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CONFORMAL YAMABE SOLITONS ON TANGENT BUNDLES WITH COMPLETE LIFTS OF SOME SPECIAL CONNECTIONS

Murat Altunbaş

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Abstract

In this paper, we investigate conformal Yamabe solitons on tangent bundles with respect to the complete lifts a semi-symmetric metric connection and a projective semi-symmetric connection using vertical and complete lifts of torqued vector fields.

Key words: tangent bundle, conformal Yamabe soliton, torse-forming vector field, complete lift

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1. Introduction and preliminaries. ***1.1. Conformal Yamabe solitons, torse forming vector fields and semi-symmetric connections.*** Let (M, g) be an n-dimensional (pseudo-)Riemannian manifold. A conformal Yamabe soliton (M, g, τ, λ) is defined by the condition

$$\mathfrak{L}_\tau g + \left[2\lambda - 2r - \left(p + \frac{2}{n}\right)\right] g = 0, \tag{1.1}$$

where $\mathfrak{L}_\tau$ is the Lie differentiation with respect to the potential field τ on M, r is the scalar curvature, λ is a real constant and p is a scalar non-dynamical field (time dependent scalar field) [1]. The conformal Yamabe soliton is called expanding if $\lambda > 0$, steady if $\lambda = 0$, and shrinking if $\lambda < 0$.

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A nowhere zero vector field τ on a (pseudo-)Riemannian manifold (M, g) is called torse-forming if

$$\nabla_U \tau = \phi U + \alpha(U)\tau, \tag{1.2}$$

where U is a vector field on M, ∇ is the Levi-Civita connection of g, ϕ is a smooth function and α is a 1-form [2]. Furthermore, the vector field τ is said to be concircular ([3,4]) if the 1-form $\alpha = 0$ in equation (1.2). Moreover, if the function $\phi = 1$ and the 1-form $\alpha = 0$, the vector field τ is called concurrent, ([5,6]). The vector field τ is called recurrent if the function $\phi = 0$ in (1.2). If $\phi = \alpha = 0$ in (1.2), then the vector field τ is said to be parallel. Chen [7] defined a new vector field, which is called a torqued vector field τ, as satisfying (1.2) with $\alpha(\tau) = 0$. In this case, ϕ is the torqued function and the 1-form α is the torqued form of τ.

From [8], a metric connection $\tilde{\nabla}$ is called semi-symmetric if its torsion tensor T is of the form

$$T(U, V) = \pi(V)U - \pi(U)V$$

for all $U, V \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of M. Here, the 1-form π is defined by

$$\pi(U) = g(U, P)$$

for a vector field P on M. The relation between a semi-symmetric metric connection $\tilde{\nabla}$ and the connection ∇ of M is given by

$$\tilde{\nabla}_U V = \nabla_U V + \pi(V)U - g(U, V)P \tag{1.3}$$

for all $U, V \in \chi(M)$.

The Riemannian curvature tensor $\tilde{R}$, the Ricci tensor $\widetilde{Ric}$ and the scalar curvature $\tilde{r}$ of the connection $\tilde{\nabla}$ are given by, respectively, [1]

$$\begin{aligned} \widetilde{R}(U, V)W &= R(U, V)W - \eta(V, W)U + \eta(U, W)V - g(V, W)LU + g(U, W)LV, \\ \widetilde{Ric}(V, W) &= Ric(V, W) - (n-2)\eta(V, W) - ag(V, W), \\ \tilde{r} &= r - 2(n-1)a, \end{aligned} \tag{1.4}$$

where η and L denote $(0, 2)$ and $(1, 1)$-type tensor fields such that

$$\eta(U, V) = g(LU, V) = (\nabla_U \pi)(V) - \pi(U)\pi(V) + \frac{1}{2}g(U, V)$$

for all $U, V, W \in \chi(M)$ and $a = Tr(\eta)$ (see also [9]).

