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NONEXISTENCE OF GLOBAL SOLUTIONS TO DOUBLE DISPERSION EQUATIONS WITH LINEAR RESTORING FORCE AND NONLINEARITIES WITH VARIABLE COEFFICIENTS

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Abstract

We investigate Cauchy problem to double dispersion equations with polynomial type nonlinearities with variable coefficients. Necessary and sufficient conditions for nonexistence of global weak solutions and nonblowing up ones are found for subcritical initial energy. For supercritical energy a sufficient condition for finite time blow up of the weak solutions, independent of the scalar product of the initial data, is developed.

Key words: double dispersion equations, linear restoring force, finite time blow up, nonlinearities with variable coefficients

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1. Introduction. We study the nonexistence of the weak solutions to the Cauchy problem for double dispersion equations with linear restoring force

$$(1)\qquad \begin{cases} u_{tt} - u_{ttxx} - u_{xx} + u_{xxxx} + u + f(x,u)_{xx} = 0, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \ u_0 \in \mathrm{H}^1(\mathbb{R}), \ (-\Delta)^{-1/2} u_0 \in \mathrm{L}^2(\mathbb{R}), \\ u_t(0,x) = u_1(x), \ u_1 \in \mathrm{L}^2(\mathbb{R}), \ (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R}). \end{cases}$$

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Here $(-\Delta)^{-l}v = \mathcal{F}^{-1}(|\xi|^{-2l}\mathcal{F}(v))$ for $l > 0$, $\mathcal{F}(v)$, $\mathcal{F}^{-1}(v)$ are the Fourier and the inverse Fourier transform, respectively.

We consider two polynomial type nonlinearities $f(x,u)$:

$$(2)\qquad f(x,u) = \sum_{k=1,k\neq s}^{m} a_k(x)|u|^{p_k-1}u + a_s(x)|u|^{p_s},\ s \in [1,m],\ m \geq 2,$$

$$(3)\qquad f(x,u) = \sum_{k=1}^{m} a_k(x)|u|^{p_k-1}u,\ m \geq 2,$$

where

$$(4)\qquad \begin{cases} 1 < p_1 < p_2 < \cdots < p_m < \infty, \\ a_k(x) \in C(\mathbb{R}),\ |a_k(x)| \leq A_k,\ \forall k \in [1,m],\ \forall x \in \mathbb{R}. \end{cases}$$

The coefficient $a_s(x)$, $s \in [1,m]$, is a sign-changing function and for every $x \in \mathbb{R}$ we assume one of conditions (5), (6) or (7)

(5) for $s = 1$: $a_j(x) \geq 0,\ \forall j \in [2,m]$;

(6) for $s \in (1,m)$: $a_j(x) \leq 0,\ \forall j \in [1,s-1];\ a_j(x) \geq 0,\ \forall j \in [s+1,m]$;

(7) for $s = m$: $a_j(x) \leq 0,\ \forall j \in [1,m-1]$.

For the special case of a single nonlinearity

$$(8)\qquad f(x,u) = a_s(x)|u|^{p_s-1}u \quad \text{or} \quad f(x,u) = a_s(x)|u|^{p_s}$$

the function $a_s(x)$ is an arbitrary nontrivial function such that (4) holds.

Equation (1) arises as a model in the nonlinear dynamics in weakly dispersive media. For example, (1) describes the transverse deflections of an elastic rod subject to axial extension. The term u_{ttxx} corresponds to the so-called rotational inertia (see [1]). Moreover, the cubic-quintic nonlinearity appears in the propagation of longitudinal strain waves in an isotropic cylindrical compressible elastic rod [2] and in shape memory alloys [3]. The double dispersion equation (1) is also a model of pulse propagation in biomembranes and nerves (see [4]).

