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PROJECTIVE EMBEDDINGS OF BALL QUOTIENTS, BIRATIONAL TO A BI-ELLIPTIC SURFACE

Azniv Kasparian

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Abstract

For a neat lattice $\Gamma < SU(1,2)$, whose quotient $\mathbb{B}/\Gamma$ is birational to a bi-elliptic surface, we compute the dimensions of the cuspidal Γ-modular forms $[\Gamma, n]_o$ and all modular forms $[\Gamma, n]$ of weight $n \geq 2$. The work provides a sufficient condition for a subspace $V \subset [\Gamma, n]$ to determine a regular projective embedding of the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ and applies this criterion to a specific example.

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Throughout, let $\mathbb{B} = SU(1,2)/S(U_1 \times U_2)$ be the complex 2-ball, $\Gamma < SU(1,2)$ be a neat lattice, $Q := \mathbb{B}/\Gamma$, $Z := \widehat{\mathbb{B}/\Gamma} = Q \coprod (\partial_\Gamma \mathbb{B}/\Gamma)$ for the cusps $\partial_\Gamma \mathbb{B}/\Gamma = \{\kappa_j \,|\, 1 \leq j \leq h\}$ and $X := (\mathbb{B}/\Gamma)' = Q \coprod D$, $D = \sum_{j=1}^{h} D_j$ be the toroidal compactification, obtained by the resolution $\rho : X \to Z$ of κ_j, $\rho^{-1}(\kappa_j) = D_j$. Consider the space $[\Gamma, n]$ of the Γ-modular forms of weight n, the subspace $[\Gamma, n]_o$ of the cuspidal forms and a rational map $\Phi_{[\Gamma,n]} = [f_0 : \cdots : f_M] : Z \dashrightarrow \mathbb{P}([\Gamma, n]) = \mathbb{P}^M(\mathbb{C})$, defined by a basis of $[\Gamma, n]$. In [1] Baily and Borel show the existence of a sufficiently large $n \in \mathbb{N}$, for which $\Phi_{[\Gamma,n]}$ is a regular embedding. There arise the

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problems of obtaining infinite series $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$, for which $\Phi_{[\Gamma, n_k]}$ are regular embeddings, and explicit generators of subspaces $V(n) \subset [\Gamma, n]$, providing regular injective $\Phi_{V(n)} : Z \to \mathbb{P}(V(n))$.

