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FINITE GROUPS WHOSE NUMBERS OF REAL-VALUED CHARACTER DEGREES OF ALL PROPER SUBGROUPS ARE AT MOST TWO

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Abstract

Finite groups with real-valued irreducible characters of prime degree are classified by Dolfi, Pacifici and Sanus. In this paper, the structures of finite groups whose all proper subgroups have at most two real-valued-irreducible-character degrees are determined.

Key words: simple group, character value, proper subgroup
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1. Introduction. All groups G that are considered are finite. The set of complex irreducible characters of G is denoted by $\mathrm{Irr}(G)$ and the set of character degrees of G is written by $\mathrm{cd}(G)$. The groups with two irreducible character degrees are classified; see [1,2]. As a generalization of [1,2], Noritzsch in [3] considered finite groups with three complex irreducible character degrees.

In the following, the **degree** means the **real-valued irreducible character degree**.

Finite groups with few character values are classified by some scholars; see [4–6] for example. In particular, Iwasaki in [7] studied the structure of a finite group G concerning the number of real-valued irreducible characters and characterized the structure of finite groups with exactly two real-valued irreducible characters.

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Let $\mathrm{Irr}_{\mathrm{rv}}(G)$ be the set of real-valued irreducible characters of a group G and let $\mathrm{cd}_{\mathrm{rv}}(G)$ be the set of real-valued-irreducible-character degrees, i.e.

$$\mathrm{cd}_{\mathrm{rv}}(G) = \{\chi(1) : \chi \in \mathrm{Irr}_{\mathrm{rv}}(G)\}.$$

Recently, the author and others considered the relation between the degrees of proper subgroups and group structure; see [8] and [9] for instance. We also consider the impact of the properties of the proper subgroups on the structure of finite group. Let $\sum G$ be the set of all proper subgroups of a group G. Then $\mathrm{cd}_{\mathrm{rv}}(G)$ is a subset of $\mathrm{cd}(G)$. Obviously an abelian group satisfies $\mathrm{cd}_{\mathrm{rv}}(G) = \mathrm{cd}(G)$. In this paper, we first consider the influence of the number of degrees on the structure of finite groups. To say in brief, we introduce the following concept.

Definition 1.1. A group G is called a TR-group if $|\mathrm{cd}_{\mathrm{rv}}(G)| \leq 2$.

Finite groups with $\mathrm{cd}(G) = \{1, m\}$ are classified ([10], Theorem 12.5). Corresponding to [1,2,10], we first show the following result.

Theorem 1.2. *A finite* TR*-group is solvable.*

In order to argue in short, we introduce the following definition.

Definition 1.3. A group G is called a PTR-group if each $H \in \sum G$ is a TR-group.

Now we have the following result.

Theorem 1.4. *Let G be a non-solvable* PTR*-group. Then G is isomorphic to either $\mathrm{PSL}_2(q)$, where either $q = 2^p$ with p a prime or $q = 3^p$ with p an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$ or $Sz(2^p)$ with p an odd prime.*

2. Basic results. In this section, some elementary results are collected.

Lemma 2.1. *Let G be a* PTR*-group.*

(1) *Let $N \in \sum G$. Then N is a* TR*-group.*

(2) *If $1 \neq N \in \sum G$ is normal in G, then G/N is a* TR*-group.*

Proof. It is easy to get from Definition of a PTR-group and the fact $\mathrm{cd}_{\mathrm{rv}}(G/N) \subset \mathrm{cd}_{\mathrm{rv}}(G)$. □

The following result can be used to shorten the proof of main theorem.

Lemma 2.2 ([9], Corollary 1)**.** *Every minimal simple group is isomorphic to one of the following minimal simple groups:*

(1) $\mathrm{PSL}_2(2^p)$ *for p a prime;*

(2) $\mathrm{PSL}_2(3^p)$ *for p an odd prime;*

(3) $\mathrm{PSL}_2(p)$*, for p any prime exceeding* 3 *such that $p^2 + 1 \equiv 0 \pmod{5}$;*

(4) $Sz(2^p)$ *for p an odd prime;*

(5) $\mathrm{PSL}_3(3)$.

