Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 76, No 7, 2023

MATHEMATICS
Complex analysis

A NOTE ON THE DIFFERENTIAL OPERATOR ON WEIGHTED FOCK SPACES

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Received on November 8, 2022
Presented by N. Nikolov, Corresponding Member of BAS, on May 30, 2023

Abstract

The differential operator $Df = f'$ is a canonical example of unbounded operators in many Banach spaces. In this study, we identify a class of weighted Fock spaces in which the differential operator is bounded. Then, we obtain some properties of the differential operator on these spaces in terms of its C^*-algebraic structure in the one and two-dimensional cases.

Key words: Fock spaces, differential operator, operator algebra, C^*-algebra

2020 Mathematics Subject Classification: 47B32, 30H20, 30D15

1. Introduction. For an analytic function f, the differential operator $Df = f'$ is an important object in operator theory and function spaces. Many basic properties such as boundedness, compactness and spectra have been extensively studied when acting on several function spaces in various domains, see for example [1–5] and the references therein. In this paper, we first identify the weighted Fock spaces $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$, $\alpha > -2$, $0 < p < \infty$. These spaces are very natural, but unlike many analytic function spaces, including the classical Fock space [6], Fock–Sobolev spaces [7], and generalized Fock spaces [4], where the weight decays faster than the Gaussian weight, the differential operator is bounded on $F^2_{(\alpha,p)}(\mathbb{C})$. This property motivates us to investigate the other properties of the differential operator on the weighted Fock spaces $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$.

DOI:10.7546/CRABS.2023.07.01

We first obtain the structure of the C^*-algebra $C^*(D)$ generated by D. It is shown that $C^*(D)$ contains the ideal of all compact operators on $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$ as the commutator ideal. Second we treat the C^*-algebra $C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$ of the partial differential operators $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ on the weighted Fock spaces $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$ using a tensor product machinery exactly along the same line as in [8]. It turns out that the structure of the C^*-algebra $C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$ is much more complicated than $C^*(D)$.

2. Main results. Let $\alpha > -2$, $0 < p < \infty$. The weighted Fock spaces $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$ consist of all entire functions f on $\mathbb{C}$ such that

$$\|f\|^2_{(\alpha,p)} := \int_{\mathbb{C}} |f(z)|^2 dA_{(\alpha,p)}(z) < \infty,$$

where

$$dA_{(\alpha,p)}(z) = \frac{1}{2\pi}|z|^\alpha e^{-p|z|} dA(z)$$

and $dA(z)$ denotes the usual Lebesgue area measure on $\mathbb{C}$.

It is clear that $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{(\alpha,p)} = \int_{\mathbb{C}} f(z)\overline{g(z)} dA_{(\alpha,p)}(z), \quad f, g \in \mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$$

and containing all polynomials of z. Note that monomials are mutually orthogonal, so the normalized monomials $e_n(z) := \frac{z^n}{\|z^n\|_{(\alpha,p)}}$ form an orthonormal basis for $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$. Some basic computations show that, for any nonnegative integer n,

$$e_n(z) = \sqrt{\frac{p^{2n+\alpha+2}}{\Gamma(2n+\alpha+2)}} z^n, \quad \alpha > -2, 0 < p < \infty, \tag{1}$$

where Γ is the Gamma function.

We first consider the differential operator $Df = f'$ on the weighted Fock spaces $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$, $\alpha > -2$, $0 < p < \infty$. This operator is bounded on $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$:

Proposition 2.1. *Let $\alpha > -2$, $0 < p < \infty$. The differential operator D is bounded on $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$.*

Proof. Using the equality

$$\begin{aligned}\|z^n\|^2_{(\alpha,p)} = \int_{\mathbb{C}} |z^n|^2 dA_{(\alpha,p)}(z) &= \frac{1}{2\pi}\int_{\mathbb{C}} |z|^{2n}|z|^{\alpha}e^{-p|z|}dA(z)\\ &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} r^{2n}r^{\alpha}e^{-pr}rdrd\theta\\ &= \int_0^{\infty} r^{2n+\alpha+1}e^{-pr}dr\\ &= \frac{\Gamma(2n+\alpha+2)}{p^{2n+\alpha+2}},\end{aligned}$$

we obtain

$$\|D(z^n)\|^2_{(\alpha,p)} = \|nz^{n-1}\|^2_{(\alpha,p)} = n^2\|z^{n-1}\|^2_{(\alpha,p)} = n^2\frac{\Gamma(2n+\alpha)}{p^{2n+\alpha}}.$$