From [10], the relation between a projective semi-symmetric connection $\bar{\nabla}$ and the connection ∇ of M is given by

$$\bar{\nabla}_U V = \nabla_U V + \psi(V)U + \psi(U)V + \mu(V)U - \mu(U)V, \tag{1.5}$$

where ψ and μ are 1-forms satisfying

$$\psi(U) = \frac{n-1}{2(n+1)}\pi(U), \quad \mu(U) = \frac{1}{2}\pi(U).$$

For the Riemannian curvature tensor $\bar{R}$, Ricci tensor $\overline{Ric}$ and the scalar curvature $\bar{r}$ of $\bar{\nabla}$, we have

$$\begin{aligned} \overline{R}(U,V)W &= R(U,V)W + \theta(U,V)W + \omega(U,W)V - \omega(V,W)U, \\ \overline{Ric}(U,V) &= Ric(U,V) + \theta(U,V) - (n-1)\omega(U,V), \\ \bar{r} &= r + Tr(\theta) - (n-1)Tr(\omega), \end{aligned} \tag{1.6}$$

where

$$\theta(U,V) = \frac{1}{2}[(\nabla_V \pi)(U) - (\nabla_U \pi)(V)],$$
$$\omega(U,V) = \frac{n-1}{2(n+1)}(\nabla_U \pi)(V) + \frac{1}{2}(\nabla_V \pi)(U) - \frac{n^2}{(n+1)^2}\pi(U)\pi(V)$$

for all $U, V, W \in \chi(M)$ [1].

1.2. Tangent bundle. In this subsection, some basic facts will be repeated about tangent bundles of differentiable manifolds that will be used in the future (for more information see [11]). Given an n-dimensional Riemannian manifold (M, g), the tangent bundle TM of the manifold M is $TM = \cup_{p \in M} T_p M$, where $T_p M$ denotes the tangent space of M at p. The manifold TM is of dimension $2n$ and has a differentiable structure induced from the manifold M by the canonical projection $\zeta \colon TM \to M$.

Let f, U, ω and F be a function, a vector field, a 1-form and a $(1,1)$-type tensor field on M, respectively. We will denote their vertical lifts by f^v, U^v, ω^v, F^v and their complete lifts by f^c, U^c, ω^c, F^c to tangent bundle TM. The following relations are satisfied:

$$\begin{gathered} (fU)^v = f^v U^v, \quad U^v f^v = 0, \quad F^v(U^v) = 0, \quad \omega^v(U^v) = 0, \\ (fU)^c = f^c U^v + f^v U^c, \quad U^c f^c = (Uf)^c, \quad F^c(U^c) = (F(U))^c, \\ \omega^c(U^c) = (\omega(U))^c, \quad U^v(f^c) = U^c(f^v) = (Uf)^v, \quad F^c(U^v) = (F(U))^v, \\ F^v(U^c) = (F(U))^v, \quad \omega^v(U^c) = \omega^c(U^v) = (\omega(U))^v. \end{gathered} \tag{1.7}$$

For the Lie brackets, we have

$$[U^v, V^v] = 0, \quad [U^v, V^c] = [U,V]^v, \quad [U^c, V^c] = [U,V]^c.$$

On the other hand, it is known that the complete lift metric g^c of the metric g is a semi-Riemannian metric on TM and it is defined by

$$\begin{aligned} & g^c(U^v, V^c) = g^c(U^c, V^v) = (g(U,V))^v, \\ & g^c(U^v, V^v) = 0, \\ & g^c(U^c, V^c) = (g(U,V))^c \text{ for all } U, V \in \chi(M). \end{aligned} \tag{1.8}$$

The complete lift connection ∇^c of a linear connection ∇ satisfies the following relations

$$(1.9) \qquad \nabla^c_{U^v} V^v = (\nabla_U V)^v, \nabla^c_{U^c} V^c = (\nabla_U V)^c, \nabla^c_{U^v} V^c = \nabla^c_{U^c} V^v = (\nabla_U V)^v.$$

For some significant recent papers about lifting theory on tangent bundles, we may refer to [12–15].

We recall the following proposition from [11], p. 62.

Proposition 1. *Let (M, g) be a Riemannian manifold and TM its tangent bundle equipped with the complete lift metric g^c. Then (TM, g^c) is a space with constant scalar curvature* 0.