Lemma 2.3. *Let G be a Frobenius group of the form $E_{p^n} : C_m$ with kernel E_{p^n}, the elementary abelian p-group of order p^n and complement C_m, the cyclic group of order m. Assume that $|G|$ is even, then G has an irreducible real character of degree m.*

Proof. It follows from Theorem 13.9(b) of [11]. □

Lemma 2.4. *Let G be a dihedral group D_{2n}. Then G is a* TR*-group.*

Proof. It is easy to see from the fact $\mathrm{cd}(G) = \{1, 2\}$. □

Lemma 2.5. *Let G be a finite group of even order.*

(1) *If $\exp(G, d)$ is odd, then at least one of the $\chi \in \mathrm{Irr}(G)$ of degree d is real. In particular, if $\exp(G, d) = 1$, then the $\chi \in \mathrm{Irr}(G)$ of degree d is real.*

(2) *If $\exp(G, d)$ is even, then for $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = d$, there is a pair $(\chi, \overline{\chi})$ which is either real or non-real.*

Proof. It follows from [12]. □

3. Solvability of TR-groups. In this section we will show the proof of Theorem 1.2.

Let $\mathrm{cd}_v(G) = \{[\chi_1, \chi_1(1)], \chi_2, \chi_2(1)], \cdots, [\chi_s, \chi_s(1)]\}$, where $\chi_i \in \mathrm{Irr}_{\mathrm{rv}}(G)$.

To prove Theorem 1.2, the following information is taken from [13].

Lemma 3.1. (1) *Let $s = p^n$, $p > 2$. Then*

if $s \equiv -1 \pmod 4$,

$$\begin{aligned}\mathrm{cd}_v(\mathrm{PSL}_2(s)) &= \{[\chi_\lambda, 1], [\chi_\mu, s], [\chi(R^a), s+1], [\chi(S^b), s-1]\},\\ \mathrm{cd}_v(\mathrm{SL}_2(s)) &= \{[\chi_\lambda, 1], [\chi_l, s], [\chi_m, s+1], [\chi_n, s+1],\\ &\qquad [\chi(R^a), s-1], [\chi(S^b), s-1]\};\end{aligned}$$

if $s \equiv 1 \pmod 4$,

$$\begin{aligned}\mathrm{cd}_v(\mathrm{PSL}_2(s)) &= \{[\chi_\lambda, 1], [\chi_\mu, s], \left[\chi_\mu, \frac{s+1}{2}\right], \left[\chi_\mu, \frac{s-1}{2}\right], [\chi(R^a), s+1],\\ &\qquad [\chi(S^b), s-1]\},\\ \mathrm{cd}_v(\mathrm{SL}_2(s)) &= \{[\chi_\lambda, 1], [\chi_\mu, s], [\chi_l, s], \left[\chi_\mu, \frac{s+1}{2}\right], \left[\chi_v, \frac{s-1}{2}\right], [\chi_m, s+1],\\ &\qquad [\chi_n, s+1], [\chi(R^a), s-1], [\chi(S^b), s-1]\}.\end{aligned}$$

(2) *Let $s = 2^n$. Then*

$$\mathrm{cd}_v(\mathrm{PSL}_2(s)) = \{[\chi_\lambda, 1], [\lambda_0, s], [\chi(R^a), s+1], [\chi(S^b), s-1]\}.$$

Proof. It follows from ([13], p. 401–403). □

Let $\exp(G, d)$ be the number of irreducible characters with the same degree d. In order to read at hand, we rewrite Theorem 1.2 here.

Theorem 3.2. *A finite* TR*-group is solvable.*

Proof. Assume that the result is not true, then we can assume that G is non-sovable, but its maximal subgroups are solvable. Now Lemma 2.2 gives the possibilities for G. Thus the following three cases are done with.

Case 1: $\mathrm{PSL}_2(q)$ for $q \geq 4$.

By Lemma 3.1, we have that $|\mathrm{cd}_{\mathrm{rv}}(G)| > 2$, a contradiction.

Case 2: $Sz(2^p)$ for p an odd prime. Let $q = 2^p$. By [14], $\exp(Sz(q), 1) = 1$, $\exp(Sz(q), q^2) = 1$ and $\exp(Sz(q), q^2 + 1) = \dfrac{q-2}{2}$. Note that $\dfrac{q-2}{2}$ is odd, so by Lemma 2.5, the irreducible characters χ_i, $i = 1, 2, 3$ are real-valued, where $\chi(1) = 1$, $\chi_2(1) = q^2$, $\chi_3(1) = q^2 + 1$. So $Sz(q)$ is a non-TR-group.