The global behaviour of the weak solutions to double dispersion equation (1) is intensively investigated. By means of the potential well method (see [5]) and the concavity method of Levine (see [6]) the case of subcritical initial energy $E(0) < d$ is studied for single nonlinearities $f(u) = a|u|^p$ and $f(u) = a|u|^{p-1}u$, $p > 1$ with constant coefficient a, see [7] and the references therein. Further on, the potential well method is extended for power-type nonlinearities with constant coefficients in [8–10].

For supercritical initial energy $E(0) > d$ sufficient conditions for finite time blow up of the weak solutions for combined power-type nonlinearities with constant coefficients are given in [9,11].

In the present article we give a complete answer of the question for finite time blow up of the weak solutions or nonexistence of blowing up solutions for equation (1), when the initial energy is subcritical and the nonlinearities (2), (3), and (8) are with variable coefficients. For supercritical initial energy we propose a new sufficient condition for nonexistence of global solutions to problem (1). This condition is independent of the sign of the scalar product $\langle u_0, u_1\rangle$ of the initial data.

2. Preliminaries. Further on we use the short notations $\|u\| = \|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R})}$, $\|u\|_1 = \|u(t,\cdot)\|_{\mathrm{H}^1(\mathbb{R})}$, $(u,v) = (u(t,x), v(t,x)) = \int_{\mathbb{R}} u(t,x)v(t,x)\,dx$,

$$\langle u,v\rangle = \langle u(t,\cdot), v(t,\cdot)\rangle = \Big((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}v\Big) + (u,v). \tag{9}$$

The function

$$u(t,x) \in \mathrm{C}([0,T_m);\mathrm{H}^1(\mathbb{R})) \cap \mathrm{C}^1([0,T_m);\mathrm{L}^2(\mathbb{R})) \cap \mathrm{C}^2([0,T_m);\mathrm{H}^{-1}(\mathbb{R}))$$

defined in the maximal existence interval $[0,T_m)$, $0 < T_m \le \infty$ is a weak solution of (1) if for every $w(x) \in H^1(\mathbb{R})$ with $(-\Delta)^{-\frac{1}{2}}w \in L^2(\mathbb{R})$ the identity

$$\Big((-\Delta)^{-\frac{1}{2}}u_t, (-\Delta)^{-\frac{1}{2}}w\Big) + (u_t,w) + \int_0^t [(u,w) + (u_x,w_x) - (f(x,u),w)]\,d\tau$$
$$= (u_1,w) + \Big((-\Delta)^{-\frac{1}{2}}u_1, (-\Delta)^{-\frac{1}{2}}w\Big)$$

holds for every $t \in [0,T_m)$. It is easy to check that the following conservation law of the energy holds for every $t \in [0,T_m)$

$$E(u(t,\cdot)) := E(t) := E(0),$$

where

$$E(u(t,\cdot)) = \frac{1}{2}\left(\langle u_t,u_t\rangle + \langle u,u\rangle + \|u_x\|^2\right) - \int_{\mathbb{R}}\int_0^{u(t,x)} f(x,y)\,dy\,dx. \tag{10}$$

We introduce the definitions of the Nehari functional $I(w)$, the potential energy functional $J(w)$, the Nehari manifold N and the critical energy constant d, which are important for the potential well method:

$$I(w) := \langle w,w\rangle + \|w_x\|^2 - \int_{\mathbb{R}} f(x,w(x))w(x)\,dx,$$
$$J(w) := \frac{1}{2}(\langle w,w\rangle + \|w_x\|^2) - \int_{\mathbb{R}}\int_0^{w(x)} f(x,z)\,dz\,dx,$$
$$N := \{w \in H^1(\mathbb{R})\colon \|w\|_1 \ne 0, I(w) = 0\}, \quad d = \inf_{w\in N} J(w).$$

The proof of nonexistence of global solutions is based on the concavity method of Levine, proposed for the wave equation in [6]. The idea of the method is to replace the blow up of $H^1(\mathbb{R})$ norm of the solution with blow up of its $L^2(\mathbb{R})$ norm.

