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REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS WITH REEB RECURRENCE CONDITION

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Abstract

In this paper we prove that the structure Lie operator of a real hypersurface in a nonflat complex space form is Reeb recurrent if and only if it is Lie Reeb recurrent, and this is equivalent to that the hypersurface is of type (A).

Key words: hypersurface, complex space form, structure Lie operator, Reeb recurrent, Lie Reeb recurrent

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1. Introduction. Let $M^n(c)$ be a complete and simply connected complex space form which is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$,
- a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$,

where c denotes the constant holomorphic sectional curvature. For convenience, a complex space from $M^n(c)$ is said to be nonflat if $c \neq 0$. A real hypersurface in $M^n(c)$ is defined as an isometrically immersion submanifold having real codimension one. On a real hypersurface in $M^n(c)$ there exists an almost contact metric structure which is denoted by (ϕ, ξ, η, g), where g is the induced metric from the ambient space. A real hypersurface is said to be Hopf when the structure vector

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field ξ is geodesic, or equivalently, ξ is an eigenvector of the shape operator A of the hypersurface everywhere. According to [1,2], a real hypersurface is said to be of type (A) if it is of type (A_1) or (A_2) in $\mathbb{C}P^n$ or of type (A_0), $(A_{1,0})$, or $(A_{1,1})$ in $\mathbb{C}H^n$.

Let L be the tensor field of type $(1,1)$ which is metrically equivalent to $\mathcal{L}_\xi g$, that is, $g(LX,Y) = (\mathcal{L}_\xi g)(X,Y)$ holds for any vector fields X, Y. According to a simple calculation, one has

$$L = \phi A - A\phi, \tag{1.1}$$

where A denotes the shape operator of the hypersurface. In general, L is called the structure Lie operator of a real hypersurface and there is a deep relationship between it and the type (A) real hypersurfaces. For example, when referring to the characterization of type (A) real hypersurfaces, the following theorem is one of the most often cited results.

Theorem 1.1 ([3,4]). *A real hypersurface in a nonflat complex space form is of type (A) if and only if the structure Lie operator vanishes identically.*

Hamada in [5] considered parallelism of L when the ambient space is $\mathbb{C}P^n(c)$, and later Yamashita in [6] generalized Hamada's results for a real hypersurface in a nonflat complex space form.

Theorem 1.2 ([5,6]). *A real hypersurface in a nonflat complex space form is of type (A) if and only if the structure Lie operator is parallel.*

Extending Theorem 1.2, the Codazzi condition for the structure Lie operator L has been considered in [7]. Also, some authors in [8,9] considered a recurrence condition for the structure Lie operator L, i.e.,

$$\nabla L = \omega \otimes L \tag{1.2}$$

for a one-form ω. Obviously, parallelism of L implies recurrence, but the converse is not valid in the general case. Following [10,11], a geometrical meaning of (1.2) is that the eigenspaces of the structure Lie operator L are parallel along any curve γ in the hypersurface, here the eigenspaces of the structure Lie operator L are said to be parallel along γ if they are invariant with respect to any parallel translation along γ.

Theorem 1.3 ([8,9]). *A real hypersurface in a nonflat complex space form is of type (A) if and only if the structure Lie operator is recurrent.*

In this paper, we aim to generalize the above three Theorems by considering Reeb recurrence condition for L, i.e.,

$$\nabla_\xi L = \omega(\xi) L \tag{1.3}$$

for certain one-form ω. Obviously, Reeb recurrence condition is much weaker than parallelism. Following again [10], a geometrical meaning of (1.3) is that the eigenspaces of the structure Lie operator L are parallel along the Reeb flow. Besides Theorems 1.2 and 1.3, Loo in [12] considered Reeb invariant condition for L, i.e.,

$$\mathcal{L}_\xi L = 0 \tag{1.4}$$

and his result was generalized by Ghosh in [13] by considering $(\mathcal{L}_\xi L)X = 0$ for any vector field X orthogonal to ξ. This leads us to investigate a weaker condition than (1.4), i.e.,

$$\mathcal{L}_\xi L = \omega(\xi)L \tag{1.5}$$

for certain one-form ω. Our main result in this paper is given as follows.