For a $\mathbb{C}$-linear space $\mathcal{W}$ and a subgroup $\mathfrak{G} \leq \mathrm{GL}(\mathcal{W})$, let $\mathcal{W}^{\mathfrak{G}}$ be the subspace of the $\mathfrak{G}$-invariants. If S is a surface, denote $\mathfrak{K}_S := \Omega_S^{2,0}$ and put K_S for the canonical divisor. Recall the $\mathbb{C}$-linear isomorphisms $j_n : [\Gamma, n] \to H^0\left(\mathbb{B}, \mathfrak{K}_{\mathbb{B}}^{\otimes n}\right)^{\Gamma} = H^0\left(Q, \mathfrak{K}_Q^{\otimes n}\right) = H^0(Z, \mathfrak{K}_Q^{\otimes n})$, $j_n(f) = f(z)(dz_1 \wedge dz_2)^{\otimes n}$, $\forall n \in \mathbb{N}$, allowing to view $[\Gamma, n] = H^0\left(Z, \mathfrak{K}_Q^{\otimes n}\right)$. Let $\mathrm{Bl} : X \to Y$ be the blow down of the rational (-1)-curves $L_i \subset X$ to a minimal surface Y, $L = \sum_{i=1}^{s} L_i$ be the exceptional divisor of Bl and $C := \sum_{j=1}^{h} C_j$ for $C_j := \mathrm{Bl}(D_j)$. For any global meromorphic section σ of $\mathfrak{K}_X(D)$, $\sigma^{\otimes n} : \mathcal{L}(n(K_X + D)) \to H^0\left(X, \mathfrak{K}_X(D)^{\otimes n}\right)$, $\sigma^{\otimes(-n)}\rho^* : H^0\left(Z, \mathfrak{K}_Q^{\otimes n}\right) \to \mathcal{L}_X(n(K_X + D))$ are $\mathbb{C}$-linear isomorphisms. The multiplicity $m_p : (\mathrm{Div}(Y), +) \to (\mathbb{Z}, +)$ at $p \in Y$ is the homomorphism, which on any irreducible curve $M \subset Y$ has value $m_p(M) = 1$ if $p \in M$ or $m_p(M) = 0$ if $p \notin M$. Let $\mathcal{L}_Y(n(K_Y + C), nC^{\mathrm{sing}}) := \{f \in \mathbb{C}(Y)^* \mid (f) + n(K_Y + C) \geq 0,\ m_p(f) + n \geq 0,\ \forall p \in C^{\mathrm{sing}}\} \cup \{0\}$ and note that $\mathrm{Bl}^* : \mathcal{L}_Y(n(K_Y + C), nC^{\mathrm{sing}}) \to \mathcal{L}_X(n(K_X + D))$ is a $\mathbb{C}$-linear isomorphism. "The transfer" $\theta_n := (\rho^*)^{-1}\sigma^{\otimes(n)}\mathrm{Bl}^* : \mathcal{L}_Y(n(K_Y + C), nC^{\mathrm{sing}}) \to j_n([\Gamma, n])$ of rational functions on Y to Γ-modular forms is introduced by HOLZAPFEL in [2] and it restricts to $\theta_n : \mathcal{L}_Y((n-1)(K_Y + C), nC^{\mathrm{sing}}) \to j_n([\Gamma, n]_o)$.