For problem (1) the blow up occurs in the norm (9). Thus we focus on the investigation of the function $\Psi(t) = \langle u(t,\cdot), u(t,\cdot)\rangle$, which satisfies the following differential equation

$$\Psi''(t) = (p_s+3)\langle u_t, u_t\rangle - 2I(u(t,\cdot)) + 2(p_s+1)\left[J(u(t,\cdot)) - E(0)\right]. \tag{11}$$

We recall our previous results for blow up of the nonnegative solutions to ordinary differential equations in the improved concavity method.

Theorem 1 ([11], Theorem 2.3, [12], Theorem 3.2)**.** *Suppose $\gamma > 1$ and $[0, T_m)$, $0 < T_m \le \infty$ is the maximal existence time interval of the nonnegative solution $\Psi(t) \in C^2([0,T_m))$ to the problem*

$$\begin{cases}\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) = \alpha\Psi^2(t) - \beta\Psi(t) + H(t), \quad t \in [0,T_m),\\ \alpha > 0,\ \beta > 0,\ H(t) \in \mathrm{C}([0,T_m)),\ H(t) \ge 0 \text{ for } t \in [0,T_m)\end{cases}$$

or to the problem

$$\begin{cases}\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) = G(t),\ t \in [0,T_m),\\ G(t) \in C([0,T_m)),\ G(t) \ge 0 \text{ for } t \in [0,T_m).\end{cases}$$

If $\Psi(t)$ blows up at T_m, then $T_m < \infty$.

Let us mention, that the properties of the Nehari functional $I(w)$ are precisely investigated for nonlinearities (2), (3) and (8) with variable coefficients in [13]. Further on, we use the following axillary proposition.

Lemma 1 ([13], Lemma 3)**.** *For each of the nonlinearities* (2)*,* (3) *or* (8) *under restriction* (4) *and for one of the assumptions* (5)*,* (6) *or* (7)*, the inequality*

$$I(w) < (p_s+1)(J(w) - d) \tag{12}$$

holds for functions $w \in H^1(\mathbb{R})$ with $\|w\|_1 \ne 0$ satisfying $I(w) < 0$.

3. Main results. Our first result is about the global behaviour of the solutions with subcritical initial energy. The success of the analysis in the potential well method, when $0 < E(0) < d$, is due to the invariance of the basic sets W and V

$$V = \{w \in H^1(\mathbb{R})\colon I(w) < 0\}, \quad W = \{w \in H^1(\mathbb{R})\colon I(w) > 0\} \cup \{0\}.$$

Theorem 2. *Suppose $u(t,x)$ is a weak solution to* (1) *with one of the nonlinearities* (2)*,* (3) *or* (8) *in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$. Let* (4) *and one of* (5)*,* (6) *or* (7) *hold.*

(i) *If $0 < E(0) < d$ and there exists some $t_0 \in [0,T_m)$ such that $u(t_0,x) \in W$, then $u(t,x) \in W$ for every $t \in [0,T_m)$;*

(ii) *If* $0 < E(0) < d$ *and there exists some* $t_0 \in [0, T_m)$ *such that* $u(t_0, x) \in V$*, then* $u(t, x) \in V$ *for every* $t \in [0, T_m)$*;*

(iii) *If* $E(0) < 0$*, then* $I(u(t, \cdot)) < 0$*, i.e.* $u(t, x) \in V$ *for every* $t \in [0, T_m)$*;*

(iv) *If* $E(0) = 0$*, then for every nontrivial solution* $u(t, x)$ *we have* $I(u(t, \cdot)) < 0$*, i.e.* $u(t, x) \in V$ *for every* $t \in [0, T_m)$*.*

Proof. From the conservation law (10) we have

(13)
$$I(u(t,\cdot)) = (p_s+1)E(0) - \frac{(p_s+1)}{2}\langle u_t(t,\cdot), u_t(t,\cdot)\rangle - \frac{p_s-1}{2}\langle u(t,\cdot), u(t,\cdot)\rangle - \frac{p_s-1}{2}\|u_x\|^2 - (p_s+1)B(u(t,\cdot))$$

with $B(t)$ defined as follows

$$B(t) := B(u(t,\cdot)) = \sum_{k=1, k\neq s}^{m} \frac{(p_k - p_s)}{(p_k+1)(p_s+1)} \int_{\mathbb{R}} a_k(x)|u(t,x)|^{p_k+1}\,dx.$$

Under assumptions of the theorem we have $B(t) \geq 0$.