The equality $\Gamma(z+1) = z\Gamma(z)$, $Re(z) > 0$ implies that

$$\Gamma(2n+\alpha+2) = (2n+\alpha+1)(2n+\alpha)\Gamma(2n+\alpha)$$

and then,

$$\begin{aligned}\|D(z^n)\|^2_{(\alpha,p)} = n^2\frac{\Gamma(2n+\alpha)}{p^{2n+\alpha}} &= \frac{n^2\Gamma(2n+\alpha+2)}{p^{2n+\alpha}(2n+\alpha+1)(2n+\alpha)}\\ &= \frac{n^2p^2}{(2n+\alpha+1)(2n+\alpha)}\|z^n\|^2_{(\alpha,p)}.\end{aligned}$$

This gives the boundedness of D. □

The special case of this result, where $\alpha = 0$ and $p = 2$, is already contained in ([4], Theorem 1.1).

Recall that a weighted unilateral shift operator T on a (complex) Hilbert space H is an operator that maps every vector in an orthonormal basis $\{e_n\}_{n\geq 0}$ into a scalar multiple of the next vector, i.e., $Te_n = w_ne_{n+1}$ for all n. Weighted unilateral shifts are described in detail in [9]. Since for any $n \geq 1$,

$$\begin{aligned}De_n = D\left(\sqrt{\frac{p^{2n+\alpha+2}}{\Gamma(2n+\alpha+2)}}z^n\right) &= \sqrt{\frac{p^{2n+\alpha+2}}{\Gamma(2n+\alpha+2)}}nz^{n-1}\\ &= \frac{np^{2n+\alpha+2/2}z^{n-1}}{\sqrt{(2n+\alpha+1)(2n+\alpha)\Gamma(2n+\alpha)}}\\ &= \frac{pn}{\sqrt{(2n+\alpha+1)(2n+\alpha)}}e_{n-1}\end{aligned}$$

and $De_0 = 0$, the differential operator D on $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$ is the weighted backward shift with the weight

$$d_n = \frac{pn}{\sqrt{(2n+\alpha+1)(2n+\alpha)}},$$

that is, $De_n = d_n e_{n-1}$, $n \geq 1$ and $De_0 = 0$. Some basic computations show that the adjoint of D is of the form

$$D^* e_n = \alpha_n e_{n+1}, \ n \geq 0,$$

which is the unilateral weighted shift with the weight

$$\alpha_n = d_{n+1} = \frac{p(n+1)}{\sqrt{(2n+\alpha+3)(2n+\alpha+2)}}.$$

To understand the structure of the C^*-algebra $C^*(D)$ generated by D, we need the following fact.

Proposition 2.2 ([10], Proposition 1)**.** *Let $\mathcal{A}$ be a C^*-algebra acting irreducibly on a Hilbert space H and having non-zero intersection with the ideal $\mathcal{K}(H)$ of all compact operators on H. Then $\mathcal{K}(H) \subset \mathcal{A}$ and $\mathcal{K}(H) \subset \mathcal{I}$ for any nontrivial ideal $\mathcal{I}$ in $\mathcal{A}$.*

Recall that a C^*-algebra $\mathcal{A}$ is called irreducible if the only closed vector subspaces of H that are invariant for $\mathcal{A}$ are 0 and H [10].

Theorem 2.3. *Let $\alpha > -2$, $0 < p < \infty$. The C^*-algebra $C^*(D)$ contains the ideal $\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$ of all compact operators on $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$. The commutator ideal of $C^*(D)$ is $\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$, in particular, the quotient $C^*(D)/\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$ is commutative.*

