leq 768$ and none of them is of the form $2^a - 1$ for any $k \geq 2$ except for $L_4 = 2^3 - 1 = 7$. So, from now on we take $n > 10$. Since the case $k = 2$ is solved in [16], we may take $k \geq 3$. If $2 \leq n \leq k$, then $L_n^{(k)} = 3 \cdot 2^{n-1}$ and hence (1.1) turns into the equation $3 \cdot 2^{n-1} = 2^a - 1$, which has no solution in integers. So, we may also eliminate the case $n \leq k$. Now, assume that $n \geq k+1$.

4.1. The case $n \geq k+1$. Assume that equation (1.1) holds. From Lemma 3 (iii), we may write (1.1) as

$$f_k(\alpha)(2\alpha-1)\alpha^{n-1} - 2^a = -e_k(n) - 1.$$

Then, we get

$$|\Lambda| < \frac{5/2}{\alpha^{n-1}},$$

where

$$\Lambda := f_k(\alpha)^{-1}(2\alpha-1)^{-1}\alpha^{-(n-1)}2^a - 1$$

since $\dfrac{1}{f_k(\alpha)(2\alpha-1)} < 1$ and $|e_k(n)| < 3/2$. Let

$$(\eta_1, \eta_2, \eta_3) = (\alpha, 2, f_k(\alpha)(2\alpha - 1)) \quad \text{and} \quad (b_1, b_2, b_3) = (-(n-1), a, -1).$$

Since $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$, we take $\mathbb{F} = \mathbb{Q}(\alpha) \subset \mathbb{R}$ with degree $d_{\mathbb{F}} = k$. From (1.1) and Lemma 3 (iv), we have the inequality

$$2^{a-1} \leq 2^a - 1 = L_n^k \leq 2\alpha^n \leq 2^{n+1}.$$

Thus, it follows that

$$a \leq n + 2. \tag{4.1}$$

Therefore, we may take $B := n + 2 > \max\{|b_1|, |b_2|, |b_3|\}$. We put

$$h(\eta_1) = \frac{\log \alpha}{k} \Rightarrow A_1 := \log \alpha,$$
$$h(\eta_2) = \log 2 \Rightarrow A_2 := k \log 2,$$
$$h(\eta_3) \leq 4 \log k \Rightarrow A_3 := 4k \log k.$$

In the above calculation, the logarithmic height of η_1 and η_2 directly follows from their definition but for the case η_3, we use the two facts, $h(f_k(\alpha)) < 3 \log k$ from Lemma 3 (ii), and $h(2\alpha - 1) < \log 3 \leq \log k$ from [1], page 147.

Assume that $\Lambda = 0$. Then, we get

$$f_k(\alpha)(2\alpha - 1)\alpha^{n-1} = 2^a.$$

By taking the absolute values of the images of both sides under any one of the automorphisms $\sigma_i : \alpha \to \alpha_i$, $i \geq 2$, and by using also Lemma 3 (i), we get

$$2^a = |f_k(\alpha_i)||2\alpha_i - 1||\alpha_i|^{n-1} \leq 3.$$

But this is a contradiction, since from (1.1) and Lemma 3 (iv), we have that

$$\alpha^{n-1} \leq L_n^{(k)} = 2^a - 1 < 2^a,$$

which implies $10 \leq n - 1 < \dfrac{\log 2}{\log(3/2)} a < 2a$. So, $\Lambda \neq 0$ and hence we may apply Theorem 2 to $|\Lambda|$ which gives us a bound for Λ as follows

$$\log |\Lambda| < -1.44 \cdot 10^{11} k^2 (1 + \log k)(1 + \log(n+2)) \log \alpha \cdot k \log 2 \cdot 4k \log k.$$

Combining this bound with the facts that

$$\log |\Lambda| < \log 5/2 - (n-1) \log \alpha, \ 1 + \log k < 2 \log k, \ (1 + \log(n+2)) < 2 \log(n+2),$$

we obtain

$$n + 2 < 1.6 \cdot 10^{12} \cdot k^4 \log^2 k \log(n+2).$$

Now, we cite the following lemma from [17].