Case (i). Suppose by contradiction that there exists some $t_1 \in [0, T_m)$, $t_1 \neq t_0$, such that $u(t_1, x) \notin W$. From the continuity of $u(t, x)$ we have a time $t_2 \neq t_1$, such that $u(t_2, x) \in \partial W$, i.e. $I(u(t_2, \cdot)) = 0$. Since $I(\lambda w) > 0$ for every w with $\|w\|_1 \neq 0$ and all sufficiently small $\lambda > 0$ it follows that $0 \notin \partial W$. Hence $u(t_2, x) \neq 0$, i.e. $\|u(t_2, \cdot)\|_1 \neq 0$ and $u(t_2, x) \in N$. From (10) we get the following impossible chain of inequalities

$$d = \inf_{w\in N} J(w) \leq J(u(t_2,\cdot)) = E(0) - \frac{1}{2}\langle u_t(t_2,\cdot), u_t(t_2,\cdot)\rangle \leq E(0) < d. \tag{14}$$

Case (ii). Suppose by contradiction that $u(t_3, x) \notin V$ for some $t_3 \in [0, T_m)$, $t_3 \neq t_0$, i.e. $I(u(t_3, \cdot)) = 0$. If $\|u(t_3, \cdot)\|_1 = 0$, i.e. $u(t_3, x) \equiv 0$, then $u(t_3, x) \in W$. From (i) we have that $u(t, x) \in W$ for every $t \in [0, T_m)$, which contradicts that $u(t_0, x) \in V$. If $\|u(t_3, x)\|_1 \neq 0$, then $u(t_3, x) \in N$, and repeating (14) for $u(t_3, x)$, we get an impossible chain of inequalities.

Case (iii). The proof follows from (13).

Case (iv). Assume $u(t, x)$ is a nontrivial solution, i.e. there exists $t_4 \in [0, T_m)$ such that $\|u(t_4, \cdot)\|_1 \neq 0$. Let (t_5, t_6) be the maximal neighbourhood of t_4 such that $\|u(t, \cdot)\|_1 \neq 0$ for $t \in (t_5, t_6)$. We will prove that $\|u(t, \cdot)\|_1 \neq 0$ for every $t \in [0, T_m)$. If not, then either $\|u(t_5, \cdot)\|_1 = 0$ or $\|u(t_6, \cdot)\|_1 = 0$. Suppose $\|u(t_5, \cdot)\|_1 = 0$. Then we can find $t_7 \in (t_5, t_6)$ such that $0 < \|u(t_7, \cdot)\|_1 < \min\left\{1, \left[4\sum_{k=s}^{m} A_k C_{p_k+1}^{p_k+1}\right]^{-\frac{1}{p_1-1}}\right\}$. Here C_{p_k+1} are the constants of the Sobolev embedding theorem $\|z\|_{L^q(\mathbb{R})} \le C_q\|z\|_1$. From (10) and the choice of time t_7 we get the impossible chain of inequalities