From (1.4) and Proposition 1, we have

Proposition 2. *Let (M, g) be an n-dimensional Riemannian manifold endowed with the semi-symmetric metric connection $\tilde{\nabla}$ defined by* (1.3)*, and let TM be its tangent bundle with the complete lift metric g^c and the complete lift of the semi-symmetric metric connection $\tilde{\nabla}^c$. Then, the scalar curvature $\tilde{r}^c$ fulfills*

$$(1.10) \qquad \tilde{r}^c = -2(n-1)a^c.$$

Similarly, from (1.6) and Proposition 1, we have

Proposition 3. *Let (M, g) be an n-dimensional Riemannian manifold endowed with the projective semi-symmetric connection $\bar{\nabla}$ defined by* (1.5)*, and let TM be its tangent bundle with the complete lift metric g^c and the complete lift of the projective semi-symmetric connection $\bar{\nabla}^c$. Then, the scalar curvature $\overline{r}^c$ fulfills*

$$(1.11) \qquad \overline{r}^c = b^c - (n-1)c^c,$$

where $b = Tr(\theta)$, $c = Tr(\omega)$.

In this paper, we study conformal Yamabe solitons on tangent bundles using vertical and complete lifts of torqued vector fields and considering the complete lifts of a semi-symmetric metric connection and a projective semi-symmetric connection. Similar problems related with Ricci solitons were examined in [16] with respect to a semi-symmetric metric connection.

2. Conformal Yamabe solitons on tangent bundles.

Case 1. The potential is a vertical lift of a torqued vector field. First, we deal with the complete lift connection ∇^c.

Theorem 1. *Let (M, g, ∇) be a Riemannian manifold with a torqued vector field τ. Then, $(TM, g^c, \nabla^c, \tau^v, \grave{\lambda}, p)$ is a conformal Yamabe soliton if τ is a Killing vector field on (M, g) and $\grave{\lambda} = \frac{1}{2}\left(p + \frac{1}{n}\right)$.*

Proof. First we compute $(\mathfrak{L}_{\tau^v} g^c)(\tilde{U}, \tilde{V})$ for all vector fields $\tilde{U}, \tilde{V}$ on TM.

From (1.2) and (1.9), we get

$$(2.1)\qquad \begin{aligned}(\mathfrak{L}_{\tau^v} g^c)(U^c, V^c) &= [g(\nabla_U \tau, V) + g(\nabla_V \tau, U)]^v \\ &= [2\phi g(U,V) + \alpha(U) g(V,\tau) + \alpha(V) g(U,\tau)]^v, \\ (\mathfrak{L}_{\tau^v} g^c)(U^c, V^v) &= 0.\end{aligned}$$

If $(TM, g^c, \nabla^c, \tau^v, \grave{\lambda}, p)$ is a conformal Yamabe soliton, then

$$(\mathfrak{L}_{\tau^v} g^c)(U^c, V^c) + \left[2\grave{\lambda} - 2r^c - \left(p + \frac{1}{n}\right)\right] g^c(U^c, V^c) = 0.$$

Using (2.1) and Proposition 1, we get

$$[2\phi g(U,V) + \alpha(U) g(V,\tau) + \alpha(V) g(U,\tau)]^v + \left[2\grave{\lambda} - \left(p + \frac{1}{n}\right)\right] (g(U,V))^c = 0.$$

This equation implies

$$(2.2)\qquad \phi g(U,V) = -\frac{1}{2}(\alpha(U) g(V,\tau) + \alpha(V) g(U,\tau)), \quad \grave{\lambda} = \frac{1}{2}\left(p + \frac{1}{n}\right).$$

Taking contraction over U and V in (2.2), we obtain

$$\phi n = -\alpha(\tau).$$

Since for a torqued vector field $\alpha(\tau) = 0$, we find $\phi = 0$. Hence, τ is a recurrent vector field. On the other hand, from [7], we know that a torqued vector field τ on a Riemannian manifold M is a Killing vector field if and only if τ is a recurrent vector field satisfying $\nabla_U \tau = \gamma(U)\tau$ and $\gamma(\tau) = 0$, where γ is a 1-form. This completes the proof. □

Secondly, we consider the complete lift of the semi-symmetric metric connection $\tilde{\nabla}^c$.