Holzapfel's [2] constructs $X_o = (\mathbb{B}/\Gamma_o)'$ with an abelian minimal model and an explicit embedding $\Phi_V : \widehat{\mathbb{B}/\Gamma_o} \to \mathbb{P}^{22}(\mathbb{C})$, $V \subset [\Gamma_o, 3]$, whose image is cut by 20 independent relations. The present work studies $[\Gamma, n]$ for $X = (\mathbb{B}/\Gamma)'$ with a bi-elliptic minimal model $Y_{d,t} = E_1 \times E_0/G_{d,t}$. Throughout, $\Lambda_j := \pi_1(E_j)$, $\zeta_d := e^{\frac{2\pi i}{d}}$, $G_{d,t} := \langle g_d \rangle \times \langle \tau_t \rangle \simeq (\mathbb{Z}_d \times \mathbb{Z}_t, +)$, $g_d := \mathcal{T}(A_d + \Lambda_1, \Lambda_0)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \zeta_d \end{smallmatrix}\right)$, $\tau_t := \mathcal{T}(B_t' + \Lambda_1, B_t'' + \Lambda_0)$ and $\pi : E_1 \times E_0 \to Y_{d,t}$ be the etale $G_{d,t}$-Galois covering. According to [3], there is a neat $\Gamma_o \lhd \Gamma$ with $\Gamma/\Gamma_o \simeq G_{d,t}$, such that $X_o := X \times_{Y_{d,t}} (E_1 \times E_0)$ is the toroidal compactification $X_o = (\mathbb{B}/\Gamma_o)'$ of $Q_o := \mathbb{B}/\Gamma_o$, $\zeta := \mathrm{Pr}_1 : X_o \to X$ is an etale $G_{d,t}$-Galois covering and $\mathrm{Bl}_o := \mathrm{Pr}_2 : X_o \to E_1 \times E_0$ contracts the rational (-1)-curves on X_o. The resolutions $\rho : X \to Z$, $\rho_o : X_o \to Z_o := \widehat{\mathbb{B}/\Gamma_o}$ are compatible with ζ and induce a $G_{d,t}$-Galois covering $\widehat{\zeta} : Z_o \to Z$, ramified at most over $\partial_\Gamma \mathbb{B}/\Gamma$. By (3) and (4) from [4], $[\Gamma, n] = [\Gamma_o, n]^{G_{d,t}}$ and $\widehat{\zeta}^* : H^0\left(Z, \mathfrak{K}_Q^{\otimes n}\right) \simeq H^0\left(Z_o, \mathfrak{K}_{Q_o}^{\otimes n}\right)^{G_{d,t}}$. If $B := \mathrm{Bl}_o(D^o)$, $\theta_n^o : \mathcal{L}_{E_1 \times E_0}(nB, nB^{\mathrm{sing}}) \to H^0\left(Z_o, \mathfrak{K}_{Q_o}^{\otimes n}\right)$, then there is a representation $\varphi_n : G_{d,t} \to \mathrm{Aut}\mathbb{C}(E_1 \times E_0)$, $\varphi_n(g)(f) := \zeta_d^{-n} g^*(f)$, $\forall g \in G_{d,t}$, $\forall f \in \mathbb{C}(E_1 \times E_0)$, so that $\pi^* : \mathcal{L}_{Y_{d,t}}(nC, nC^{\mathrm{sing}}) \to \mathcal{L}_{E_1 \times E_0}(nB, nB^{\mathrm{sing}})^{\varphi_n(G_{d,t})}$ is a $\mathbb{C}$-linear isomorphism and $\widehat{\zeta}^*\theta_n = \theta_n^o \pi^*$.