Proof. We first show that the operator $DD^* - D^*D$ is compact. Since D^* is the unilateral weighted shift with the weight

$$\alpha_n = \frac{p(n+1)}{\sqrt{(2n+\alpha+3)(2n+\alpha+2)}},$$

for $n \geq 1$,

$$(DD^* - D^*D)e_n = (\alpha_n^2 - \alpha_{n-1}^2)e_n,$$

and

$$\begin{aligned} \alpha_n^2 - \alpha_{n-1}^2 &= \frac{p^2(n+1)^2}{(2n+\alpha+3)(2n+\alpha+2)} - \frac{p^2 n^2}{(2n-2+\alpha+3)(2n-2+\alpha+2)} \\ &= \frac{p^2(n+1)^2}{(2n+\alpha+3)(2n+\alpha+2)} - \frac{p^2 n^2}{(2n+\alpha+1)(2n+\alpha)} \end{aligned}$$

goes to zero as $n \to \infty$. This implies that the operator $DD^* - D^*D$ is compact by ([9], Proposition 4). On the other hand, ([11], Theorem 1) tells us that D^* is irreducible, so that the C^*-algebra $C^*(D)$ is irreducible. Since $DD^* - D^*D$ is compact and contained in the irreducible C^*-algebra $C^*(D)$, the ideal $\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$ of all compact operators on $\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})$ is contained in $C^*(D)$ and $\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})) \subset \mathcal{I}$ for any nontrivial ideal $\mathcal{I}$ in $C^*(D)$ by Proposition (2.2). Then, the commutator ideal of $C^*(D)$ is $\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$ and the quotient algebra $C^*(D)/\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$ is commutative. $\square$

Since the quotient $C^*(D)/\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$ is commutative, by the Gelfand–Naimark Theorem ([12], p. 92, Theorem 4.29), $C^*(D)/\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$ is isometrically $*$-isomorphic to the space $C(\mathbb{M})$ of all continuous functions defined in the maximal ideal space $\mathbb{M}$ of $C^*(D)/\mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C}))$. From this it follows that the following sequence

$$0 \longrightarrow \mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})) \xrightarrow{j} C^*(D) \xrightarrow{\pi} C(\mathbb{M}) \longrightarrow 0 \tag{2}$$

is short exact.

We can extend the results in the one-dimensional case to the two-dimensional complex space $\mathbb{C}^2$. To do this, we consider the partial differential operators $\dfrac{\partial}{\partial z_1}$ and $\dfrac{\partial}{\partial z_2}$ on $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$.

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{p} = (p_1, p_2)$ be multi-indexes such that $\alpha_i > -2$ and $0 < p_i < \infty$, $i = 1, 2$. The weighted Fock spaces $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$ consist of all entire functions f on $\mathbb{C}^2$ such that

$$\|f\|^2_{(\boldsymbol{\alpha},\boldsymbol{p})} = \int_{\mathbb{C}^2} |f(z_1, z_2)|^2 dA_{(\boldsymbol{\alpha},\boldsymbol{p})}(z_1, z_2) < \infty,$$

where $dA_{(\boldsymbol{\alpha},\boldsymbol{p})}(z_1, z_2) = \dfrac{1}{(2\pi)^2}|z_1|^{\alpha_1}|z_2|^{\alpha_2}e^{-p_1|z_1|}e^{-p_2|z_2|}dA(z_1)dA(z_2)$. Observe that $dA_{(\boldsymbol{\alpha},\boldsymbol{p})}(z_1, z_2) = dA_{(\alpha_1,p_1)}(z_1)dA_{(\alpha_2,p_2)}(z_2)$.

It is clear that $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{(\boldsymbol{\alpha},\boldsymbol{p})} = \int_{\mathbb{C}^2} f(z_1, z_2)\overline{g(z_1, z_2)}dA_{(\boldsymbol{\alpha},\boldsymbol{p})}(z_1, z_2), \quad f, g \in \mathcal{F}^2_{(\alpha,p)}(\mathbb{C}^2)$$

and containing all polynomials of z_1, z_2.

For any multi-index (m, n) such that $m, n \geq 0$ and $z = (z_1, z_2) \in \mathbb{C}^2$, let

$$e_{(m,n)}(z) = \sqrt{\frac{p_1^{2m+\alpha_1+2}p_2^{2n+\alpha_2+2}}{\Gamma(2m + \alpha_1 + 2)\Gamma(2n + \alpha_2 + 2)}}z_1^m z_2^n, \tag{3}$$

where $0 < p_i < \infty$, $\alpha_i > -2$, $i = 1, 2$ and Γ is the Gamma function. Similar to the one variable case, it is easy to show that the set $\{e_{(m,n)}\}_{m,n\geq 0}$ is an orthonormal basis for $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$.