Lemma 4. *Let $s \geq 1$ and $T > (4s^2)^s$. Then we have*

$$\frac{x}{(\log x)^s} < T \Rightarrow x < 2^s T (\log T)^s.$$

We take $s = 1$ and $T := 1.6 \cdot 10^{12} \cdot k^4 \log^2 k$. Then

$$\begin{aligned}\log T &< \log(1.6) + 12 \log 10 + 4 \log k + 2 \log(\log k) \\ &< 60 \log k.\end{aligned}$$

Therefore, from Lemma 4, we get that

$$n + 2 < 2(1.6 \cdot 10^{12} \cdot k^4 \log^2 k) 60 \log k$$

or

$$n < 2 \cdot 10^{14} k^4 \log^3 k. \tag{4.2}$$

Now, we treat the cases $k \leq 180$ and $k > 180$ separately.

4.2. The case $3 \leq k \leq 180$. Let

$$\Gamma := \log(1 + \Lambda) = -(n-1) \log \alpha + a \log 2 - \log(f_k(\alpha)(2\alpha - 1)).$$

Note that for $n \geq 10$, we have that $|\Lambda| < \dfrac{5/2}{\alpha^{n-1}} < 0.1$. Therefore, by taking the parameters $r := 0.1$ and $\theta := \Lambda$ in Lemma 2, we find

$$|\Gamma| < -\frac{\log(1 - 0.1)}{0.1} |\Lambda| < \frac{3}{\alpha^{n-1}}.$$

So, we have

$$0 < \left| \frac{\Gamma}{\log 2} \right| = |(n-1)\tau_k - a + \mu_k| < 3/\alpha^{n-1} \log 2, \tag{4.3}$$

where

$$\mu_k := \frac{\log(f_k(\alpha)(2\alpha - 1))}{\log 2} \quad \text{and} \quad \tau_k := \frac{\log \alpha}{\log 2}.$$

For each $3 \leq k \leq 180$, we take $M_k := \lfloor 6.5 \cdot 10^{14} k^4 \rfloor > n - 1$. Note that, for each $3 \leq k \leq 180$, the number τ_k is irrational. For each k, we find a convergent p_i/q_i of the continued fraction of irrational number τ_k, such that $q_i > 6M_k$. Then, we calculate

$$\epsilon_k := ||\mu_k q_i|| - M_k ||\tau_k q_i||$$

for each $k \in \{3, \ldots, 180\}$. If $\epsilon_k < 0$, then we repeat the same calculations for q_{i+1}. We find an appropriate q_i, such that $\epsilon_k > 0$. In fact, $0.000161 < \epsilon_{115} \leq \epsilon_k$ for each k.

Thus, by taking $A := \dfrac{1}{\log 2}$ and $B := \alpha$, from Lemma 1, we find an upper bound on $n-1$, say $N(k)$, for each $3 \le k \le 180$. Some of these bounds are $N(3) = 72$, $N(10) = 74$, $N(50) = 89$, $N(100) = 105$, $N(150) = 155$ and $N(180) = 187$. By (4.1), all parameters are bounded. By using these bounds of $n-1$ and a, we write a short programme in Maple to check that there are no variables satisfying (1.1) in the range $3 \le k \le 180$, when $n \ge 10$.

4.3. The case $k \ge 180$. We cite the following lemma from [2], Lemma 1.

Lemma 5. *Let $k \ge 2$ and n be positive integers such that $k/2 \le n-1 < 2^{k/2}$. Then*

$$L_n^{(k)} = 3 \cdot 2^{n-2}(1 + \zeta(n,k)), \quad \textit{where} \quad |\zeta(n,k)| < \frac{6}{2^{k/2}}.$$

For $k > 180$, from (4.2), the inequality $k/2 \le n-1 < 2^{k/2}$ holds. Therefore, we combine the estimate coming from Lemma 5 together with (1.1) and hence we get that

$$|2^a - 3 \cdot 2^{n-2}| < 18\frac{2^{n-2}}{2^{k/2}} + 1,$$

or

$$|2^{a-n+2}3^{-1} - 1| < \frac{6}{2^{k/2}} + \frac{1/3}{2^{n-2}} < \frac{7}{2^{k/2}}. \tag{4.4}$$

But this is impossible, since for any $t \in \mathbb{Z}$ and for any $k \ge 180$, it is easy to check that

$$0.33 < \left|\frac{2^t}{3} - 1\right| \quad \text{and} \quad \frac{7}{2^{k/2}} < 10^{-25}.$$

Thus, we conclude that equation (1.1) has no solutions when $k \ge 180$. This completes the proof.

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