In [4] Holzapfel obtains $\dim[\Gamma_o, n]_o$ and $\dim[\Gamma_o, n]$ for $n \geq 2$, if Γ_o is a neat lattice, whose quotient $\mathbb{B}/\Gamma_o$ is birational to an abelian surface.

Theorem 1. *For any $n > 1$ there holds* $\dim[\Gamma, n] = \dim[\Gamma, n]_o + h$, $\dim[\Gamma, n]_o = 3e(X)\binom{n}{2}$, *where $e(X)$ is the Euler number of X. Moreover,* $[\Gamma, 1]_o = \{0\}$, $\dim[\Gamma, 1] \leq h$ *and there are $\omega_{i[n]} \in [\Gamma, n]$, $n \geq 2$,*

$$(1) \qquad \omega_{i[n]}(\kappa_j) = \delta_{ij} = \begin{cases} 1 & \text{for } 1 \leq i = j \leq h, \\ 0 & \text{for } 1 \leq i \neq j \leq h, \end{cases}$$

which complete any $\mathbb{C}$-basis of $[\Gamma, n]_o$ to a $\mathbb{C}$-basis of $[\Gamma, n]$.

Proof. For a sheaf $\mathcal{L} \to S$, put $h^i(S, \mathcal{L}) := \dim H^i(S, \mathcal{L})$, $h^{i,0}(S) := \dim H^i(S, \mathcal{O}_S)$ and recall that $\chi(\mathcal{L}) := \sum_{i=0}^{2}(-1)^i h^i(X, \mathcal{L})$. If Y is a bi-elliptic surface, then $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 0$, $\mathfrak{K}_Y^{\otimes d} = \mathcal{O}_Y$, whereas $[\mathrm{Bl}^*(K_Y)]^2 = \mathrm{Bl}^*(K_Y).L = \mathrm{Bl}^*(K_Y).D = 0$, $K_X^2 = [\mathrm{Bl}^*(K_Y) + L]^2 = L^2 = -s$. Theorem 1.3 [5] asserts that $L_i.D = 4$, $\forall 1 \leq i \leq s$, so that $K_X.D = L.D = 4s$ and $(K_X + D)^2 = K_X^2 + K_X.D = 3s$. Denote $\mathfrak{S}_n := \mathfrak{K}_X(D)^{\otimes n}$, $\mathfrak{T}_n := \mathfrak{S}_n \otimes \mathcal{O}_X(-D)$, $\forall n \in \mathbb{N}$ and note that $\dim[\Gamma, 1]_o = h^0(X, \mathfrak{T}_1) = h^{2,0}(X) = h^{2,0}(Y) = 0$. Hirzebruch–Riemann–Roch Theorem for $X = (\mathbb{B}/\Gamma)'$ gives $\chi(\mathfrak{S}_n) = \chi(\mathfrak{T}_n) = 3e(X)\binom{n}{2}$, $\forall n \in \mathbb{N}$, as in the proof of Proposition 2.2 [4]. By Serre duality, $h^2(X, \mathfrak{S}_n) = h^0(X, \mathfrak{K}_X \otimes \mathfrak{S}_n^*) = \dim \mathcal{L}_X(-(n-1)(\mathrm{Bl}^* K_Y + L) - nD) = 0$, $\forall n \in \mathbb{N}$, due to $\deg(\mathrm{Bl}^*(K_Y)) = 0$, $\deg(L), \deg(D) \in \mathbb{N}$. Since $\mathfrak{T}_n$, $n > 1$ are acyclic by Proposition 2.3 [4], one has $\chi(\mathfrak{T}_n) = h^0(X, \mathfrak{T}_n) = \dim[\Gamma, n]_o$. As in [4], $(\mathfrak{S}_1 \otimes \mathcal{O}_D)^{\otimes(n-1)} = \mathcal{O}_D$ and there is an exact sequence $0 \to H^0(X, \mathfrak{T}_n) \to H^0(X, \mathfrak{S}_n) \to H^0(D, \mathcal{O}_D) \to 0$, $\forall n \geq 2$ (cf.(26) [4]). Thus, $h^0(D, \mathcal{O}_D) = h$ and $\dim[\Gamma, n] = h^0(X, \mathfrak{S}_n) = \dim[\Gamma, n]_o + h$.

If $V_i := \{\omega \in [\Gamma, n] \,|\omega(\kappa_j) = 0, \forall 1 \leq j \leq i\}$, $1 \leq i \leq h$, then $V_0 := [\Gamma, n] \supseteq V_1 \supseteq \cdots \supseteq V_i \supseteq V_{i+1} \supseteq \cdots \supseteq V_{h-1} \supseteq V_h =: [\Gamma, n]_o$ is a flag with $\dim(V_{i-1}/V_i) \leq 1$, as far as $\forall \omega'_{i-1}, \omega''_{i-1} \in V_{i-1} \setminus V_i$ satisfy $\omega'_{i-1}(\kappa_i)\omega''_{i-1} - \omega''_{i-1}(\kappa_i)\omega'_{i-1} \in V_i$. Thus, $\dim[\Gamma, 1] \leq h + \dim[\Gamma, 1]_o = h$. If $n \geq 2$, then $\dim(V_{i-1}/V_i) = 1$, $\forall 1 \leq i \leq h$, due to $\dim[\Gamma, n] = h + \dim[\Gamma, n]_o$. By an induction on h, let $\omega'_1, \ldots, \omega'_{h-1} \in [\Gamma, n]$ have $\omega'_i(\kappa_j) = \delta_{ij}$, $\forall 1 \leq i, j \leq h-1$ and $\omega'_h \in V_{h-1} \setminus V_h$. Then $\omega_{h[n]} := (\omega'_h(\kappa_h))^{-1}\omega'_h$ and $\omega_{i[n]} := \omega'_i - \omega'_i(\kappa_h)\omega_{h[n]}$, $1 \leq i \leq h-1$ satisfy (1). If $\sum_{i=1}^{h} c_i(\omega_{i[n]} + [\Gamma, n]_o) = [\Gamma, n]_o$, $c_i \in \mathbb{C}$, then evaluating at κ_j one gets $c_j = 0$ and the linear independence of $\omega_{i[n]} + [\Gamma, n]_o$, $1 \leq i \leq h$. □

Lemma 2. *If* $\mathrm{Bl} : X \to Y_{d,t}$, *then any subspace* $V(dn) \subseteq [\Gamma, dn]$, $V(dn) \ni \theta_{dn}(1), \omega_{1[d]}^n, \ldots, \omega_{h[d]}^n$ *gives a regular* $\Phi_{V(dn)} : Z \to \mathbb{P}(V(dn))$.