Proposition 2.4. *The partial differential operators* $\dfrac{\partial}{\partial z_1}$ *and* $\dfrac{\partial}{\partial z_2}$ *are bounded on* $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$.

Proof. It is enough to show the boundedness of $\frac{\partial}{\partial z_1}$. With basic computations we obtain for any multi-index (m, n) such that $m, n \geq 0$,

$$\begin{aligned}\left\| \frac{\partial}{\partial z_1} (z_1^m z_2^n) \right\|^2_{(\boldsymbol{\alpha}, \boldsymbol{p})} &= m^2 \|z_1^{m-1} z_2^n\|^2_{(\boldsymbol{\alpha}, \boldsymbol{p})} \\ &= m^2 \frac{\Gamma(2m + \alpha_1)\Gamma(2n + \alpha_2 + 2)}{p_1^{2m+\alpha_1} p_2^{2n+\alpha_2+2}}.\end{aligned}$$

The equality $\Gamma(z + 1) = z\Gamma(z)$, $Re(z) > 0$ implies

$$\Gamma(2m + \alpha_1 + 2) = (2m + \alpha_1 + 1)(2m + \alpha_1)\Gamma(2m + \alpha_1)$$

and

$$\begin{aligned}\left\| \frac{\partial}{\partial z_1} (z_1^m z_2^n) \right\|^2_{(\boldsymbol{\alpha}, \boldsymbol{p})} &= m^2 \frac{\Gamma(2m + \alpha_1)\Gamma(2n + \alpha_2 + 2)}{p_1^{2m+\alpha_1} p_2^{2n+\alpha_2+2}} \\ &= \frac{m^2 \Gamma(2m + \alpha_1 + 2)\Gamma(2n + \alpha_2 + 2)}{p_1^{2m+\alpha_1} (2m + \alpha_1 + 1)(2m + \alpha_1) p_2^{2n+\alpha_2+2}} \\ &= \frac{m^2 p_1^2}{(2m + \alpha_1 + 1)(2m + \alpha_1)} \|z_1^m z_2^n\|^2_{(\boldsymbol{\alpha}, \boldsymbol{p})}.\end{aligned}$$

This gives the boundedness of $\frac{\partial}{\partial z_1}$. □

As in the one-variable case, we can consider the structure of the C^*-algebra $C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$ of the partial differential operators $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ on $\mathcal{F}^2_{(\boldsymbol{\alpha}, \boldsymbol{p})}(\mathbb{C}^2)$. To study this structure, we use a tensor product machinery exactly on the same line as in [8]. In [8], Douglas and Howe study Toeplitz operators on the Hardy spaces of the bidisc reducing the bidisc case to the classical one-variable case by using the device of the tensor product C^*-algebra. To this end, we need to recall the following facts.

Let H and K be two given Hilbert spaces. On the algebraic tensor product $H \otimes K$ of H and K, there is a unique inner product $\langle ., . \rangle$ satisfying the following equation

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle_H \langle y_1, y_2 \rangle_K$$

for all $x_1, x_2 \in H$ and $y_1, y_2 \in K$ (see [13], p. 185). For any $T \in B(H)$ and $S \in B(K)$ there is a unique operator $T \hat{\otimes} S \in B(H \otimes K)$ satisfying the following equation:

$$(T \hat{\otimes} S)(x \otimes y) = Tx \otimes Sy.$$

For any two C^*-algebras $A \subset B(H)$ and $B \subset B(K)$ the algebraic tensor product

$A \odot B$ is defined to be the linear span of operators of the form $T\hat{\otimes}S$ i.e.

$$A \odot B = \left\{ \sum_{j=1}^{n} T_j \hat{\otimes} S_j : T_j \in A, \quad S_j \in B \right\}.$$

The algebraic tensor product $A \odot B$ becomes a $*$-algebra with multiplication

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2$$

and involution

$$(x \otimes y)^* = x^* \otimes y^*.$$

However, there might be more than one norm making the closure of $A \odot B$ into a C^*-algebra. A C^*-algebra A is called "nuclear" if for any C^*-algebra B there is a unique pre C^*-algebra norm on the algebraic tensor product $A \odot B$ of A and B. We will always denote by $A \otimes B$ the C^*-algebra obtained by completing the algebraic tensor product of A and B with respect to this norm.

