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GEOMETRY OF CR-SLANT WARPED PRODUCTS IN NEARLY TRANS-SASAKIAN MANIFOLDS

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Abstract

A CR-slant warped product is considered a generalization of a CR-warped product. It was firstly introduced and studied in the field of Kaehler geometry. In this paper, we study CR-slant warped products in nearly trans-Sasakian manifolds. We obtain a general inequality related to the second fundamental form and the warping function in such warped products. The equality case is discussed.

Key words: warped products, CR-warped product, CR-slant warped product, slant submanifolds, nearly Sasakian, nearly Kenmotsu, nearly trans-Sasakian manifolds

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1. Introduction. CR-submanifolds of a Kaehler manifold were first introduced by Bejancu [1]. Later, these submanifolds of Kaehler manifolds were studied by Chen [2,3] for further geometry. The CR -submanifolds of a Sasakian manifold were studied by Kobayashi [4], Yano and Kon [5]. Beside the CR-submanifolds, Chen introduced the idea of CR-warped products and obtained several fundamental results in his seminal paper [6]. In [7], Gherghe introduced the nearly trans-Sasakian structure of type (α, β) on an almost contact metric

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manifold $\tilde{M}$ which generalizes the trans-Sasakian structure in the same sense as nearly Sasakian generalizes Sasakian ones.

In this paper, we study a new class of submanifolds in nearly trans-Sasakian manifolds introduced as a CR-slant warped product in [8] and a skew CR-warped product in [9]. We prove the following main results in this paper.

Theorem 1.1. *There is no proper CR-slant warped product submanifold $B_1 \times_f M_T$ in a proper nearly trans-Sasakian manifold $\tilde{M}$, where $B_1 = M_\perp \times M_\theta$ is a hemi-slant product of an anti-invariant submanifold $M_\perp$ and a proper slant submanifold M_θ, while M_T is an invariant submanifold of $\tilde{M}$.*

On the other hand, we find that there exist several CR-slant warped products of the type $B_2 \times_f M_\perp$, where $B_2 = M_T \times M_\theta$ is a semi-slant product in a nearly trans-Sasakian manifold $\tilde{M}$ and we prove Chen's first inequality as:

Theorem 1.2. *Every $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$ mixed geodesic CR-slant warped product submanifold $M = B_2 \times_f M_\perp$ of a nearly trans-Sasakian manifold $\tilde{M}$ satisfying the following inequalities:*

(i) *If ξ is tangent to M_T, then*

$$\|h\|^2 \geq 2q\left(\|\nabla^T(\ln f)\|^2 + \alpha^2 - \beta^2\right) + q\cot^2\theta\|\nabla^\theta(\ln f)\|^2.$$

(ii) *If ξ is tangent to M_θ, then*

$$\|h\|^2 \geq 2q\|\nabla^T(\ln f)\|^2 + q\cot^2\theta\left(\|\nabla^\theta(\ln f)\|^2 - \beta^2\right),$$

where $q = \dim(M_\perp)$, whereas $\nabla^T(\ln f)$ and $\nabla^\theta(\ln f)$ are the gradient components of the warping function along M_T and M_θ, respectively.

(iii) *If the equality cases hold, then B_2 is totally geodesic in $\tilde{M}$ and $M_\perp$ is a totally umblical submanifold of $\tilde{M}$. Moreover, M is also $\mathfrak{D} \oplus \mathfrak{D}^\theta$ mixed totally geodesic in $\tilde{M}$, where $\mathfrak{D}$ and $\mathfrak{D}^\theta$ denote to invariant and proper slant distributions, respectively.*

2. Preliminaries. An almost contact metric manifold is a smooth manifold $\tilde{M}$ of an odd dimension $(2n+1)$ endowed with a structure (φ, ξ, η, g) given by a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying [10]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for all vector fields X, Y on $\tilde{M}$. From this definition, it follows that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, φ is skew-symmetric with respect to g, so that the bilinear form $\Phi(X, Y) := g(X, \varphi Y)$ defines a 2-form on $\tilde{M}$, called the fundamental 2-form. An almost contact metric manifold $\tilde{M}$ such that $d\eta = 2\Phi$ is called a contact metric manifold. In this case, η is a contact form, that is, $\eta \wedge (d\eta)^n \neq 0$ everywhere on $\tilde{M}$.