Case 3: $\mathrm{PSL}_3(3)$.

From ([15], p. 13), we have that χ_1, χ_2, χ_3 are real-valued characters with $\chi_1(1) = 1$, $\chi_2(1) = 12$ and $\chi_3(1) = 13$. So $\mathrm{PSL}_3(3)$ is a non-TR-group.

It follows from the above three cases that G is solvable. □

4. Non-solvable PTR-groups. In this section, the structures of PTR-groups are determined. Denote by $\max G$ the set of all maximal subgroups of G with respect to its subgroup order divisibility.

The following information is used [15,16].

Lemma 4.1. $\mathrm{cd}_v(A_4) = \{[\chi_1, 1], [\chi_4, 3]\}$;

$\mathrm{cd}_v(S_4) = \{[\chi_1, 1], [\chi_2, 1], [\chi_3, 2], [\chi_4, 3], [\chi_5, 3]\}$;

$\mathrm{cd}_v(A_5) = \{[\chi_1, 1], [\chi_2, 3], [\chi_3, 3], [\chi_4, 4], [\chi_5, 5]\}$.

Lemma 4.2. *Let G be a non-abelian* PTR*-group. Then one of the following holds:*

(1) *G is isomorphic to $\mathrm{PSL}_2(q)$, where either $q = 2^p$ with p a prime or $q = 3^p$ with p an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod 8$;*

(2) *G is isomorphic to $Sz(2^p)$ with p an odd prime.*

Proof. By Lemma 2.1, $H \in \sum G$, H is a TR-group, so by Lemma 1.2, H is solvable. It follows that G is a non-solvable group but its subgroups are solvable. So by Lemma 2.2, the following five cases are considered.

T a b l e 1

$\mathrm{PSL}_2(q)$, $q \geq 5$ ([17], Chap. II, Theorem 8.27)

	$\max \mathrm{PSL}_2(q)$	Condition
$\mathcal{C}_1$	$E_q : C_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$\mathcal{C}_2$	$D_{2(q-1)/k}$	$q \notin \{5, 7, 9, 11\}$
$\mathcal{C}_3$	$D_{2(q+1)/k}$	$q \notin \{7, 9\}$
$\mathcal{C}_5$	$\mathrm{PSL}_2(q_0).(k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
$\mathcal{C}_6$	S_4	$q = p \equiv \pm 1 \pmod 8$
	A_4	$q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$
$\mathcal{S}$	A_5	$q \equiv \pm 1 \pmod{10}, F_q = F_p[\sqrt{5}]$

Case 1: $\mathrm{PSL}_2(2^p)$ for p a prime.

Then $k = 1$ and let $q = 2^p$,

$$\max \mathrm{PSL}_2(q) = \{E_q : C_{q-1}, D_{2(q\pm1)}\}.$$

We see from Lemmas 2.3 and 2.4 that $E_q : C_{q-1}$ and $D_{2(q\pm1)}$ are TR-groups, so $\mathrm{PSL}_2(q)$ is a PTR-group and G is isomorphic to $\mathrm{PSL}_2(q)$.

Case 2: $\mathrm{PSL}_2(3^p)$ for p an odd prime.

Then $k = 2$. Let $q = 3^p$. Then $\max \mathrm{PSL}_2(q)$ contains possibly $E_q : C_{(q-1)/2}$, $D_{q\pm1}$, $\mathrm{PSL}_2(3)$ and A_5 as its elements. Notice that $\mathrm{PSL}_2(3) \cong A_4$, so Lemmas 2.3, 2.4 and 4.1 force that $E_q : C_{(q-1)/2}$, $D_{q\pm1}$ and $\mathrm{PSL}_2(3)$ are TR-groups but A_5 is a non-TR-group. It follows from Table 1 that $\mathrm{PSL}_2(q)$ is a PTR-group with $q \not\equiv \pm1 \pmod{10}$.

Case 3: $\mathrm{PSL}_2(p)$, for p any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod 5$.

