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ON A COMPLEX POLYNOMIAL AND ITS DERIVATIVE

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Abstract

In this paper, we prove some Turán-type inequalities for the polynomials all of whose zeros lie on a ray emanating from the origin, or symmetrically placed along this ray and having all their zeros in $|z| \leq K$, $K \geq 1$. Our results are the best possible, and examples of equality cases have been presented.
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1. Introduction. The class $\mathbb{H}_n^-$ of *Hurwitz polynomials* is defined as follows. $\mathbb{H}_n^-$ is the set of all polynomials $P(z)$ of degree less than or equal to n such that:

(i) $P(z)$ has only real coefficients;

(ii) all the zeros of $P(z)$ lie in the half-plane $\Re(z) \leq 0$.

This class $\mathbb{H}_n^-$ of Hurwitz polynomials has its own intrinsic beauty and is quite significant, because they represent the characteristic equations of stable linear systems in many areas such as control systems, signal processing, circuits analysis, and systems theory. Algebraically, the simplest necessary condition for a Hurwitz polynomial is that all its coefficients be non-negative. This is the core of any approach to the study of Hurwitz polynomials, which is also seen in this work.

Recently the author [1] derived some Erdős–Lax type inequalities for Hurwitz polynomials and this motivated us to find the analogues for Turán-type inequalities for the same class of Hurwitz polynomials having all its zeros in a disc of radius greater than or equal to one. In this paper we derived some results of this kind, which improve upon all the corresponding inequalities for a general class of polynomials with the same restriction on the location of zeros. We extend the

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obtained result to the class of polynomials having all their zeros in any given closed half plane, but symmetrically placed along a ray perpendicularly contained in this closed half-plane. Before stating our results let us look at few fundamental inequalities for polynomials.

The well-known BERNSTEIN's inequalities [2] on polynomials state that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| \tag{1.1}$$

and

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)|, \tag{1.2}$$

whenever $R \geq 1$.

The inequality (1.1) is a direct consequence of Bernstein's Theorem on the derivative of a trigonometric polynomial [3] and the inequality (1.2) follows from the maximum modulus theorem (see [4], Corollary 12.1.3). For the class of polynomials having no zeros inside the unit circle, it was ERDŐS [5] who conjectured, and later proved by LAX [6] that, if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{1.3}$$

Equality holds in (1.3) if all zeros of $P(z)$ lie on the circle $|z| = 1$.

A sharpened version of (1.3) was recently proved by the author [7] (see also [8]).

On the other hand, for the class of polynomials having all their zeros inside the closed unit circle, it was TURÁN [9] who proved that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{1.4}$$

Equality holds in (1.4) if all zeros of $P(z)$ lie on the circle $|z| = 1$. A Turán-type result of the form (1.2) can be seen in a recently published paper by the author [10].

It was MALIK [11], who generalized (1.3) to the class of polynomials having no zeros in the disc $|z| < K$, $K \geq 1$ by proving that, if $P(z)$ is a polynomial of degree n having no zeros in the disc $|z| < K, K \geq 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+K} \max_{|z|=1}|P(z)|. \tag{1.5}$$

The result is the best possible for $P(z) = (z + K)^n$.

On the other hand, GOVIL [12] generalized the inequality (1.4) to the class of polynomials having all their zeros in the disc $|z| \le K$, $K \ge 1$ by proving that, if $P(z)$ is a polynomial of degree n having all its zeros in the disc $|z| \le K$, $K \ge 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+K^n} \max_{|z|=1} |P(z)|. \tag{1.6}$$

The result is the best possible and equality holds for $P(z) = z^n + K^n$. A generalization of (1.6) in L^p settings may be seen in a paper due to the author [13].

For the class of polynomials vanishing in $|z| \le K$, $K \le 1$, the estimate for the maximum of $|P'(z)|$ on the unit circle was obtained by Malik [11], who proved that, if $P(z)$ is a polynomial of degree n having all its zeros in the disc $|z| \le K, K \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+K} \max_{|z|=1} |P(z)|. \tag{1.7}$$

The result is the best possible for $P(z) = (z+K)^n$.

Malik used the generalization of Erdős–Lax inequality to establish (1.7) and later in 1973 Govil [12] reproved using the real part of the logarithmic derivative of $P(z)$. Just for a change in approach, let us present an alternative proof to the inequality (1.7).

Since the result for $K = 1$ follows by continuity, let us assume that $K < 1$. Then by Gauss–Lucas Theorem it follows that $P'(z)$ has all its zeros in $K < 1$. Therefore if z is any complex number on $|z| = 1$ then we have

$$\left|\frac{nP(z)}{P'(z)}\right| = \left|\frac{nP(z)}{P'(z)} - z + z\right| \le \left|\frac{nP(z)}{P'(z)} - z\right| + 1$$

and therefore on $|z| = 1$

$$\left|\frac{nP(z)}{P'(z)}\right| \le 1 + \left|\frac{1-\alpha}{\alpha}\right|, \tag{1.8}$$

where $\alpha = \dfrac{zP'(z)}{nP(z)}$.