A nearly trans-Sasakian manifold of type (α, β) is an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$ such that

$$(\tilde{\nabla}_X \varphi)Y + (\tilde{\nabla}_Y \varphi)X = \alpha(2g(X,Y) - \eta(X)Y - \eta(Y)X) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X), \tag{2.2}$$

for all vector fields X, Y on $\tilde{M}$. It is known that a trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly-Sasakian [11] or nearly Kenmotsu [12] or nearly cosymplectic [10] if $\beta = 0$ or $\alpha = 0$ or $\alpha = 0 = \beta$, respectively.

Let M be an m-dimensional manifold isometrically immersed into a Riemannian manifold $\tilde{M}$ and denote by the same symbol g the induced metric on M. Let $\Gamma(TM)$ be the Lie algebra of the vector fields on M and $\Gamma(T^\perp M)$ be the set of all vector fields normal to M. If we denote by ∇ and $\tilde{\nabla}$, the Levi-Civita connections of M and $\tilde{M}$, respectively, then the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$, h is the second fundamental form and A_N is the shape operator (corresponding to the normal vector field N) for the immersion of M into $\tilde{M}$. These are related by $g(h(X,Y),N) = g(A_N X, Y)$. For any X tangent to M and N normal to M, we write

$$\varphi X = TX + FX, \tag{2.4}$$

where TX (respectively, FX) is the tangential (respectively, normal) component of φX. Then, T is an endomorphism of the tangent bundle TM and F is a normal bundle valued 1-form on TM. Invariant and anti-invariant submanifolds are Reimannian submanifolds with $F = 0$ and $T = 0$, respectively. For any $p \in M$, $\{e_1, \ldots, e_m, \ldots, e_{2n+1}\}$ is an orthonormal frame of $T_p\tilde{M}$ such that $e_1, \ldots, e_m$ are tangent to M at p and $e_{m+1}, \ldots, e_{2n+1}$ normal to M. Then,

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)) = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i,e_j), e_r)^2. \tag{2.5}$$

As a generalization of invariant and anti-invariant submanifolds, B.-Y. Chen introduced slant submanifolds of almost Hermitian manifolds [13, 14]. Later, Lotta [15] and Cabrerizo et al. [16] in separate articles extended this study to almost contact metric manifolds.

A submanifold M tangent to the structure vector field ξ is said to be slant if for each non-zero tangent vector X which is not proportional to ξ_p, the angle $\theta(X)$ (called, *slant angle*) between φX and T_pM is constant, that is, θ is independent of the choice of $X \in T_pM$ and $p \in M$.

Now, we recall the following useful result of [16] in the same sense for almost Hermitian case introduced by CHEN [13,14].

Theorem 2.1 ([16]). *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that $T^2 = \lambda(-I + \eta \otimes \xi)$. Furthermore, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.*

The following relations are easily obtained from Theorem 2.1. For any $X, Y \in \Gamma(TM)$,

$$g(TX, TY) = \cos^2\theta\,[g(X,Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$g(FX, FY) = \sin^2\theta\,[g(X,Y) - \eta(X)\eta(Y)]. \tag{2.7}$$

A submanifold M of an almost contact metric manifold $\tilde{M}$ is hemi-slant if there exists a pair orthogonal distributions $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ such that the tangent bundle of M splits as $TM = \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \xi$, where $\mathfrak{D}^\perp$ is an anti-invariant distribution and $\mathfrak{D}^\theta$ is a proper slant distribution on M. A submanifold M is a semi-slant submanifold of $\tilde{M}$ if TM splits as $TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \xi$, where $\mathfrak{D}$ is an invariant distribution on M. Hemi-slant and semi-slant submanifolds are the generalizations of slant, CR, invariant, and anti-invariant submanifolds but there is no direct relation between hemi-slant and semi-slant submanifolds.

On the other hand, the warped product manifold M of two Reimannian (or semi-Riemannian) manifolds M_1 and M_2 is the product manifold, denoted $M_1 \times_f M_2$, such that the metric of the product manifold is

$$g(X,Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi_1)^2 g_2(\pi_{2*}X, \pi_{2*}Y), \tag{2.8}$$

where $f : M_1 \to (0,\infty)$ and $\pi_1 : M \to M_1$, $\pi_2 : M \to M_2$ are projection maps given by $\pi_1(p,q) = p$ and $\pi_2(p,q) = q$ for any $(p,q) \in M_1 \times M_2$ and $*$ denotes the symbol for tangent map. The function f is called the warping function, and if f is constant, then M is simply a Riemannian product manifold. It is known that for any vector field X on M_1 and a vector field Z on M_2, we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{2.9}$$

where ∇ is the Levi-Civita connection on M. Further, notice that M_1 is totally geodesic and M_2 is totally umbilical of M [6].