$$\begin{aligned}\frac{1}{2}\|u(t_7,\cdot)\|_1^2 &\le \int_{\mathbb{R}} u(t_7,x)f(x,u(t_7,x))\,dx \le \sum_{k=s}^{m} A_k \int_{\mathbb{R}} |u(t_7,x)|^{p_k+1}\,dx \\ &\le \sum_{k=s}^{m} A_k C_{p_k+1}^{p_k+1}\|u(t_7,\cdot)\|_1^{p_k+1} = \|u(t_7,\cdot)\|_1^2 \sum_{k=s}^{m} A_k C_{p_k+1}^{p_k+1}\|u(t_7,\cdot)\|_1^{p_k-1} \\ &\le \|u(t_7,\cdot)\|_1^2\|u(t_7,\cdot)\|_1^{p_1-1}\sum_{k=s}^{m} A_k C_{p_k+1}^{p_k+1} < \frac{1}{4}\|u(t_7,\cdot)\|_1^2.\end{aligned}$$

Hence $\|u(t,\cdot)\|_1 \neq 0$ for every $t \in [0,T_m)$. The assertion (iv) holds from (13). The proof of Theorem 2 is completed. □

Theorem 3. *Consider problem* (1) *with one of the nonlinearities* (2), (3) *or* (8). *Suppose* $0 < E(0) < d$, (4) *and one of* (5), (6) *or* (7) *hold. Then*

(i) *for* $u_0 \in V$, *problem* (1) *has no global in time weak solution;*

(ii) *for* $u_0 \in W$, *problem* (1) *has no blowing up weak solution.*

Proof. Case (i). Suppose by contradiction that (1) has a weak solution $u(t,x)$ for every $t \ge 0$ and define $\Psi(t) = \langle u(t,\cdot), u(t,\cdot)\rangle$. Since $u_0 \in V$, from Theorem 2(ii) we have $u(t,x) \in V$, i.e. $I(u(t,\cdot)) < 0$ and $\|u(t,\cdot)\|_1 \neq 0$ for every $t \ge 0$. From (11) and the definition of d we get the inequalities

$$\begin{aligned}\Psi''(t) &= (p_s+3)\langle u_t(t,\cdot),u_t(t,\cdot)\rangle - 2I(u(t,\cdot)) + 2(p_s+1)\left[J(u(t,\cdot)) - E(0)\right] \\ &> (p_s+3)\langle u_t(t,\cdot),u_t(t,\cdot)\rangle + 2(p_s+1)(d-E(0)) \\ &\ge 2(p_s+1)(d-E(0)) > 0. \qquad (15)\end{aligned}$$

By integrating (15) twice, we obtain

$$\Psi(t) \ge (p_s+1)(d-E(0))t^2 + \Psi'(0)t + \Psi(0), \text{ hence } \lim_{t\to\infty}\Psi(t) = \infty. \qquad (16)$$

Moreover, using (11), we get that function $\Psi(t)$ satisfies the equation

$$\Psi''(t)\Psi(t) - \frac{p_s+3}{4}\Psi'^2(t) = G(t),$$

where

$$\begin{aligned}G(t) &= (p_s+3)\{\langle u_t(t,\cdot),u_t(t,\cdot)\rangle\langle u(t,\cdot),u(t,\cdot)\rangle - \langle u_t(t,\cdot),u(t,\cdot)\rangle^2\} \\ &\quad + 2\langle u(t,\cdot),u(t,\cdot)\rangle\{(p_s+1)(J(u(t,\cdot)) - d) - I(u(t,\cdot)) + (p_s+1)(d-E(0))\}.\end{aligned}$$

From (12) we have $G(t) > 0$. We apply Theorem 1 with $\gamma = \frac{p_s+3}{4} > 1$ since $\Psi(t)$ blows up at $T_m = \infty$ from (16). We conclude that $u(t,x)$ blows up for a finite time, which contradicts our assumption.