But

$$\alpha = \frac{1}{n}\sum_{k=1}^{n} \frac{z}{z-z_k} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1-w_k},$$

where $w_k = \dfrac{z_k}{z}$, $1 \le k \le n$.

Let us denote $\alpha_k = \dfrac{1}{1-w_k}$, $1 \le k \le n$, and therefore $\alpha = \dfrac{1}{n}\sum_{k=1}^{n} \alpha_k$.

Observe that each α_k, $1 \le k \le n$ is in the disc $|\dfrac{1-z}{z}| \le K$ having the centre $\left(\dfrac{1}{1-K^2}, 0\right)$ and radius $\dfrac{K}{1-K^2}$, since $|w_k| \le K$. As a convex linear combination of α_k's, the complex number α is also in the same disc and therefore we should have $\left|\dfrac{1-\alpha}{\alpha}\right| \le K$. Using this fact in (1.8) we get

$$\left|\frac{nP(z)}{P'(z)}\right| \le 1+K,$$

which proves inequality (1.7).

It is quite natural to test the significance of Malik's bound (1.7) whenever $K \ge 1$. This paper presents some results in this direction. Although Govil's bound is the best one for the polynomials having all zeros in the disc $|z| \le K$ whenever $K \ge 1$, at the same time it makes us think; for what class of polynomials for which the bound $\dfrac{n}{1+K}$ remains valid whenever $K \ge 1$? In this paper, we make an attempt to answer this question by considering Hurwitz polynomials. In fact, in Section 2 we prove a Turán-type inequality for a Hurwitz polynomial satisfying the hypotheses of (1.6) which involves the real linear combination of the bounds $\dfrac{1}{1+K}$ and $\dfrac{1}{1+K^2}$. This bound is sharper than the standard bound $\dfrac{n}{1+K^n}$. Some of its generalizations are also presented.

If $P(z)$ is a polynomial of degree n, then the *polar derivative of* $P(z)$ *with respect to a complex number* α is defined as

$$D_\alpha\{P(z)\} = nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha\{P(z)\}$ is a polynomial of degree at most $n-1$, and it is a generalization of the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha\{P(z)\}}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \le R,\ R > 0$. For more information on polar derivatives of polynomials, one can refer to monographs Marden [14], Rahman and Schmeisser [4], or Milovanović et al. [15]. For the latest developments on Bernstein-type inequalities for polar derivatives of polynomials we refer to a recently published book by Gardner et al. [16].

Bernstein-type inequalities for complex polynomials have been extended widely from 'ordinary derivative' to 'polar derivative' of complex polynomials and in this paper also we extend all our results to the polar derivative of a complex polynomial as well.

2. Main results and proofs. Let us derive our first result for the class of Hurwitz polynomials, which will in turn sharpen the inequality (1.6), and many of its generalizations that appeared in the literature (see [16]).

Lemma 2.1. *Let $P(z) \in \mathbb{H}_n^-$ be a polynomial of degree n. If $P(z)$ has all its zeros in the disc $|z| \le K$, $K \ge 1$, then*

$$\max_{|z|=1} |P'(z)| \ge \left(\frac{m}{1+K} + \frac{n-m}{1+K^2} \right) \max_{|z|=1} |P(z)|, \tag{2.1}$$

where m is the number of real zeros, and $n-m$ is the number of non-real zeros of $P(z)$. The result is the best possible and equality in (2.1) holds for $P(z) = (z+K)^m(z^2+K^2)^p$, where $m+2p=n$.

Proof. Let $P(z) \in \mathbb{H}_n^-$ of degree n having all its zeros in $|z| \le K$, $K \ge 1$. Without loss of generality let us assume that the constant term in $P(z)$ is 1. Since the coefficients of $P(z)$ are all real, its zeros are either real negative or will appear in conjugate pairs in the closed left half-plane. Denote the real zeros of $P(z)$ by $\{-c_i | 1 \le i \le m\}$, and the non-real zeros by $\{z_k,\ \overline{z_k} | 1 \le k \le p\}$, where

$$z_k = r_k e^{i\theta_k}$$

and $\dfrac{\pi}{2} \le \theta_k < \pi$. If we write

$$Q_k(z) = (1 - z/z_k)(1 - z/\overline{z_k}),$$

then a simple calculation shows that

$$Q_k(z) = 1 - 2\cos\theta_k \frac{z}{r_k} + \frac{z^2}{r_k^2}.$$

The restriction on θ_k makes $-1 < \cos\theta_k \le 0$, and hence $0 \le (-\cos\theta_k) = a_k < 1$. Therefore