Proposition 4. *Let (M, g, ∇, τ) be a Riemannian manifold with a torqued vector field τ. Then, $(TM, g^c, \tilde{\nabla}^c, \tau^v, \tilde{\lambda})$ is a conformal Yamabe soliton if the following conditions hold*

$$\phi = \frac{1-n}{n}\pi(\tau) \quad \text{and} \quad \tilde{\lambda} = \frac{1}{2}\left(\left(p + \frac{1}{n}\right) - 4(n-1)a^c\right).$$

Proof. We compute $(\widetilde{\mathfrak{L}}_{\tau^v} g^c)(\tilde{U}, \tilde{V})$ for all vector fields $\tilde{U}$, $\tilde{V}$ on TM. From (1.3) and (1.9), we get

$$(2.3)\qquad \begin{aligned}(\widetilde{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^c) &= [g(\tilde{\nabla}^c_U \tau, V) + g(\tilde{\nabla}^c_V \tau, U)]^v \\ &= [2\phi g(U,V) + \alpha(U) g(V,\tau) + \alpha(V) g(U,\tau) \\ &\quad + 2\pi(\tau) g(U,V) - (g(U,\tau)\pi(V) + g(V,\tau)\pi(U))]^v, \\ (\widetilde{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^v) &= 0.\end{aligned}$$

If $(TM, g^c, \tilde{\nabla}^c, \tau^v, \tilde{\lambda})$ is a conformal Yamabe soliton, we have

$$(\widetilde{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^c) + \left[2\tilde{\lambda} - 2\widetilde{r}^c - \left(p + \frac{1}{n}\right)\right] g^c(U^c, V^c) = 0.$$

Using (2.3) and Proposition 2, we obtain

$$[2\phi g(U,V) + \alpha(U)g(V,\tau) + \alpha(V)g(U,\tau) + 2\pi(\tau)g(U,V) - (g(U,\tau)\pi(V) + g(V,\tau)\pi(U))]^v + \left[2\tilde{\lambda} + 4(n-1)a^c - \left(p + \frac{1}{n}\right)\right](g(U,V))^c = 0.$$

This equation implies

$$(2.4)\qquad \begin{aligned} 2\phi g(U,V) + \alpha(U)g(V,\tau) + \alpha(V)g(U,\tau) + 2\pi(\tau)g(U,V) \\ - (g(U,\tau)\pi(V) + g(V,\tau)\pi(U)) = 0, \\ 2\tilde{\lambda} + 4(n-1)a^c - \left(p + \frac{1}{n}\right) = 0. \end{aligned}$$

Taking contraction over U and V in (2.4), we obtain

$$2\phi n + 2\alpha(\tau) + 2(n-1)\pi(\tau) = 0.$$

Since for a torqued vector field $\alpha(\tau) = 0$, we find $\phi = \frac{1-n}{n}\pi(\tau)$. This finishes the proof. □

Finally, we study the complete lift of the projective semi-symmetric connection $\bar{\nabla}^c$.

Proposition 5. *Let (M, g, ∇, τ) be a Riemannian manifold with a torqued vector field τ. Then, $(TM, g^c, \bar{\nabla}^c, \tau^v, \bar{\lambda})$ is a conformal Yamabe soliton if the following conditions hold*

$$\phi = \frac{1-n}{n}\pi(\tau) \quad \textit{and} \quad \bar{\lambda} = \frac{1}{2}\left(\left(p + \frac{1}{n}\right) + 2(b^c - (n-1)c^c)\right).$$

Proof. We compute $(\overline{\mathfrak{L}}_{\tau^v} g^c)(\tilde{U}, \tilde{V})$ for all vector fields $\tilde{U}, \tilde{V}$ on TM. From (1.5) and (1.9), we obtain

(2.5)
$$\begin{aligned} (\overline{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^c) &= [g(\overline{\nabla}_U \tau, V) + g(\overline{\nabla}_V \tau, U)]^v \\ &= \left[2\left(\phi + \frac{n}{n+1}\pi(\tau)\right)g(U,V) + \left(\alpha(U) - \frac{1}{n+1}\pi(U)\right)g(V,\tau) \right. \\ &\quad \left. + \left(\alpha(V) - \frac{1}{n+1}\pi(V)\right)g(U,\tau)\right]^v, \end{aligned}$$
$$(\overline{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^v) = 0.$$