Theorem 1.4. *On a real hypersurface in a nonflat complex space form, the following statements are equivalent*:

- The hypersurface is of type (A).
- The structure Lie operator is Reeb recurrent.
- The structure Lie operator is Lie Reeb recurrent.

Clearly, Theorem 1.4 is a nice extension of Theorems 1.1–1.3.

2. Preliminaries. Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of the metric g, respectively. Then the Gauss and Weingarten formulas are given, respectively, as follows:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX \tag{2.1}$$

for any $X, Y \in \mathfrak{X}(M)$, where A denotes the shape operator of M in $M^n(c)$. For any vector field $X \in \mathfrak{X}(M)$, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi. \tag{2.2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.4}$$

for any $X, Y \in \mathfrak{X}(M)$.

Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla} J = 0$) of $M^n(c)$ and using (2.1), (2.2) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for any $X, Y \in \mathfrak{X}(M)$. From (2.6), one sees that the structure vector field ξ is principal if and only if the hypersurface is Hopf.

Let R be the Riemannian curvature tensor of M. Because $M^n(c)$ is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations of M in $M^n(c)$ are given, respectively, as follows:

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &- 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\} \tag{2.8}$$

for any $X, Y \in \mathfrak{X}(M)$.

In this paper, all manifolds are assumed to be connected and of class C^∞ and all hypersurfaces are assumed to be oriented.

3. Proofs of main theorems. First, we prove that the Reeb recurrence condition for the structure Lie operator and its vanishing condition on a real hypersurface are equivalent.

Lemma 3.1. *If the structure Lie operator of a real hypersurface in a nonflat complex space form is Reeb recurrent, then it vanishes identically.*

Proof. By assumption of this Lemma, substituting (1.1) into (1.3) yields

$$(\nabla_\xi\phi)AY + \phi((\nabla_\xi A)Y) - (\nabla_\xi A)\phi Y - A((\nabla_\xi\phi)Y) = \omega(\xi)(\phi AY - A\phi Y)$$

for any vector field Y. Putting (2.5) into the above equality gives

$$2\eta(AY)A\xi - \eta(A^2Y)\xi - \eta(Y)A^2\xi + \phi((\nabla_\xi A)Y) - (\nabla_\xi A)\phi Y = \omega(\xi)(\phi AY - A\phi Y). \tag{3.1}$$

Taking the inner product of this equality with ξ, we obtain

$$2\eta(AY)\eta(A\xi) - \eta(A^2Y) - \eta(Y)\eta(A^2\xi) - g((\nabla_\xi A)\xi, \phi Y) + \omega(\xi)\eta(A\phi Y) = 0.$$

In the above equality setting $Y = \xi$ we get

$$\eta^2(A\xi) = \eta(A^2\xi). \tag{3.2}$$

Then, the hypersurface must be Hopf. In fact, suppose the hypersurface is not Hopf, then there exists at least a point at which $A\xi - \eta(A\xi)\xi$ is not vanishing. Thus, working at this point, we are able to set

$$\beta = \|A\xi - \eta(A\xi)\xi\| \neq 0.$$

Substituting this into (3.2) gives $\eta^2(A\xi) = \|A\xi\|^2 = \eta^2(A\xi) + \beta^2$, and this implies $\beta = 0$. So we arrive at a contradiction. Now we write $A\xi = \eta(A\xi)\xi$ and use it in (3.1) to obtain

$$\phi((\nabla_\xi A)Y) - (\nabla_\xi A)\phi Y = \omega(\xi)(\phi AY - A\phi Y). \tag{3.3}$$

From [2], Theorem 2.1, in this case we see that $\eta(A\xi)$ is a constant. According to the Codazzi equation (2.8), with the aid of (2.6), setting $X=\xi$, we have

$$(\nabla_\xi A)Y = (\nabla_Y A)\xi + \frac{c}{4}\phi Y = \eta(A\xi)\phi AY - A\phi AY + \frac{c}{4}\phi Y.$$