$$Q_k(z) = 1 + 2a_k \frac{z}{r_k} + \frac{z^2}{r_k^2}$$

is a polynomial of degree 2 with non-negative coefficients. It can be easily verified that

$$\frac{Q_k'(1)}{Q_k(1)} = \frac{2\frac{1}{r_k}a_k + \frac{2}{r_k^2}}{1 + 2\frac{1}{r_k}a_k + \frac{1}{r_k^2}} \ge \frac{2}{1+K^2}.$$

Now therefore $P(z)$ can be expressed as

$$P(z) = \prod_{i=1}^{m} (1 + z/c_i) \prod_{k=1}^{p} Q_k(z), \tag{2.2}$$

where $m + 2p = n$. The fact that $Q_k(z)$ is a polynomial of degree 2 with only real non-negative coefficients, and this together with (2.2) implies that $P(z)$ is a polynomial with only real non-negative coefficients. Therefore it follows that $\max_{|z|=1}|P(z)| = P(1)$ and $\max_{|z|=1}|P'(z)| = P'(1)$, and thus we have

$$\begin{aligned}\frac{\max_{|z|=1}|P'(z)|}{\max_{|z|=1}|P(z)|} &= \frac{P'(1)}{P(1)} \\ &= \sum_{i=1}^{m}\frac{1}{1+c_k} + \sum_{k=1}^{p}\frac{Q_k'(1)}{Q_k(1)} \\ &\geq \frac{m}{1+K} + \frac{2p}{1+K^2},\end{aligned}$$

which is nothing but (2.1), and hence the proof is complete. □

Lemma 2.1 expects us to know the exact number of real zeros, and non-real zeros of a given polynomial. To avoid this requirement, as a consequence of Lemma 2.1 by noting that $\frac{1}{1+K} \geq \frac{1}{1+K^2}$ whenever $K \geq 1$, we get the following result.

Corollary 2.2. *Let $P(z) \in \mathbb{H}_n^-$ be a polynomial of degree n. If $P(z)$ has all it zeros in the disc $|z| \leq K, K \geq 1$, then*

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+K^2}\max_{|z|=1}|P(z)|. \tag{2.3}$$

The result is the best possible and equality in (2.3) *holds for $P(z) = (z^2+K^2)^{n/2}$, where n is even.*

Similarly, we can observe the case where $P(z) \in \mathbb{H}_n^-$ contains only real zeros, and therefore the case $n = m$ in Lemma 2.1 gives the following result.

Corollary 2.3. *Let $P(z) \in \mathbb{H}_n^-$ be a polynomial of degree n having only real zeros. If $P(z)$ has all its zeros in $(-\infty, -K]$ where $K \geq 1$, then*

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+K}\max_{|z|=1}|P(z)|. \tag{2.4}$$

The result is the best possible and equality in (2.4) *holds for $P(z) = (z+K)^n$.*

Remark 2.4. It may be easily seen that all our results stated above in this section sharpen the bound $\frac{n}{1+K^n}$ given in (1.6).

If $P(z)$ is a polynomial that satisfies (2.1), then it is easy to verify that $P(e^{i\lambda}z)$ where λ is a real number also satisfies (2.1). Therefore, if (2.1) is valid for all polynomials $P(z)$ in the class $\mathbb{H}_n^-$, then it is also valid for polynomials all of whose zeros lie on a ray emanating from the origin, say L, as well as polynomials whose zeros are symmetrically placed along this ray L. Thus, although (2.1) is generally not valid for the hypotheses of (1.6), the class of polynomials to which the results of this note hold is large enough to contain many of the interesting and known classes of polynomials. Thus the above results can be extended to the class of polynomials having all their zeros in any given closed half-plane, but symmetrically placed along the ray perpendicularly contained in this closed half-plane.

Let $\mathbb{H}_n^L$ be a class of polynomials $P(z)$ all of whose zeros lie on a ray L emanating from the origin or lie symmetrically placed along L. Note that all the zeros of a polynomial in $\mathbb{H}_n^L$ lie in the same half-plane that is perpendicular to the ray L. For example, if L is the non-positive real axis, then $\mathbb{H}_n^L = \mathbb{H}_n^-$. Observe that the polynomials in $\mathbb{H}_n^L$ have all their coefficients on a ray and this would be the necessary condition for any polynomial to be in $\mathbb{H}_n^L$ like the non-negativity of coefficients for polynomials in $\mathbb{H}_n^-$.