Case (ii). Suppose by contradiction that $u(t,x)$, defined in $[0,T_m)$, $0 \le T_m \le \infty$, blows up at T_m. Since $u_0 \in W$, from Theorem 2(i) we get $u(t,x) \in W$ and hence $I(u(t,\cdot)) \ge 0$ for $t \in [0,T_m)$. According to (13) the estimate

$$\langle u(t,\cdot), u(t,\cdot)\rangle + \|u_x(t,\cdot)\|^2 \le \frac{2(p_s+1)}{p_s-1}E(0) < \infty$$

holds for every $t \in [0,T_m)$, and $u(t,x)$ does not blow up at T_m. The proof of Theorem 3 is completed. □

Corollary 1. *Suppose* (4) *and one of assumptions* (5), (6) *or* (7) *holds for problem* (1) *with one of nonlinearities* (2), (3) *or* (8). *Then problem* (1)

(i) *with* $E(0) < 0$, *has no global in time weak solution;*

(ii) *with* $E(0) = 0$, *except the trivial solution, has no global in time weak solution.*

The proof of Corollary 1 is similar to the proof of Theorem 3 and we omit it.

In the case of supercritical initial energy $E(0) \ge d$ we have the following nonexistence result.

Theorem 4. *Suppose* (4) *and one of assumptions* (5), (6) *or* (7) *holds. Then problem* (1) *with one of nonlinearities* (2), (3) *or* (8) *has no global in time weak solution if*

$$0 < E(0) < \frac{\sqrt{p_s-1}}{p_s+1}\langle u_0, u_1\rangle + \frac{p_s-1}{2(p_s+1)}\langle u_0, u_0\rangle . \tag{17}$$

Proof. Suppose by contradiction that $u(t,x)$ exists for every $t \ge 0$. From (10) and (11), the function $\Psi(t) = \langle u(t,\cdot), u(t,\cdot)\rangle$ satisfies the equation

$$\Psi''(t) = (p_s-1)\Psi(t) - 2(p_s+1)E(0) + G_1(t), \tag{18}$$

where

$$G_1(t) = (p_s+3)\langle u_t, u_t\rangle + (p_s-1)\|u_x\|^2 + 2(p_s+1)B(t) \ge 0.$$

For $\alpha = p_s - 1 > 0$, $\beta = 2(p_s+1)E(0) > 0$ the solution of (18) is given by

$$\begin{aligned} \Psi(t) = {} & \frac{\beta}{\alpha} + \frac{1}{\sqrt{\alpha}}\int_0^t G_1(\tau)\sinh(\sqrt{\alpha}(t-\tau))\,d\tau \\ & + \frac{1}{2}\left\{\Psi(0) + \frac{1}{\sqrt{\alpha}}\Psi'(0) - \frac{\beta}{\alpha}\right\}e^{\sqrt{\alpha}t} + \frac{1}{2}\left\{\Psi(0) - \frac{1}{\sqrt{\alpha}}\Psi'(0) - \frac{\beta}{\alpha}\right\}e^{-\sqrt{\alpha}t}. \end{aligned} \tag{19}$$

Since condition (17) is equivalent to $\Psi(0)+\frac{1}{\sqrt{\alpha}}\Psi'(0)-\frac{\beta}{\alpha}>0$, it follows from (19) that

$$\lim_{t\to\infty}\Psi(t)=\infty. \tag{20}$$

Straightforward computations give us the equality

$$\Psi''(t)\Psi(t)-\frac{p_s+3}{4}\Psi'^2(t)=\alpha\Psi^2(t)-\beta\Psi(t)+H(t), \tag{21}$$

where

$$H(t)=(p_s+3)\left\{\langle u_t(t,\cdot),u_t(t,\cdot)\rangle\langle u(t,\cdot),u(t,\cdot)\rangle-\langle u(t,\cdot),u_t(t,\cdot)\rangle^2\right\}$$
$$+(p_s-1)\|u_x\|^2\langle u(t,\cdot),u(t,\cdot)\rangle+2(p_s+1)\langle u(t,\cdot),u(t,\cdot)\rangle B(t)\geq 0.$$

From (20), (21) and Theorem 1 it follows that $\Psi(t)$ blows up for a finite time, which contradicts our assumption. Theorem 4 is proved. □

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