A well-known theorem of Takesaki asserts that any commutative C^*-algebra is nuclear ([13], p. 205). An extension of a C^*-algebra by nuclear C^*-algebras is nuclear, i.e. if A, B and C are C^*-algebras such that the following sequence

$$0 \to A \xrightarrow{j} B \xrightarrow{\pi} C \to 0$$

is short exact and A and C are nuclear, then B is also nuclear ([13], p. 212). For any separable Hilbert space H the C^*-algebra of all compact operators $K(H)$ on H is nuclear ([13], p. 183 and 196). For any separable Hilbert spaces H_1 and H_2 we have from ([8], p. 207)

$$K(H_1 \otimes H_2) = K(H_1) \otimes K(H_2). \tag{4}$$

Lemma 2.5. *For any multi-indexes* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ *and* $\boldsymbol{p} = (p_1, p_2)$ *such that* $\alpha_i > -2$ *and* $0 < p_i < \infty$, $i = 1, 2$, *the weighted Fock space* $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$ *can be identified as the tensor product of the spaces* $\mathcal{F}^2_{(\alpha_1,p_1)}(\mathbb{C})$ *and* $\mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C})$.

Proof. We define a map U from $\mathcal{F}^2_{(\alpha_1,p_1)}(\mathbb{C}) \otimes \mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C})$ onto $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$ such that

$$U(e_m(z_1) \otimes e_n(z_2)) = e_{(m,n)}(z_1, z_2),$$

where e_m, e_n, and $e_{(m,n)}$ are orthonormal bases of $\mathcal{F}^2_{(\alpha_1,p_1)}(\mathbb{C})$, $\mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C})$ and $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$, respectively, and defined as in equations (1) and (3). Then, for any

$f(z_1) = \sum_{m=1}^{\infty} c_m z_1^m$, $g(z_2) = \sum_{n=1}^{\infty} d_n z_2^n \in \mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C})$, we have

$$\begin{aligned}
U(f \otimes g) &= U\left(\left(\sum_{m=1}^{\infty} c_m z_1^m\right) \otimes (\sum_{n=1}^{\infty} d_n z_2^n)\right) \\
&= U\left(\left(\sum_{m=1}^{\infty} c_m z_1^m \frac{\Gamma(2m+\alpha_1+2)}{p_1^{2m+\alpha_1+2}} \frac{p_1^{2m+\alpha_1+2}}{\Gamma(2m+\alpha_1+2)}\right)\right. \\
&\left.\otimes \left(\sum_{n=1}^{\infty} d_n z_2^n \frac{\Gamma(2n+\alpha_2+2)}{p_2^{2n+\alpha_2+2}} \frac{p_2^{2n+\alpha_2+2}}{\Gamma(2n+\alpha_2+2)}\right)\right) \\
&= U\left(\sum_{m,n} c_m d_n \frac{\Gamma(2m+\alpha_1+2)}{p_1^{2m+\alpha_1+2}} \frac{\Gamma(2n+\alpha_2+2)}{p_2^{2n+\alpha_2+2}} \left(\frac{p_1^{2m+\alpha_1+2}}{\Gamma(2m+\alpha_1+2)} z_1^m\right.\right. \\
&\left.\left.\otimes \frac{p_2^{2n+\alpha_2+2}}{\Gamma(2n+\alpha_2+2)} z_2^n\right)\right) \\
&= U\left(\sum_{m,n} c_m d_n \frac{\Gamma(2m+\alpha_1+2)}{p_1^{2m+\alpha_1+2}} \frac{\Gamma(2n+\alpha_2+2)}{p_2^{2n+\alpha_2+2}} (e_m(z_1) \otimes e_n(z_2))\right) \\
&= \sum_{m,n} c_m d_n \frac{\Gamma(2m+\alpha_1+2)}{p_1^{2m+\alpha_1+2}} \frac{\Gamma(2n+\alpha_2+2)}{p_2^{2n+\alpha_2+2}} e_{(m,n)}(z_1, z_2) \\
&= \sum_{m,n} c_m d_n z_1^m z_2^n = \left(\sum_{m=1}^{\infty} c_m z_1^m\right)\left(\sum_{n=1}^{\infty} d_n z_2^n\right) = f(z_1)g(z_2).
\end{aligned}$$