3. Basic results and definitions. Throughout the paper we assume that the structure vector field ξ tangent to the submanifold and the manifolds and functions are smooth otherwise stated. First we prove the following useful results.

Theorem 3.1. *There do not exist any proper warped product submanifold $M = B \times_f M_T$ such that ξ is tangent to the invariant submanifold M_T, where B is any Riemannian submanifold of a nearly trans-Sasakian manifold $\tilde{M}$, unless $\tilde{M}$ is nearly α-Sasakian.*

Proof. From (2.2), we find

$$-\varphi\tilde{\nabla}_X\xi + \tilde{\nabla}_\xi\varphi X - \varphi\tilde{\nabla}_\xi X = -\alpha X - \beta\varphi X,$$

for any $X \in \Gamma(TB)$. Using (2.3) and (2.9), we get

$$-\varphi h(X,\xi) + \tilde{\nabla}_\xi\varphi X = -\alpha X - \beta\varphi X.$$

Taking the inner product with φX, we obtain

$$g(\tilde{\nabla}_\xi\varphi X, \varphi X) = -\beta\|X\|^2. \tag{3.1}$$

Since g is Riemannian and $g(\varphi X, \varphi X) = g(X, X)$, then by taking the covariant derivative with respect to ξ, we find

$$g(\tilde{\nabla}_\xi\varphi X, \varphi X) = g(\tilde{\nabla}_\xi X, X) = X(\ln f)g(\xi, X) = 0. \tag{3.2}$$

Then, from (3.1) and (3.2), we find $\beta = 0$ and hence, $\tilde{M}$ is α-Sasakian. □

In the next result, we consider ξ is tangent to the base manifold of the warped product $M = B \times_f M_T$.

Theorem 3.2. *There do not exist any proper warped product submanifold $M = B \times_f M_T$ such that ξ is tangent to the base manifold B of a nearly trans-Sasakian manifold $\tilde{M}$, unless $\tilde{M}$ is nearly β-Kenmotsu.*

Proof. For any $X \in \Gamma(\mathfrak{D})$, from (2.2) we have

$$-\varphi\tilde{\nabla}_X\xi + \tilde{\nabla}_\xi\varphi X - \varphi\tilde{\nabla}_\xi X = -\alpha X - \beta\varphi X.$$

Then, from (2.3) and (2.9), we derive

$$\xi(\ln f)\varphi X = -\alpha X - \beta\varphi X, \quad 2\varphi h(X,\xi) = h(\varphi X, \xi).$$

In the tangential part, taking the inner product with X, we obtain $\alpha\|X\|^2 = 0$. Since X is non-zero, hence we achieve the result. □

The proof of Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2.

Now, we have the following general result.

Theorem 3.3. *There do not exist any proper warped product submanifold $M = B \times_f M_\perp$ such that ξ is tangent to the anti-invariant submanifold $M_\perp$, where B is any Riemannian submanifold of a nearly trans-Sasakian manifold $\tilde{M}$, unless $\tilde{M}$ is nearly α-Sasakian.*

Proof. For any $X \in \Gamma(TB)$, we have from (2.2), (2.3) and (2.9)

$$-2\varphi h(X,\xi) + \tilde{\nabla}_\xi \varphi X = -\alpha X - \beta\varphi X.$$

Taking the inner product with φX and using (2.1), we find

$$g(\tilde{\nabla}_\xi X, X) = -\beta\|X\|^2.$$

Again, using (2.9), we get $\beta = 0$. □

Now, we consider ξ is tangent to the base manifold B of the warped product $M = B \times_f M_\perp$ and we have the following useful lemma.