If $(TM, g^c, \bar{\nabla}^c, \tau^v, \bar{\lambda})$ is a conformal Yamabe soliton, then

$$(\bar{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^c) + \left[2\bar{\lambda} - 2\bar{r}^c - \left(p + \frac{1}{n}\right)\right] g^c(U^c, V^c) = 0.$$

Using (2.5) and Proposition 3, we get

$$\begin{aligned}\Bigg[2\left(\phi + \frac{n}{n+1}\pi(\tau)\right) g(U,V) + \left(\alpha(U) - \frac{1}{n+1}\pi(U)\right) g(V,\tau) \\ + \left(\alpha(V) - \frac{1}{n+1}\pi(V)\right) g(U,\tau)\Bigg]^v \\ + \left[2\bar{\lambda} - 2(b^c - (n-1)c^c) - \left(p + \frac{1}{n}\right)\right](g(U,V))^c = 0.\end{aligned}$$

This equation implies

$$\begin{aligned}(2.6)\qquad & 2\left(\phi + \frac{n}{n+1}\pi(\tau)\right) g(U,V) + \left(\alpha(U) - \frac{1}{n+1}\pi(U)\right) g(V,\tau) \\ & + \left(\alpha(V) - \frac{1}{n+1}\pi(V)\right) g(U,\tau) = 0, \\ & 2\bar{\lambda} - 2(b^c - (n-1)c^c) - \left(p + \frac{1}{n}\right) = 0.\end{aligned}$$

Taking contraction over U and V in (2.6), we obtain

$$\left(\phi + \frac{n}{n+1}\pi(\tau)\right) n = \frac{1}{n+1}(\pi(\tau) - \alpha(\tau)).$$

Since for a torqued vector field $\alpha(\tau) = 0$, we find $\phi = \frac{1-n}{n}\pi(\tau)$. This finishes the proof. □

Case 2. The potential is a complete lift of a torqued vector field. In this last subsection, we repeat the previous process when the potential is considered the complete lift of a torqued vector field.

Proposition 6. *Let (M, g, ∇, τ) be a Riemannian manifold with a torqued vector field τ. Then, $(TM, g^c, \nabla^c, \tau^c, \grave{\lambda}, p)$ is a conformal Yamabe soliton if the constant $\grave{\lambda}$ satisfies*

$$\grave{\lambda} = \frac{1}{2}\left(-2\phi + p + \frac{1}{n}\right).$$

Proof. First we compute $(\mathfrak{L}_{\tau^c} g^c)(\tilde{U}, \tilde{V})$ for all vector fields $\tilde{U}$, $\tilde{V}$ on TM

$$\begin{aligned}(2.7)\qquad (\mathfrak{L}_{\tau^c} g^c)(U^c, V^c) &= [g(\nabla_U \tau, V) + g(\nabla_V \tau, U)]^c \\ &= [2\phi g(U,V) + \alpha(U) g(V,\tau) + \alpha(V) g(U,\tau)]^c, \\ (\mathfrak{L}_{\tau^c} g^c)(U^c, V^v) &= [g(\nabla_U \tau, V) + g(\nabla_V \tau, U)]^v \\ &= [2\phi g(U,V) + \alpha(U) g(V,\tau) + \alpha(V) g(U,\tau)]^v.\end{aligned}$$

If $(TM, g^c, \nabla^c, \tau^c, \grave{\lambda}, p)$ is a conformal Yamabe soliton, then

$$(\mathfrak{L}_{\tau^c} g^c)(U^c, V^c) + \left[2\grave{\lambda} - 2r^c - \left(p + \frac{1}{n}\right)\right] g^c(U^c, V^c) = 0.$$

Using (2.7) and Proposition 1, we obtain

$$[2\phi g(U, V) + \alpha(U) g(V, \tau) + \alpha(V) g(U, \tau)]^c + \left[2\grave{\lambda} - \left(p + \frac{1}{n}\right)\right] (g(U, V))^c = 0.$$

This equation implies

$$2\phi g(U, V) + \alpha(U) g(V, \tau) + \alpha(V) g(U, \tau) + \left(2\grave{\lambda} - \left(p + \frac{1}{n}\right)\right) (g(U, V))^c = 0. \tag{2.8}$$