From a direct calculation, this reduces to

$$\phi((\nabla_\xi A)Y) = -\eta(A\xi)AY + \eta^2(A\xi)\eta(Y)\xi - \phi A\phi AY + \frac{c}{4}\phi^2 Y \tag{3.4}$$

and

$$(\nabla_\xi A)\phi Y = \eta(A\xi)\phi A\phi Y - A\phi A\phi Y + \frac{c}{4}\phi^2 Y. \tag{3.5}$$

Now, substituting the above two equalities into (3.3) gives

$$\begin{aligned} &-\eta(A\xi)AY + \eta^2(A\xi)\eta(Y)\xi - \phi A\phi AY - \eta(A\xi)\phi A\phi Y + A\phi A\phi Y \\ &\qquad = \omega(\xi)(\phi AY - A\phi Y). \end{aligned} \tag{3.6}$$

Recall from [2], Lemma 2.2 that on any Hopf hypersurface, the following equality is valid

$$A\phi A = \frac{1}{2}\eta(A\xi)(A\phi + \phi A) + \frac{c}{4}\phi. \tag{3.7}$$

From (3.7), with the aid of the Hopf condition and direct calculations, we have

$$\phi A\phi A = \frac{1}{2}\eta(A\xi)\phi A\phi - \frac{1}{2}\eta(A\xi)A + \frac{1}{2}\eta^2(A\xi)\eta\otimes\xi + \frac{c}{4}\phi^2 \tag{3.8}$$

and

$$A\phi A\phi = -\frac{1}{2}\eta(A\xi)A + \frac{1}{2}\eta^2(A\xi)\eta\otimes\xi + \frac{1}{2}\eta(A\xi)\phi A\phi + \frac{c}{4}\phi^2. \tag{3.9}$$

Substituting the above two equalities (3.8) and (3.9) into (3.6), we obtain

$$\eta(A\xi)\phi A\phi Y = \eta^2(A\xi)\eta(Y)\xi - \eta(A\xi)AY - \omega(\xi)\phi AY + \omega(\xi)A\phi Y \tag{3.10}$$

for any vector field Y. Taking the action of ϕ on (3.10), with the aid of the Hopf condition and (2.3), we have

$$\omega(\xi)\phi A\phi Y = \eta(A\xi)\omega(\xi)\eta(Y)\xi + \eta(A\xi)\phi AY - \eta(A\xi)A\phi Y - \omega(\xi)AY. \tag{3.11}$$

Multiplying (3.10) by $\omega(\xi)$ yields an equality; the subtraction of this equality from (3.11) multiplied by $\eta(A\xi)$, eliminating $\phi A\phi$, we obtain

$$(\eta^2(A\xi) + \omega^2(\xi))(\phi A - A\phi) = 0.$$

Suppose there exists a point at which $\omega(\xi)$ is not zero, from the above equality we have $A\phi = \phi A$. If $\omega(\xi) = 0$ at some point, then the above equality reduces to

$$\eta^2(A\xi)(\phi A - A\phi) = 0.$$

Recall that $\eta(A\xi)$ is a constant. If it is nonzero, it follows again from the above equality that $A\phi = \phi A$. If it is zero, from (3.8) and (3.9) we have

$$A\phi A\phi = \phi A\phi A,$$

which is used in (3.1), together with (3.4) and (3.5), yielding $\nabla_\xi L = 0$. Recall that Ghosh in [13], Proposition 4.1 proved that when $(\nabla_\xi L)X = 0$ for any vector field X orthogonal to ξ, the hypersurface must be of type (A) in nonflat complex space forms. In this case, according to Theorem 1.1, we have $L = 0$. All in all, in any case, we always have $L = 0$ at each point of the hypersurface. Now this completes the proof. □

Lemma 3.2. *If the structure Lie operator of a real hypersurface in a nonflat complex space form is Lie Reeb recurrent, then it vanishes identically.*

Proof. By assumption of this Lemma, substituting (1.1) into (1.5) yields

$$\begin{aligned}(\nabla_\xi\phi)AY + \phi((\nabla_\xi A)Y) - (\nabla_\xi A)\phi Y - A((\nabla_\xi\phi)Y)\\ = \omega(\xi)(\phi AY - A\phi Y) - A^2Y + \eta(AY)A\xi - \phi A^2\phi Y\end{aligned}$$

for any vector field Y. Putting (2.5) into the above equality gives

$$\begin{aligned}2\eta(AY)A\xi - \eta(A^2Y)\xi - \eta(Y)A^2\xi + \phi((\nabla_\xi A)Y) - (\nabla_\xi A)\phi Y\\ = \omega(\xi)(\phi AY - A\phi Y) - A^2Y + \eta(AY)A\xi - \phi A^2\phi Y.\end{aligned} \tag{3.12}$$