In view of the above, we obtain the generalizations of Lemma 2.1, Corollaries 2.2 and 2.3 to the polynomials in $\mathbb{H}_n^L$ as stated below.

Theorem 2.5. *Let $P(z) \in \mathbb{H}_n^L$ be a polynomial of degree n. If $P(z)$ has all its zeros in the disc $|z| \leq K, K \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{m}{1+K} + \frac{n-m}{1+K^2} \right) \max_{|z|=1} |P(z)|, \tag{2.5}$$

where m is the number of zeros of $P(z)$ on the ray L emanating from the origin, and $n-m$ is the number of zeros of $P(z)$ lying outside L but symmetrically placed along L.

Again as a consequence of Theorem 2.5, we can easily obtain the following two results.

Corollary 2.6. *Let $P(z) \in \mathbb{H}_n^L$ be a polynomial of degree n. If $P(z)$ has all its zeros in the disc $|z| \leq K$, $K \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+K^2} \max_{|z|=1} |P(z)|. \tag{2.6}$$

Corollary 2.7. *Let $P(z) \in \mathbb{H}_n^L$ be a polynomial of degree n having all its zeros on the ray L of length K, $K \geq 1$. Then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+K} \max_{|z|=1} |P(z)|. \tag{2.7}$$

Remark 2.8. As is easily seen, Theorem 2.5, Corollaries 2.6, and 2.7 are sharp and equality holds for the examples given in Lemma 2.1, Corollaries 2.2, and 2.3, respectively.

As mentioned in Section 1, it is quite natural to look for an extension of Lemma 2.1 to the one in more generalized form involving the polar derivative of a polynomial with the same restrictions which is given below.

Theorem 2.9. *Let $P(z) \in \mathbb{H}_n^L$ be a polynomial of degree n. If $P(z)$ has all its zeros in the disc $|z| \leq K, K \geq 1$, then for any complex number α with $|\alpha| \geq K^2$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \left[\left(\frac{m(|\alpha|-K)}{1+K} + \frac{(n-m)(|\alpha|-K^2)}{1+K^2} \right) \right] \max_{|z|=1} |P(z)|, \tag{2.8}$$

where m is the number of zeros of $P(z)$ on the ray L emanating from the origin, and $n-m$ is the number of zeros of $P(z)$ lying outside L but symmetrically placed along L.

Proof. Note that for any complex number α with $|\alpha| \geq 1$, and using the well-known inequality (see [17])

$$|nP(z) - zP'(z)| + |P'(z)| \leq n \max_{|z|=1} |P(z)|,$$

we have on $|z| = 1$

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &= |nP(z) - zP'(z) + \alpha P'(z)| \\ &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)| \\ &\geq |\alpha||P'(z)| - (n \max_{|z|=1} |P(z)| - |P'(z)|) \\ &= (|\alpha| + 1)|P'(z)| - n \max_{|z|=1} |P(z)|. \end{aligned}$$

Therefore

$$\max_{|z|=1} |D_\alpha P(z)| \geq (|\alpha| + 1) \max_{|z|=1} |P'(z)| - n \max_{|z|=1} |P(z)|.$$

Since $P(z) \in \mathbb{H}_n^L$ is of degree n and satisfies the hypotheses of Theorem 2.5, $P(z)$ satisfies the inequality (2.5) and using this in the above inequality we have

$$\max_{|z|=1} |D_\alpha P(z)| \geq (|\alpha| + 1) \left(\frac{m}{1+K} + \frac{n-m}{1+K^2} \right) \max_{|z|=1} |P(z)| - n \max_{|z|=1} |P(z)|,$$

which by simplification yields the required inequality, and thus the proof is complete. □

As an immediate consequence of Theorem 2.9, we get the following result.

Corollary 2.10. *Let $P(z) \in \mathbb{H}_n^L$ be a polynomial of degree n. If $P(z)$ has all its zeros in the disc $|z| \leq K, K \geq 1$, then for any complex number α with $|\alpha| \geq K^2$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K^2}{1+K^2} \right) \max_{|z|=1} |P(z)|. \tag{2.9}$$

It is once again an easy exercise to obtain the following special case of Theorem 2.9 for the polynomial $P(z) \in \mathbb{H}_n^L$ having all its zeros on the ray L, which is given below.

Corollary 2.11. *Let $P(z) \in \mathbb{H}_n^L$ be a polynomial of degree n having all its zeros on the ray L of length K, $K \geq 1$. Then for any complex number α with $|\alpha| \geq K$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K}{1+K} \right) \max_{|z|=1} |P(z)|. \tag{2.10}$$

Remark 2.12. Dividing (2.8), (2.9), (2.10) by $|\alpha|$ and taking $|\alpha| \to \infty$ we get (2.5), (2.6), (2.7), respectively.

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