Taking contraction over U and V in (2.8), we obtain

$$2\phi n + 2\alpha(\tau) + \left(2\grave{\lambda} - \left(p + \frac{1}{n}\right)\right) n = 0.$$

Since for a torqued vector field $\alpha(\tau) = 0$, we find $\grave{\lambda} = \frac{1}{2}\left(-2\phi + p + \frac{1}{n}\right)$. This completes the proof. □

Proposition 7. *Let (M, g, ∇, τ) be a Riemannian manifold with a torqued vector field τ. Then, $(TM, g^c, \tilde{\nabla}^c, \tau^c, \tilde{\lambda})$ is a conformal Yamabe soliton if the constant $\tilde{\lambda}$ satisfies*

$$\tilde{\lambda} = \frac{-1}{n}[n\phi + (n-1)\pi(\tau)] - \frac{1}{2}\left[4(n-1)a^c + p + \frac{1}{n}\right].$$

Proof. We compute $(\widetilde{\mathfrak{L}}_{\tau^c} g^c)(\tilde{U}, \tilde{V})$ for all vector fields $\tilde{U}$, $\tilde{V}$ on TM.

$$\begin{aligned}(\widetilde{\mathfrak{L}}_{\tau^c} g^c)(U^c, V^c) &= [g(\tilde{\nabla}_U \tau, V) + g(\tilde{\nabla}_V \tau, U)]^c \\ &= [2\phi g(U, V) + \alpha(U) g(V, \tau) + \alpha(V) g(U, \tau) \\ &\quad + 2\pi(\tau) g(U, V) - (g(U, \tau)\pi(V) + g(V, \tau)\pi(U)]^c, \\ (\widetilde{\mathfrak{L}}_{\tau^c} g^c)(U^c, V^v) &= [g(\tilde{\nabla}_U \tau, V) + g(\tilde{\nabla}_V \tau, U)]^v \\ &= [2\phi g(U, V) + \alpha(U) g(V, \tau) + \alpha(V) g(U, \tau) \\ &\quad + 2\pi(\tau) g(U, V) - (g(U, \tau)\pi(V) + g(V, \tau)\pi(U))]^v.\end{aligned} \tag{2.9}$$

If $(TM, g^c, \tilde{\nabla}^c, \tau^c, \tilde{\lambda})$ is a conformal Yamabe soliton, then

$$(\widetilde{\mathfrak{L}}_{\tau^c} g^c)(U^c, V^c) + \left[2\tilde{\lambda} - 2\tilde{r}^c - \left(p + \frac{1}{n}\right)\right] g^c(U^c, V^c) = 0. \tag{2.10}$$

Using (2.9) and Proposition 2, we get

$$[2\phi g(U,V)+\alpha(U)g(V,\tau)+\alpha(V)g(U,\tau)+2\pi(\tau)g(U,V)-(g(U,\tau)\pi(V) + g(V,\tau)\pi(U))]^c+\left[2\tilde{\lambda}+4(n-1)a^c-\left(p+\frac{1}{n}\right)\right](g(U,V))^c=0.$$

This equation implies

$$2\phi g(U,V)+\alpha(U)g(V,\tau)+\alpha(V)g(U,\tau)+2\pi(\tau)g(U,V)-(g(U,\tau)\pi(V) + g(V,\tau)\pi(U))+\left(2\tilde{\lambda}+4(n-1)a^c-\left(p+\frac{1}{n}\right)\right)g(U,V)=0. \tag{2.11}$$

Taking contraction over U and V in (2.11), we obtain

$$2\phi n+2\alpha(\tau)+2(n-1)\pi(\tau)+\left[2\tilde{\lambda}+4(n-1)a^c-\left(p+\frac{1}{n}\right)\right]n=0.$$

Since for a torqued vector field $\alpha(\tau)=0$, we find $\tilde{\lambda}=\frac{-1}{n}[n\phi+(n-1)\pi(\tau)]-\frac{1}{2}\left[4(n-1)a^c+p+\frac{1}{n}\right]$. This finishes the proof. □

Proposition 8. *Let (M,g,∇,τ) be a Riemannian manifold with a torqued vector field τ. Then, $(TM,g^c,\bar{\nabla}^c,\tau^c,\bar{\lambda})$ is a conformal Yamabe soliton if the constant $\bar{\lambda}$ satisfies*

$$\bar{\lambda}=\frac{-1}{2n}\left[2\left(\phi+\frac{n}{n+1}\pi(\tau)\right)n-\frac{1}{n+1}\pi(\tau)\right]+\frac{1}{2}\left[2(b^c-(n-1)c^c)+p+\frac{1}{n}\right].$$

Proof. We compute $(\overline{\mathfrak{L}}_{\tau^c}g^c)(\tilde{U},\tilde{V})$ for all vector fields $\tilde{U}$, $\tilde{V}$ on TM.