Lemma 3.1. *Let $M = B \times_f M_\perp$ be a warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$ such that the vector field ξ is tangent to the base manifold B. Then, we have $\xi(\ln f) = \beta$.*

Proof. For any $Z \in \Gamma(\mathfrak{D}^\perp)$, we have from (2.2), (2.3) and (2.9)

$$-2\xi(\ln f)\varphi Z - 2\varphi h(\xi, Z) + \tilde{\nabla}_\xi \varphi Z = -\alpha Z - \beta\varphi Z.$$

Taking the inner product with φZ and using the Riemannian metric property, we find $\xi(\ln f) = \beta$. □

Now, we define CR-slant submanifolds.

Definition 3.1. A submanifold M tangent to the structure vector field ξ of an almost contact metric manifold $\tilde{M}$ is called a CR-slant warped product if it is a warped product of the forms $M = B_1 \times_f M_T$, or $B_2 \times_f M_\perp$ where $B_1 = M_\perp \times M_\theta$ and $B_2 = M_T \times M_\theta$ are the Riemannian products in $\tilde{M}$.

In this paper, we study two types of CR-slant warped product submanifolds of $B_1 \times_f M_T$ and $B_2 \times_f M_\perp$ in a nearly trans-Sasakian manifold $\tilde{M}$, where $B_1 = M_\perp \times M_\theta$ is a hemi-slant product and $B_2 = M_T \times M_\theta$ is a semi-slant product in $\tilde{M}$.

We use the following conventions in the paper for the corresponding tangent spaces of M_T, $M_\perp$ and M_θ are $\mathfrak{D}$, $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively. Also, we choose the vector fields $X, Y \in \Gamma(\mathfrak{D})$, $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $U, V \in \Gamma(\mathfrak{D}^\theta)$.

In both cases of CR-slant warped products, the tangent space of M is decomposed as follows:

$$TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus <\xi>, \tag{3.3}$$

where $\mathfrak{D}$ is an invariant distribution, $\mathfrak{D}^\perp$ is an anti-invariant distribution and $\mathfrak{D}^\theta$ is a proper slant distribution and $<\xi>$ is the 1-dimensional distribution spanned by the structure vector field ξ. Clearly, we observe that if ξ along $\mathfrak{D}^\theta$ then the CR-slant warped product $M = B \times_f N^\theta$ is trivial as follows: Since ξ is killing, then, for any $X \in \Gamma(TB)$ and $\xi \in \Gamma(\mathfrak{D}^\theta)$, and by using (2.2) and (2.9), we find $X(\ln f) = 0$, that is, f is constant on M.

Furthermore, the normal bundle $T^\perp M$ is decomposed as

$$T^\perp M = \varphi\mathfrak{D}^\perp \oplus F\mathfrak{D}^\theta \oplus \mu, \tag{3.4}$$

where μ is the invariant normal subbundle of $T^\perp M$ under φ.

4. Proof of Theorem 1.2. To prove the main theorem, first we have the following useful lemmas.

Lemma 4.1. *Let $M = B_2 \times_f M_\perp$ be a CR-slant warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$ such that $B_2 = M_T \times M_\theta$ and $\xi \in \Gamma(TB_2)$. Then, for any $X, Y \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we have*

(i) $g(h(X, Y), \varphi Z) = 0$,

(ii) $g(h(X, Z), \varphi W) = -[\varphi X(\ln f) + \alpha\eta(X)]g(Z, W)$.

Proof. Statements (i) and (ii) directly follow from [17]. □

Now, for any $U \in \Gamma(\mathfrak{D}^\theta)$ and $W \in \Gamma(\mathfrak{D}^\perp)$, from (2.2), (2.3) and (2.9), we have

$$\begin{aligned} -A_{\varphi W}U + \nabla^\perp_U \varphi W - 2U(\ln f)\varphi W - 2\varphi h(U, W) + TU(\ln f)W \\ + h(TU, W) - A_{FU}W + \nabla^\perp_{FU}W = -\alpha\eta(U) - \beta\eta(U)\varphi W. \end{aligned} \tag{4.1}$$

Using (4.1), we have the following lemma.

Lemma 4.2. *Let $M = B_2 \times_f M_\perp$ be a CR-slant warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$ such that $B_2 = M_T \times M_\theta$ and $\xi \in \Gamma(TB_2)$. Then, for any $X \in \Gamma(\mathfrak{D})$, $W \in \Gamma(\mathfrak{D}^\perp)$ and $U, V \in \Gamma(\mathfrak{D}^\theta)$, we have*

(i) $g(h(U, V), \varphi W) = g(h(V, W), FU)$,

(ii) $g(h(X, U), \varphi W) = -g(h(X, W), FU)$.