Proof. By $\mathrm{Bl}(L) = C^{\mathrm{sing}}$, $\mathrm{Bl}(D) = C$, one has $X \setminus L \equiv Y_{d,t} \setminus C^{\mathrm{sing}}$, $X \setminus$

$(D \cup L) \equiv Y_{d,t} \setminus C$. The rational curves $M_j := \rho(L_j) \subset Z$ have $M_j^{\mathrm{sing}}, M_i \cap M_j \subseteq \partial_\Gamma \mathbb{B}/\Gamma$, $\forall 1 \leq i \neq j \leq h$. By $D_j^2 < 0$, $C_j^2 = 0$ one has $\kappa_j \in M := \sum_{i=1}^{h} M_i$ and $Z \setminus M \equiv X \setminus (D \cup L) \equiv Y_{d,t} \setminus C$. Due to $1 \in \theta_{dn}^{-1}(V(dn))$, $\Phi_{V(dn)}|_{Z\setminus M} \equiv \Phi_{\theta_{dn}^{-1}(V(dn))}|_{Y_{d,t}\setminus C}$ is regular. If $M_i^o := M_i \setminus (\partial_\Gamma \mathbb{B}/\Gamma)$, then $\Phi_{V(dn)}|_{Z\setminus\left(\sum_{i=1}^{h} M_i^o\right)}$ is regular by $\omega_{i[d]}^n \in V(dn)$. Fix $M_i^o = M_1^o$, let $\varphi_0, \ldots, \varphi_r$ be a basis of $V(dn)$, $Z(\varphi_i) := \{x \in Q \,|\, \varphi_i(x) = 0\}$. One has to show $\Sigma := M_1^o \cap [\cap_{i=0}^r Z(\varphi_i)] = \varnothing$. Due to $M_1 \neq M_1^o$, there is $\varphi_i|_{M_1} \not\equiv 0$ and the sets $\Sigma \subseteq \Sigma_i := M_1^o \cap Z(\varphi_i)$ are discrete. Let $p \in \Sigma$ and $T(p) \subset Q$ be such a tubular neighborhood of $Z(\varphi_i)$ that $T(p) \cap \Sigma_i = \{p\}$, $T(p) \cap M_j^o = \varnothing$, $\forall j \geq 2$. The relatively closed and bounded $Z(\varphi_i) \cap T(p)$ is compact and has connected $T(p) \setminus Z(\varphi_i)$. By Hartogs' Extension Theorem, $\frac{\varphi_j}{\varphi_i} : T(p) \setminus Z(\varphi_i) \to \mathbb{C}$, $j \neq i$ have holomorphic extensions $\frac{\varphi_j}{\varphi_i} : T(p) \to \mathbb{C}$. The contradiction yields $\Sigma = \varnothing$. $\square$

By Holzapfel's [4], the abelian minimal model of a neat $X_o = (\mathbb{B}/\Gamma_o)'$ is a product $E_1 \times E_0$ of isogeneous elliptic curves and $E_1 \simeq E_0$ if E_j have CM by $\mathbb{Q}(\sqrt{-m})$, $m \in \mathbb{N}$. Combining with the classification of the bi-elliptic surfaces, if a neat $X = (\mathbb{B}/\Gamma)'$ has bi-elliptic minimal model $Y_{d,t} = E_1 \times E_0/G_{d,t}$, then $E_1 = E_0 = E$ except for $G_{d,t} \in \{G_{2,1}, G_{2,2}\}$, $\mathrm{End}(E_1) = \mathrm{End}(E_0) = \mathbb{Z}$. Moreover, $E = \mathbb{C}/\mathcal{O}_{\sqrt{-m}}$ for the integers ring $\mathcal{O}_m$ of $\mathbb{Q}(\sqrt{-m})$, $m \in \{1,3\}$ except for $\mathrm{Aut}_{\mathrm{gr}} E = \{\pm 1\}$.