Then $k = 2$ and $q = p$. By Table 1, $\max \mathrm{PSL}_2(q)$ has possibly $E_q : C_{(q-1)/2}$, $D_{q\pm1}$, S_4 and A_4 as its members. By Lemmas 2.3, 2.4 and 4.1, $E_q : C_{(q-1)/2}$ and $D_{q\pm1}$ are TR-group but S_4 is a non-TR-group. By Table 1, we have $q = p \not\equiv \pm1 \pmod 8$, so $q = p \equiv 3, 5, 7 \pmod 8$. Hypothesis $p^2 + 1 \equiv 0 \pmod 5$ gives that $q = p \equiv \pm3 \pmod 8$. Thus similarly as Case 1, we have that $\mathrm{PSL}_2(q)$ is a PTR-group with $q = p \equiv \pm3 \pmod 8$.

Case 4: $Sz(2^p)$ for p an odd prime.

Let $q = 2^p$. Then from ([18], p. 385), we have $\max Sz(q) = \{E_q^{1+1} : C_{q-1}, D_{2(q-1)}, (q \pm \sqrt{2q} + 1) : 4\}$. Note that

$$\mathrm{cd}_{\mathrm{rv}}(E_q^{1+1} : C_{q-1}) = \{1, q-1\},\ \mathrm{cd}_{\mathrm{rv}}(D_{2(q-1)}) = \{1, 2\},$$

$$\mathrm{cd}_{\mathrm{rv}}((q \pm \sqrt{2q} + 1) : 4) = \{1, 4\},$$

so for all $H \in \max Sz(q)$, H is a TR-group by Lemmas 2.3 and 2.4. It follows that $Sz(q)$ is a PTR-group, so G is isomorphic to $Sz(q)$ with $q = 2^p$, p an odd prime.

Case 5: $\mathrm{PSL}_3(3)$.

By ([15], p. 13), $S_4 \in \max \mathrm{PSL}_3(3)$. Note from Lemma 4.1 that $\mathrm{cd}_{\mathrm{rv}}(S_4) = \{1, 2, 3\}$, so S_4 is a non-TR-group. It follows that $\mathrm{PSL}_3(3)$ is a non-PTR-group. □

For reading easily, we rewrite Theorem 1.4 here.

Theorem 4.3. *Let G be a non-solvable* PTR*-group. Then G is isomorphic to either $\mathrm{PSL}_2(q)$, where either $q = 2^p$ with p a prime or $q = 3^p$ with p an odd prime and $q \equiv \pm3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm3 \pmod 8$ or $Sz(2^p)$ with p an odd prime.*

Proof. The non-solvability of G gives that there is a normal series

$$1 \leq H \leq K \leq G$$

such that K/H is isomorphic to a direct product of isomorphic non-abelian simple groups and $|\mathrm{Out}(K/H)||G/K|$, where $\mathrm{Out}(M)$ denotes the outer-automorphism group of a group M; see [19].

Let $\mathfrak{S}$ be the set of groups consisting of the following groups:

(i) $\mathrm{PSL}_2(q)$, where either $q = 2^p$ with p a prime or $q = 3^p$ with p an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod 8$;

(ii) $Sz(2^p)$ with p an odd prime.

Then

$$K/H \text{ is isomorphic to } \underbrace{S \times S \times \cdots \times S}_{m \text{ times}}$$

with $S \in \mathfrak{S}$. If $m \geq 2$, then let $M \in \max S$, $M \times \underbrace{S \times \cdots \times S}_{m-1 \text{ times}}$ is a maximal subgroup of $\underbrace{S \times S \times \cdots \times S}_{m \text{ times}}$. Here two cases are dealt with: if $S = \mathrm{PSL}_2(q)$, then by Lemma 3.1, $|\mathrm{cd}_{\mathrm{rv}}(S)| \geq 4$; if $S = Sz(q)$, then by Lemma 2.5 and [14], $\mathrm{cd}_{\mathrm{rv}}(Sz(q)) \supset \{1, q^2, q^2+1\}$(we also can get this result from Case 2 of the proof of Theorem 1.2). It follows that

$$|\mathrm{cd}_{\mathrm{rv}}(M \times \underbrace{S \times \cdots \times S}_{m-1 \text{ times}})| \geq 4,$$

so $M \times \underbrace{S \times \cdots \times S}_{m-1 \text{ times}}$ is a non-TR-group. Now $m = 1$ and

$$G \text{ is a non-almost simple group}$$

as when $S < G$, $|\mathrm{cd}_{\mathrm{rv}}(S)| \geq 4$. In what follows, we will divide the proof into two cases in view of $S \in \mathfrak{S}$.

Case 1: $\mathrm{PSL}_2(q)$, where either $q = 2^p$ with p a prime or $q = 3^p$ with p an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod 8$.