Proof. Taking the inner product in (4.1) with $V \in \Gamma(\mathfrak{D}^\theta)$, we find

$$2g(h(U, W), FV) = g(h(U, V), \varphi W) + g(h(V, W), FU). \tag{4.2}$$

By polarization identity, we obtain

$$2g(h(V, W), FU) = g(h(U, V), \varphi W) + g(h(U, W), FV). \tag{4.3}$$

From (4.2) and (4.3), we deduce the first statement of the lemma. If we take the inner product in (4.1) with $X \in \Gamma(\mathfrak{D})$, then we get the second statement. □

Furthermore, the useful equation (4.1) gives

Lemma 4.3. *Let $M = B_2 \times_f M_\perp$ be a CR-slant warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$ such that $B_2 = M_T \times M_\theta$ and $\xi \in \Gamma(TB_2)$. Then, for any $U \in \Gamma(\mathfrak{D}^\theta)$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we have*

(i) $g(h(Z, W), FU) - g(h(U, Z), \varphi W) = [TU(\ln f) + \alpha\eta(U)]g(Z, W)$,

(ii) $g(h(Z, W), FTU) - g(h(TU, Z), \varphi W) = \cos^2\theta[-U(\ln f) + \beta\eta(U)]g(Z, W)$.

Proof. Taking the inner product in (4.1) with $Z \in \Gamma(\mathfrak{D}^{\perp})$, we derive taking the inner product in (4.1) with $V \in \Gamma(\mathfrak{D}^{\theta})$, we find

$$(4.4)\quad 2g(h(U,W),\varphi Z) - g(h(U,Z),\varphi W) - g(h(Z,W),FU) = -[TU(\ln f) + \alpha\eta(U)]g(Z,W).$$

Using the polarization identity, we find

$$(4.5)\quad 2g(h(U,Z),\varphi W) - g(h(U,W),\varphi Z) - g(h(Z,W),FU) = -[TU(\ln f) + \alpha\eta(U)]g(Z,W).$$

From (4.4) and (4.5), we get (i). Second statement follows from (i) by interchanging U with TU and using Lemma 3.1. □

Let $M = B_2 \times_f M_{\perp}$ be an m-dimensional CR-slant warped product submanifold of a $(2n+1)$-dimensional nearly trans-Sasakian manifold $\tilde{M}$ such that $B = M_T \times M_{\theta}$, a Riemannian product of invariant and proper slant submanifolds of $\tilde{M}$. Conveniently, we use the tangent space of M_T, $M_{\perp}$ and M_{θ}, respectively, by $\mathfrak{D}$, $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$. If $\dim\mathfrak{D} = 2p+1$, $\dim\mathfrak{D}^{\perp} = q$ and $\dim\mathfrak{D}^{\theta} = 2s$, then the tangent bundle TM is spanned by the following orthonormal frame fields $\mathfrak{D} = \mathrm{Span}\{e_1, \ldots, e_p, e_{p+1} = \varphi e_1, \ldots, e_{2p} = \varphi e_p, e_{2p+1} = \xi\}$, $\mathfrak{D}^{\theta} = \mathrm{Span}\{e_{2p+2} = e_1^*, \ldots, e_{2p+s+1} = e_s^*,\ e_{2p+s+2} = e_{s+1}^* = \sec\theta\, Te_1^*, \ldots, e_{2p+1+2s} = e_{2s}^* = \sec\theta\, Te_s^*\}$, and $\mathfrak{D}^{\perp} = \mathrm{Span}\{e_{2p+2+2s} = \hat{e}_1, \ldots, e_m = \hat{e}_q\}$. Moreover, the normal subbundles of $T^{\perp}M$ are spanned by $\varphi\mathfrak{D}^{\perp} = \mathrm{Span}\{e_{m+1} = \varphi\hat{e}_1, \ldots, e_{m+q} = \varphi\hat{e}_q\}$, $F\mathfrak{D}^{\theta} = \mathrm{Span}\{e_{m+q+1} = \csc\theta F e_1^*, \ldots, e_{m+q+s} = \csc\theta F e_s^*, e_{m+q+s+1} = \csc\theta\sec\theta FTe_1^*, \ldots, e_{n+q+2s} = \csc\theta\sec\theta FTe_s^*\}$, and $\mu = \mathrm{Span}\{e_{m+q+2s+1}, \ldots, e_{2n+1}\}$.