By Lemma 3 [6], any elliptic curve $B_j \subset E_1 \times E_0$ is of the form $B_j = E(\alpha_j, \beta_j) + (u_j + \Lambda_1, v_j + \Lambda_0) := \{(\alpha_j c + u_j + \Lambda_1, \beta_j c + v_j + \Lambda_0) \,|\, c \in \mathbb{C}\} = \{(u + \Lambda_1, v + \Lambda_0) \,|\, \beta_j(u - u_j) - \alpha_j(v - v_j) \in \beta_j \Lambda_1 + \alpha_j \Lambda_0\}$ for some $(\alpha_j, \beta_j) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $u_j, v_j \in \mathbb{C}$. Proposition 4 [6] shows that $C_j := \pi(B_j) \subset Y_{d,t}$ is smooth iff $B_j = E_1 \times (v_j + \Lambda_0)$ or $B_j = (u_j + \Lambda_0) \times E_0$.

Recall Weierstrass' $\sigma(z) = \sigma_\Lambda(z) := z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right)^{\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$, associated with $(\Lambda, +) \simeq (\mathbb{Z}^2, +)$. If $\Lambda := \mathcal{O}_{\sqrt{-m}}$ is the integers ring of $\mathbb{Q}(\sqrt{-m})$ for $m \in \{1,3\}$ and $B_j \subset E \times E$ is an elliptic curve, defined over $\mathbb{Q}(\sqrt{-m})$, then there exist relatively prime $\alpha_j, \beta_j \in \mathcal{O}_{\sqrt{-m}}$ with $B_j = E(\alpha_j, \beta_j) + (u_j + \Lambda, v_j + \Lambda)$.

From now on, for a a rational function $f : N \dashrightarrow \mathbb{C}$ on a curve or a surface N, denote by $(f)_\infty^N$ the pole divisor of f on N.

Lemma 3. *Let* $E = \mathbb{C}/\Lambda$ *with* $\Lambda = \mathcal{O}_{\sqrt{-m}}$, $m \in \{1,3\}$ *and* $B_j = E(\alpha_j, \beta_j) + (u_j + \Lambda, v_j + \Lambda) \subset E \times E$, $1 \leq j \leq k$ *be smooth elliptic curves with* $\alpha_j, \beta_j \in \mathcal{O}_{\sqrt{-m}}$, $\alpha_j \mathcal{O}_{\sqrt{-m}} + \beta_j \mathcal{O}_{\sqrt{-m}} = \mathcal{O}_{\sqrt{-m}}$. *If* $\sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} \beta_j = 0$ *and* $\gamma_j := -\beta_j u_j + \alpha_j v_j$, *then for* $\forall \mu \in \mathbb{C} \setminus \mathcal{O}_{\sqrt{-m}}$, $f(u,v) := \prod_{j=1}^{k} \frac{\sigma(\beta_j u - \alpha_j v + \gamma_j + \mu)}{\sigma(\beta_j u - \alpha_j v + \gamma_j)} : \mathbb{C}^2 \dashrightarrow \mathbb{C}$ *is* $(\Lambda \times \Lambda, +)$*-periodic and induces an abelian function* $f : E \times E \dashrightarrow \mathbb{C}$ *with* $(f)_\infty^{E\times E} = \sum_{j=1}^{k} B_j$.