This shows that U can be unitarily extended to any f, g, that is, $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$ is isometrically isomorphic to $\mathcal{F}^2_{(\alpha_1,p_1)}(\mathbb{C}) \otimes \mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C})$. □

The short-exact sequence (2)

$$0 \longrightarrow \mathcal{K}(\mathcal{F}^2_{(\alpha,p)}(\mathbb{C})) \xrightarrow{j} C^*(D) \xrightarrow{\pi} C(\mathbb{M}) \longrightarrow 0$$

shows us that $C^*(D)$ is nuclear since $\mathcal{K}(\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}))$ is nuclear and $C(\mathbb{M})$ is commutative and hence nuclear. Therefore all the C^*-algebras in this paper will be nuclear.

Following the approach in [8], we identify the C^*-algebra $C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$ with $C^*(D) \otimes C^*(D)$ corresponding to the identification of $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2)$ with $\mathcal{F}^2_{(\alpha_1,p_1)}(\mathbb{C}) \otimes \mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C})$.

For simplicity, the spaces $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2), \mathcal{F}^2_{(\alpha_1,p_1)}(\mathbb{C}), \mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C})$ will be denoted by $\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}, \mathcal{F}^2_{(\alpha_1,p_1)}, \mathcal{F}^2_{(\alpha_2,p_2)}$, respectively. Similarly, the ideals $\mathcal{K}(\mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}(\mathbb{C}^2))$, $\mathcal{K}(\mathcal{F}^2_{(\alpha_1,p_1)}(\mathbb{C}))$ and $\mathcal{K}(\mathcal{F}^2_{(\alpha_2,p_2)}(\mathbb{C}))$ will be denoted by $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$, respectively.

We can define the partial operators $\dfrac{\partial}{\partial z_1}$ and $\dfrac{\partial}{\partial z_2}$ as

$$\frac{\partial}{\partial z_1} f(z_1, z_2) = D \otimes I = Df(z_1, \cdot)$$

and

$$\frac{\partial}{\partial z_2} f(z_1, z_2) = I \otimes D = Df(\cdot, z_2)$$

for any $f \in \mathcal{F}^2_{(\boldsymbol{\alpha},\boldsymbol{p})}$, respectively. Since $C^*\left(\dfrac{\partial}{\partial z_1}, \dfrac{\partial}{\partial z_2}\right)$ is generated by $\dfrac{\partial}{\partial z_1}$ and $\dfrac{\partial}{\partial z_2}$, and $C^*(D)$ is nuclear,

$$C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) = C^*(D) \otimes C^*(D).$$

Moreover, we know from Equation (4) that

$$\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2.$$

We now recall the following facts.

Theorem 2.6 ([13], Theorem 6.5.2, p. 211)**.** *Let $\mathcal{J}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ be C^*-algebras and suppose that*

$$0 \longrightarrow \mathcal{J} \xrightarrow{j} \mathcal{A} \xrightarrow{\pi} \mathcal{B} \longrightarrow 0$$

is a short exact sequence of C^*-algebras. Suppose also that $\mathcal{B} \otimes \mathcal{D}$ has a unique C^*-norm (this is the case if $\mathcal{B}$ or $\mathcal{D}$ is nuclear). Then

$$0 \longrightarrow \mathcal{J} \otimes \mathcal{D} \xrightarrow{j\otimes 1} \mathcal{A} \otimes \mathcal{D} \xrightarrow{\pi\otimes 1} \mathcal{B} \otimes \mathcal{D} \longrightarrow 0$$

is a short exact sequence of C^*-algebras.