Now, we use the above lemmas to proof Theorem 1.2 as follows:

Proof. From the definition of h, we have

$$\begin{aligned}
\|h\|^2 &= \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}\left(g(h(e_i,e_j),e_r)\right)^2, \\
&= \sum_{r=m+1}^{m+q}\sum_{i,j=1}^{m}\left(g(h(e_i,e_j),\varphi e_r)\right)^2 + \sum_{r=m+q+1}^{m+q+2s}\sum_{i,j=1}^{m}\left(g(h(e_i,e_j),Fe_r)\right)^2 \\
(4.6)\qquad &+ \sum_{r=m+q+2s+1}^{2n+1}\sum_{i,j=1}^{m}\left(g(h(e_i,e_j),e_r)\right)^2.
\end{aligned}$$

Leaving the μ-components term in (4.6), we find

$$\|h\|^2 \geq \sum_{r=1}^{q}\sum_{i,j=1}^{2p+1}\left(g(h(e_i,e_j),\varphi\hat{e}_r)\right)^2 + 2\sum_{j,r=1}^{q}\sum_{i=1}^{2p+1}\left(g(h(e_i,\hat{e}_j),\varphi\hat{e}_r)\right)^2$$

$$+\sum_{i,j,r=1}^{q}\left(g(h(\hat{e}_i,\hat{e}_j),\varphi\hat{e}_r)\right)^2+\sum_{r=1}^{q}\sum_{i,j=1}^{2s}\left(g(h(e_i^*,e_j^*),\varphi\hat{e}_r)\right)^2$$

$$+2\sum_{r=1}^{q}\sum_{i=1}^{2p+1}\sum_{j=1}^{2s}\left(g(h(e_i,e_j^*),\varphi\hat{e}_r)\right)^2+2\sum_{i,r=1}^{q}\sum_{j=1}^{2s}\left(g(h(\hat{e}_i,e_j^*),\varphi\hat{e}_r)\right)^2$$

$$+2\sum_{r=1}^{2s}\sum_{i=1}^{q}\sum_{j=1}^{2p+1}\left(g(h(\hat{e}_i,e_j),Fe_r^*)\right)^2+2\sum_{i,r=1}^{2s}\sum_{j=1}^{2p+1}\left(g(h(e_i^*,e_j),Fe_r^*)\right)^2$$

$$+2\sum_{i,r=1}^{2s}\sum_{j=1}^{q}\left(g(h(e_i^*,\hat{e}_j),Fe_r^*)\right)^2+\sum_{r=1}^{2s}\sum_{i,j=1}^{2p+1}\left(g(h(e_i,e_j),Fe_r^*)\right)^2$$

$$+\sum_{r=1}^{2s}\sum_{i,j=1}^{q}\left(g(h(\hat{e}_i,\hat{e}_j),Fe_r^*)\right)^2+\sum_{i,j,r=1}^{2s}\left(g(h(e_i^*,e_j^*),Fe_r^*)\right)^2. \tag{4.7}$$

Here, we find that there is no relation for the third, seventh, ninth and eleventh terms of (4.7) in terms of the warping functions. Then, by using Lemma 4.1, Lemma 4.2 and Lemma 4.3 with the assumed mixed totally condition, we find

$$\|h\|^2\geq q\sum_{r=1}^{2p+1}\left(\varphi e_r(\ln f)+\alpha\eta(e_r)\right)^2+qcsc^2\theta\sum_{r=1}^{s}\left(Te_r^*(\ln f)\right)^2$$

$$+q\cot^2\theta\sum_{r=1}^{s}\left(e_r^*(\ln f)\right)^2+4\sum_{r=1}^{2s}\sum_{i=1}^{q}\sum_{j=1}^{2p+1}\left(g(h(\hat{e}_i,e_j),Fe_r^*)\right)^2. \tag{4.8}$$

Leaving the last positive term of (4.8), then we have the following two cases:

Case (i): when ξ is tangent to M_T, we find

$$\|h\|^2\geq q\|\varphi\nabla^T(\ln f)\|^2+q\alpha^2+2q\sum_{r=1}^{2p+1}\left(\varphi e_r(\ln f)\alpha\eta(r)\right)^2$$

$$+q\csc^2\theta\|T\nabla^\theta(\ln f)\|^2-q\csc^2\theta\sum_{r=s+1}^{2s}\left(Te_r^*(\ln f)\right)^2$$

$$+q\cot^2\theta\sum_{r=1}^{s}\left(e_r^*(\ln f)\right)^2,$$

$$=2q\left(\|\nabla^T(\ln f)\|^2+\alpha-\beta^2\right)+q\cot^2\theta\|\nabla^\theta(\ln f)\|^2$$

$$-q\csc^2\theta\sum_{r=1}^{s}g\left(e_{r+s}^*,T\nabla^\theta(\ln f)\right)^2+q\cot^2\theta\sum_{r=1}^{s}\left(e_r^*(\ln f)\right)^2. \tag{4.9}$$

Hence, inequality (i) follows from (4.9). Similarly, when ξ is tangent to M_θ, case (ii), then from the orthonormal frame fields, (4.8) takes the form

$$\|h\|^2\geq q\|\varphi\nabla^T(\ln f)\|^2+q\csc^2\theta\|T\nabla^\theta(\ln f)\|^2$$

$$
\begin{aligned}
&- q\csc^2\theta \sum_{r=s+1}^{2s} (Te_r^*(\ln f))^2 + q\cot^2\theta \sum_{r=1}^{s} (e_r^*(\ln f))^2 \\
&= 2q\|\nabla^T(\ln f)\|^2 + q\cot^2\theta \left(\|\nabla^\theta(\ln f)\|^2 - \beta^2\right) \\
&- q\sec^2\theta\csc^2\theta \sum_{r=1}^{s} g\left(Te_r^*, T\nabla^\theta(\ln f)\right)^2 + q\cot^2\theta \sum_{r=1}^{s} (e_r^*(\ln f))^2, \qquad (4.10)
\end{aligned}
$$

which provides inequality (ii).

Now, we discuss the equality cases of (i) and (ii) together. From the leaving third term in (4.6), we find $h(X,Y)$ has no components in μ for all X, Y tangent to M, that is,

$$h(TM, TM) \notin \mu. \qquad (4.11)$$

The third leaving term in (4.7) with (4.11) implies

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\perp) \in F\mathfrak{D}^\theta. \qquad (4.12)$$

Also, from leaving seventh term in (4.7) with (4.11) we find

$$h(\mathfrak{D}, \mathfrak{D}) \in \varphi\mathfrak{D}^\perp. \qquad (4.13)$$

Similarly, from the leaving ninth term in (4.7) together with (4.11), we obtain

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) \in \varphi\mathfrak{D}^\perp. \qquad (4.14)$$

Also, by the leaving eleventh term in (4.7) with (4.11), we get

$$h(\mathfrak{D}, \mathfrak{D}^\theta) \in \varphi\mathfrak{D}^\perp. \qquad (4.15)$$

Furthermore, Lemma 4.1 (i) implies

$$h(\mathfrak{D}, \mathfrak{D}) \notin \varphi\mathfrak{D}^\perp. \qquad (4.16)$$

Then, from (4.13) and (4.16), we conclude that $h(\mathfrak{D}, \mathfrak{D}) = \{0\}$. Also, Lemma 4.2 (i) implies

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) \notin \varphi\mathfrak{D}^\perp. \qquad (4.17)$$

Then, (4.14) and (4.17) imply

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) = \{0\}. \qquad (4.18)$$

Further, from Lemma 4.2 (ii) and the assumption $h(\mathfrak{D}^\perp, \mathfrak{D}^\theta) = \{0\}$, we find

$$h(\mathfrak{D}, \mathfrak{D}^\perp) = \{0\}, \quad h(\mathfrak{D}, \mathfrak{D}^\theta) \notin \varphi\mathfrak{D}^\perp. \qquad (4.19)$$

From (4.15) and (4.19), we conclude that $h(\mathfrak{D}, \mathfrak{D}^\theta) = \{0\}$. Then, from the above relation and the fact that B_2 is totally geodesic and M_θ is totally umbilical in M [6], we conclude that B_2 is a totally geodesic submanifold of $\tilde{M}$, while $M_\perp$ is a totally umbilical submanifold of $\tilde{M}$. Hence, the proof is complete. □

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