Based on q, two subcases are done with.

Subcase 1: $q = 2^p$ with p a prime.

In this case, $k = 1$ and

$$\mathrm{GL}_2(q) \cong \mathrm{SL}_2(q) \cong \mathrm{PSL}_2(q);$$

see [20]. We know from ([15], p. xvi) that the Schur multiplier of $\mathrm{PSL}_2(q)$ with $q \geq 8$ is trivial and the Schur multiplier of $\mathrm{PSL}_2(4)$ is of order 2.

Let $q = 4$, then G/H is isomorphic to $\mathrm{PSL}_2(4)$. Now by ([17], Chap. II, Theorem 6.10), $G'/H \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$. Note that $[G', H] \leq C_2$, where $C_2 = Z(\mathrm{SL}_2(5))$, so either $G \cong \mathrm{PSL}_2(4) \times H$ when $[G', H] = 1$ or $G \cong \mathrm{SL}_2(5) * H$ when $[G', H] = C_2$.

Let $G \cong \mathrm{PSL}_2(4) \times H$. If $H > 1$, then $\mathrm{PSL}_2(4) < G$, so G is a non-PTR-group as $|\mathrm{cd}_{\mathrm{rv}}(\mathrm{PSL}_2(4))| = 4$. Thus $H = 1$ and $G \cong \mathrm{PSL}_2(4)$.

T a b l e 2

$\mathrm{SL}_2(q)$, $q \geq 4$ ([18], p. 377)

$\max \mathrm{SL}_2(q)$	Condition
$E_q : C_{q-1}$	
$Q_{2(q-1)}$	$q \neq 5, 7, 9, 11$; q odd
	N1 if $q = 7, 11$; N2 if $q = 9$
$D_{2(q-1)}$	q even
$Q_{2(q+1)}$	$q \neq 7, 9$; q odd
	N1 if $q = 7$; N2 if $q = 9$
$D_{2(q+1)}$	q even
$\mathrm{SL}_2(q_0).2$	$q = q_0^2$, q odd
$\mathrm{SL}_2(q_0)$	$q = q_0^r$, q odd, r odd prime
$\mathrm{PSL}_2(q_0)$	$q = q_0^r$, q even, $q_0 \neq 2$, r prime
$2.S_4$	$q = p \equiv \pm 1 \pmod 8$
$2.A_4$	$q = p \equiv \pm 3, 5, \pm 13 \pmod{40}$
	N1 if $q = p \equiv \pm 11, \pm 19 \pmod{40}$
$2.A_5$	$q = p \equiv \pm 1 \pmod{10}$
	$q = p^2$, $p \equiv \pm 3 \pmod{10}$

where N1 maximal under $< \delta >$ with $|\delta| = (q-1, 2)$;
N2 maximal under subgroups not contained in $< \varphi >$
with $|\varphi| = e$, $q = p^e$, p a prime.

Let $G \cong \mathrm{SL}_2(5) * H$. Then $H \geq C_2$. If $H > C_2$, then $\mathrm{SL}_2(5) < G$, so by Lemma 3.1, $|\mathrm{cd}_{\mathrm{rv}}(\mathrm{SL}_2(5))| = 5$ and $\mathrm{SL}_2(5)$ is a non-TR-group. Thus $H = C_2$ and $G \cong \mathrm{SL}_2(5) * H = \mathrm{SL}_2(5)$. By Table 2, $2.A_4 \in \max \mathrm{SL}_2(5)$. By [16], $\mathrm{cd}_{\mathrm{rv}}(2.A_4) = \{1, 2, 3\}$, and so $2.A_4$ is a non-TR-group. It follows that $\mathrm{SL}_2(5)$ is a non-PTR-group.

Now let $q \geq 8$, then $G \cong \mathrm{PSL}_2(q) \times H$ as above similar arguments. If $H \neq 1$, then $\mathrm{PSL}_2(q) < G$ is a TR-group, a contradiction to Lemma 3.1. Thus $H = 1$.

So in this subcase, G is isomorphic to $\mathrm{PSL}_2(q)$ with $q = 2^p$, and p a prime.

Subcase 2: $q = 3^p$ with p an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod 8$.