Proposition 2.7 ([8], Proposition 3)**.** *If X is a compact Hausdorff space and $\mathcal{B}$ is a C^*-algebra, then $C(X) \otimes \mathcal{B}$ is naturally isomorphic to the C^*-algebra $C(X, \mathcal{B})$ of continuous functions from X to $\mathcal{B}$. Moreover, if $\sum_{i=1}^{n} f_i \otimes B_i$ in $C(X) \otimes \mathcal{B}$ correspond to F in $C(X, \mathcal{B})$, then $F(x) = \sum_{i=1}^{n} f_i(x) B_i$.*

Applying Theorem 2.6 to the short exact sequence (2), we set the following commutative diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_1 \otimes \mathcal{K}_2 & \overset{j\otimes 1}{\longrightarrow} & C^*(D) \otimes \mathcal{K}_2 & \overset{\pi\otimes 1}{\longrightarrow} & C(\mathbb{M}) \otimes \mathcal{K}_2 & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle 1\otimes j} & & \downarrow {\scriptstyle 1\otimes j} & & \downarrow {\scriptstyle 1\otimes j} & & \\
0 & \longrightarrow & \mathcal{K}_1 \otimes C^*(D) & \overset{j\otimes 1}{\longrightarrow} & C^*(D) \otimes C^*(D) & \overset{\pi\otimes 1}{\longrightarrow} & C(\mathbb{M}) \otimes C^*(D) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle 1\otimes \pi} & & \downarrow {\scriptstyle 1\otimes \pi} & & \downarrow {\scriptstyle 1\otimes \pi} & & \\
0 & \longrightarrow & \mathcal{K}_1 \otimes C(\mathbb{M}) & \overset{j\otimes 1}{\longrightarrow} & C^*(D) \otimes C(\mathbb{M}) & \overset{\pi_1\otimes 1}{\longrightarrow} & C(\mathbb{M}) \otimes C(\mathbb{M}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Consideration of the diagonal map shows that the commutator ideal of $C^*(D) \otimes C^*(D)$ is the subspace spanned by $C^*(D) \otimes \mathcal{K}_2$ and $\mathcal{K}_1 \otimes C^*(D)$ and the corresponding quotient is $C(\mathbb{M}) \otimes C(\mathbb{M})$ which is naturally isomorphic to $C(\mathbb{M} \times \mathbb{M})$. We are primarily interested, however, in the dual sequence

$$
\begin{aligned}
0 \longrightarrow \mathcal{K}_1 \otimes \mathcal{K}_2 \overset{\alpha}{\longrightarrow} C^*(D) \otimes C^*(D) \\
\overset{(\pi\otimes 1)\oplus(1\otimes\pi)}{\longrightarrow} (C(\mathbb{M}) \otimes C^*(D)) \oplus (C^*(D) \otimes C(\mathbb{M})) \longrightarrow 0,
\end{aligned}
$$

where $\alpha = (1 \otimes j)(j \otimes 1) = (j \otimes 1)(1 \otimes j)$ which is exact. The proof involves routine diagram chasing.

We now have the following result.

Proposition 2.8. *There exist $*$-homomorphisms γ_{z_1} and γ_{z_2} from $C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$ onto $C(\mathbb{M}, C^*(D))$ such that $\gamma_{z_1}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right) = F$ and $\gamma_{z_2}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right) = G$, where $(F\hat{f})(z_1) = \frac{\partial}{\partial z_1} f(z_1, \cdot)$ and $(G\hat{f})(z_2) = \frac{\partial}{\partial z_2} f(\cdot, z_2)$ and $\hat{f} \in \mathbb{M}$. Moreover, under natural identification of $C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$ with $C^*(D) \otimes C^*(D)$, $C(\mathbb{M}) \otimes C^*(D)$ with $C(\mathbb{M}, C^*(D))$ and $C^*(D) \otimes C(\mathbb{M})$ with $C(\mathbb{M}, C^*(D))$, then $\pi \otimes 1 = \gamma_{z_1}$ and $1 \otimes \pi = \gamma_{z_2}$.*

The proof can be obtained similar to the proof of Proposition 3.1 in [14]. So, we shall omit it.

We now restate preceding the exactness result as follows:

Theorem 2.9. *The sequence*

$$
0 \longrightarrow \mathcal{K} \overset{i}{\longrightarrow} C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) \overset{\gamma_{z_1}\oplus\gamma_{z_2}}{\longrightarrow} C(\mathbb{M}, C^*(D)) \oplus C(\mathbb{M}, C^*(D)) \longrightarrow 0
$$

is exact.

In other words, the quotient algebra $C^*\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)/\mathcal{K}$ is isometrically $*$-isomorphic to $C(\mathbb{M}, C^*(D)) \oplus C(\mathbb{M}, C^*(D))$.

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