Taking the inner product of this equality with ξ, we obtain

$$\eta(AY)\eta(A\xi) - \eta(Y)\eta(A^2\xi) - g((\nabla_\xi A)\xi, \phi Y) + \omega(\xi)\eta(A\phi Y) = 0.$$

In the above equality setting $Y = \xi$ we get

$$\eta^2(A\xi) = \eta(A^2\xi). \tag{3.13}$$

Then, the hypersurface must be Hopf. Notice that (3.13) is the same with (3.2). Here we omit the proof because it is the same as that shown in Lemma 3.1. Applying $A\xi = \eta(A\xi)\xi$ in (3.12) we have

$$\begin{aligned}\phi((\nabla_\xi A)Y) - (\nabla_\xi A)\phi Y\\ = \omega(\xi)(\phi AY - A\phi Y) - A^2Y + \eta^2(A\xi)\eta(Y)\xi - \phi A^2\phi Y.\end{aligned} \tag{3.14}$$

So, the application of (3.4) and (3.5) into (3.14) gives

$$-\eta(A\xi)AY-\phi A\phi AY-\eta(A\xi)\phi A\phi Y+A\phi A\phi Y = \omega(\xi)(\phi AY-A\phi Y)-A^2Y-\phi A^2\phi Y. \tag{3.15}$$

We remark that (3.7) is still valid in this case, and consequently so are (3.8) and (3.9). Substituting the two equalities into (3.15) we obtain

$$-\eta(A\xi)AY-\eta(A\xi)\phi A\phi Y=\omega(\xi)(\phi AY-A\phi Y)-A^2Y-\phi A^2\phi Y. \tag{3.16}$$

Recall that $\eta(A\xi)$ is a constant on a Hopf hypersurface. Let $Y\in\{\xi\}^{\perp}$ be a unit eigenvector of the shape operator, i.e., $AY=\lambda Y$. From (3.7) we have

$$(2\lambda-\eta(A\xi))A\phi Y=\frac{2\lambda\eta(A\xi)+c}{2}\phi Y. \tag{3.17}$$

From (3.16) we obtain an equality, and taking the inner product of the resulting equality with ϕY and using (2.4) we get

$$\omega(\xi)(g(AY,Y)-g(A\phi Y,\phi Y))=0. \tag{3.18}$$

If there exits a point at which $\omega(\xi)\neq 0$, by continuity there exits an open subset on which $\omega(\xi)\neq 0$. Working on this subset, the combination of (3.18) and (3.17) gives

$$\lambda^2-\eta(A\xi)\lambda-\frac{c}{4}=0. \tag{3.19}$$

According to this, we observe that λ must be a constant. So, the hypersurface is a Hopf hypersurface with constant principal curvatures, and the number of distinct principal curvatures is two or three with the principal subspaces being ϕ-invariant. Following [1], Theorem 6.19, we obtain $L=0$. In fact, checking all possibilities of the principal curvatures satisfying (3.19) for Hopf hypersurfaces with constant principal curvatures in [1,2], we observe that the hypersurface must be of type (A). On the other hand, when $\omega(\xi)=0$, (1.5) is nothing but a Reeb invariant condition and in this case the proof has been proved by Loo in [12], Proposition 4.1 (see also [13]). All in all, we always have $L=0$ at each point of the hypersurface and this completes the proof. □

Proof of Theorem 1.4. The proof follows immediately from Lemmas 3.1 and 3.2 and Theorem 1.1. □

According to Theorem 1.4, Reeb recurrence and recurrence conditions for the structure Lie operator are equivalent. The situation is much different when the structure Lie operator was replaced by some other operators. We refer the reader for some results related to the Reeb recurrence and recurrences conditions for shape operators in [14], for Ricci operators in [15,16] and for the structure Jacobi operators in [17].

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