(2.12)

$$\begin{aligned}(\overline{\mathfrak{L}}_{\tau^c}g^c)(U^c,V^c)&=[g(\overline{\nabla}_U\tau,V)+g(\overline{\nabla}_V\tau,U)]^c\\&=\left[2(\phi+\frac{n}{n+1}\pi(\tau))g(U,V)+\left(\alpha(U)-\frac{1}{n+1}\pi(U)\right)g(V,\tau)\right.\\&\quad\left.+\left(\alpha(V)-\frac{1}{n+1}\pi(V)\right)g(U,\tau)\right]^c,\\(\overline{\mathfrak{L}}_{\tau^c}g^c)(U^c,V^v)&=[g(\overline{\nabla}_U\tau,V)+g(\overline{\nabla}_V\tau,U)]^v\\&=\left[2\left(\phi+\frac{n}{n+1}\pi(\tau)\right)g(U,V)+\left(\alpha(U)-\frac{1}{n+1}\pi(U)\right)g(V,\tau)\right.\\&\quad\left.+\left(\alpha(V)-\frac{1}{n+1}\pi(V)\right)g(U,\tau)\right]^v.\end{aligned}$$

If $(TM,g^c,\bar{\nabla}^c,\tau^c,\bar{\lambda})$ is a conformal Yamabe soliton, then

$$(\overline{\mathfrak{L}}_{\tau^c}g^c)(U^c,V^c)+\left[2\bar{\lambda}-2\overline{r}^c-\left(p+\frac{1}{n}\right)\right]g^c(U^c,V^c)=0.$$

Using (2.12) and Proposition 3, we find

$$\left[2\left(\phi+\frac{n}{n+1}\pi(\tau)\right)g(U,V)+\left(\alpha(U)-\frac{1}{n+1}\pi(U)\right)g(V,\tau)\right.$$
$$\left.+\left(\alpha(V)-\frac{1}{n+1}\pi(V)\right)g(U,\tau)\right]^c$$
$$+\left[2\bar{\lambda}-2(b^c-(n-1)c^c)-\left(p+\frac{1}{n}\right)\right](g(U,V))^c=0.$$

This equation implies

$$(2.13)\quad 2\left(\phi+\frac{n}{n+1}\pi(\tau)\right)g(U,V)+\left(\alpha(U)-\frac{1}{n+1}\pi(U)\right)g(V,\tau)$$
$$+\left(\alpha(V)-\frac{1}{n+1}\pi(V)\right)g(U,\tau)+\left[2\bar{\lambda}-2(b^c-(n-1)c^c)-\left(p+\frac{1}{n}\right)\right]g(U,V)=0.$$

Taking contraction over U and V in (2.13), we obtain

$$\bar{\lambda}=\frac{-1}{2n}\left[2\left(\phi+\frac{n}{n+1}\pi(\tau)\right)n+\alpha(\tau)-\frac{1}{n+1}\pi(\tau)\right]$$
$$+\frac{1}{2}\left[2(b^c-(n-1)c^c)+p+\frac{1}{n}\right].$$

Since for a torqued vector field $\alpha(\tau)=0$, we find

$$\bar{\lambda}=\frac{-1}{2n}\left[2\left(\phi+\frac{n}{n+1}\pi(\tau)\right)n-\frac{1}{n+1}\pi(\tau)\right]+\frac{1}{2}\left[2(b^c-(n-1)c^c)+p+\frac{1}{n}\right].$$

This finishes the proof. □

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Department of Mathematics
Faculty of Sciences and Arts
Erzincan Binali Yıldırım University
24100 Erzincan, Turkey
e-mail: maltunbas@erzincan.edu.tr