Then by ([15], p. xvi), the Schur multiplier of $\mathrm{PSL}_2(q)$ with $q \neq 9$ odd is of order two. Note that G is not an almost simple group, so $G/H \cong \mathrm{PSL}_2(q)$, where $q = 3^p$ with p an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod 8$. Similarly as Subcase 1, we can get that G is isomorphic to $\mathrm{PSL}_2(q) \times H$ or $\mathrm{SL}_2(q) * H$.

Let $G \cong \mathrm{PSL}_2(q) \times H$. Let $H \neq 1$. Then $\mathrm{PSL}_2(q) < G$ and $\mathrm{PSL}_2(q)$ is a TR-group, a contradiction to Lemma 3.1. Now $H = 1$ and $G \cong \mathrm{PSL}_2(q)$ as wanted.

Let $G \cong \mathrm{SL}_2(q) * H$. Then $H \geq C_2$, where $C_2 = Z(\mathrm{SL}_2(q))$. If $H > C_2$, then $\mathrm{SL}_2(q) \in \sum G$ and by Lemma 2.1, $\mathrm{SL}_2(q)$ is a TR-group, a contradiction to Lemma 3.1. So $H = C_2$ and $G \cong \mathrm{SL}_2(q) * H = \mathrm{SL}_2(q)$.

If $q = 3^p$, then by Table 2, $\mathrm{SL}_2(3) \in \sum \mathrm{SL}_2(3^p)$. We know that $\mathrm{SL}_2(3) \cong S_4$,

so by [16] or Lemma 3.1, $\mathrm{cd_{rv}}(\mathrm{SL}_2(3)) = \{1, 2, 3\}$. It follows that $\mathrm{SL}_2(3)$ is a non-TR-group and $\mathrm{SL}_2(3^p)$ is a non-PTR-group.

If $q = p$ is a prime, then by Table 2, the groups $2.A_4$, $2.S_4$ and $2.A_5$ are involved possibly in $\max \mathrm{SL}_2(q)$, so there is no prime satisfying the three inequations

$$\begin{aligned} q = p &\not\equiv 1 \pmod 8 \\ q = p &\not\equiv 1 \pmod{10} \\ q = p &\not\equiv \pm 3, 5, \pm 13 \pmod{40}. \end{aligned}$$

Thus $\mathrm{SL}_2(q)$ with q a prime is a non-PTR-group.

Case 2: $Sz(2^p)$ with p an odd prime.

Now we get from ([15], p. xvi) that the Schur multiplier of $Sz(2^p)$ is trivial when $q > 8$ and the Schur multiplier of $Sz(q)$ is of order 4 when $q = 8$. As G is not an almost simple group, we have that $G/H \cong Sz(q)$.

If $q = 8$, then we obtain from ([15], p. 28) that

$$\begin{aligned} 2.2^{3+3} : 7 &\in \sum 2.Sz(8) \\ 2^2.2^{3+3} : 7 &\in \sum 2^2.Sz(8). \end{aligned}$$

By [16], $\mathrm{cd_{rv}}(2.2^{3+3} : 7) = \{1, 7, 8\} = \mathrm{cd_{rv}}(2^2.2^{3+3} : 7)$. It follows that if G has a subgroup $2.Sz(8)$ or $2^2.Sz(8)$, then $2.Sz(8)$ has a non-TR-subgroup $2.2^{3+3} : 7$ and $2^2.Sz(8)$ has a non-TR-subgroup $2^2.2^{3+3} : 7$, respectively. Thus G is not a semisimple group. We see that G is also a non-almost simple group, so G is isomorphic to $Sz(8) \times H$. If $H \neq 1$, then by ([15], p. 28), $\mathrm{cd_{rv}}(Sz(8)) = \{1, 35, 64, 65, 91\}$, so $Sz(8)$ is a non-TR-group, a contradiction to Lemma 2.1. So $H = 1$ and G is isomorphic to $Sz(8)$ as needed.

If $q > 8$, then $G/H \cong Sz(q)$ as G is not an almost simple group. We see that the Schur multiplier of $Sz(q)$ is trivial, so $[G', H] = 1$ and G is isomorphic to $Sz(q) \times H$. If $H \neq 1$, then $Sz(q) \in \sum G$, so $Sz(q)$ is a TR-group. We know from [14] and Lemma 2.5 that $\mathrm{cd_{rv}}(Sz(q)) \supset \{1, q^2, q^2+1\}$, so $Sz(q)$ is a non-TR-group. Now $H = 1$ gives that G is isomorphic to $Sz(q)$ as desired. □

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