Proof. There is a homomorphism $\eta = \eta_\Lambda : (\Lambda, +) \to (\mathbb{C}, +)$ and a map $\varepsilon : \Lambda \to \{\pm 1\}$ with $\varepsilon(\lambda) = 1$ for $\lambda \in 2\Lambda$ or $\varepsilon(\lambda) = -1$ for $\lambda \in \Lambda \setminus 2\Lambda$, such that $\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda) e^{\eta(\lambda)\left(z+\frac{\lambda}{2}\right)}$ for $\forall \lambda \in \Lambda$, $\forall z \in \mathbb{C}$. Thus, $\frac{f(u+\lambda,v)}{f(u,v)} = e^{\left[\sum_{j=1}^{k} \eta(\beta_j\lambda)\right]\mu}$, $\frac{f(u,v+\lambda)}{f(u,v)} = e^{-\left[\sum_{j=1}^{k} \eta(\alpha_j\lambda)\right]\mu}$ for $\forall \lambda \in \Lambda$. Legendre's equality $\eta(\omega_{\sqrt{-m}}) - \omega_{\sqrt{-m}}\eta(1) = 2\pi\sqrt{-1}$ for $\omega_{\sqrt{-1}} := \sqrt{-1}$, $\omega_{\sqrt{-3}} = e^{\frac{2\pi i}{6}}$, $\mathcal{O}_{\sqrt{-m}} = \mathbb{Z} + \omega_{\sqrt{-m}}\mathbb{Z}$ implies $\eta(\beta_j\lambda) = \beta_j\lambda\eta(1) + \frac{4\pi}{(5-m)\sqrt{m}}\left(\beta_j\lambda - \overline{\beta_j\lambda}\right)$, so that $\sum_{j=1}^{k}\eta(\beta_j\lambda) = \left(\sum_{j=1}^{k}\beta_j\right)\left\{\eta(1) + \frac{4\pi}{(5-m)\sqrt{m}}\right\}\lambda - \overline{\left(\sum_{j=1}^{k}\beta_j\right)\frac{4\pi}{(5-m)\sqrt{m}}}\overline{\lambda} = 0$, $\frac{f(u+\lambda,v)}{f(u,v)} = 1$. By $(\sigma_\Lambda)_0^{\mathbb{C}} = \Lambda$ follows $(f)_\infty^{E\times E} = \sum_{j=1}^{k} B_j$. $\square$

Let $\mathrm{Irr}(\Delta)$ be the set of the irreducible components of a divisor Δ.

Proposition 4. *Let* $Y_{d,t} = E \times E / G_{d,t}$ *with* $E = \mathbb{C}/\Lambda$, $\Lambda = \mathcal{O}_{\sqrt{-m}}$, $m \in \{1, 3\}$, $m_p(C) \geq 2$ *for* $\forall p \in C^{\mathrm{sing}}$ *and* $C_1 = \pi((u_0 + \Lambda_1) \times E)$, $C_2 = \pi(E \times (v_0 + \Lambda_0))$ *be intersecting smooth irreducible components of* C *with* $\left|\mathrm{Irr}(\pi^{-1}(C_2))\right| = dt$ *(always* $\left|\mathrm{Irr}(\pi^{-1}(C_1))\right| = dt$*). Then:*

(i) $\psi_1(u) := \prod_{k\in\mathbb{Z}_d}\prod_{l\in\mathbb{Z}_t} \frac{\sigma\left(u - u_0 - kA_d - lB'_t + e^{\frac{2\pi i k}{d}}\mu_1\right)}{\sigma\left(u - u_0 - kA_d - lB'_t\right)}$ *for* $\mu_1 \in \mathbb{C}\setminus\Lambda$, $\left(e^{\frac{2\pi i}{d}} - 1\right)\mu_1 \in \Lambda$, $d \in \{2,3,4\}$,

$\psi_1(u) = \prod_{k\in\mathbb{Z}_3} \frac{\sigma\left(u - u_0 - kA_6 + e^{\frac{2\pi i k}{3}}\mu_1\right)\sigma\left(u - u_0 - (k+3)A_6 + e^{\frac{2\pi i k}{3}}\mu_1\right)}{\sigma\left(u - u_0 - kA_6\right)\sigma\left(u - u_0 - (k+3)A_6\right)}$ *for* $\mu_1 \in \mathbb{C}\setminus\mathcal{O}_{\sqrt{-3}}$, $\left(e^{\frac{2\pi i}{3}} - 1\right)\mu_1 \in \mathcal{O}_{\sqrt{-3}}$, $d = 6$,

$\psi_2(v) = \prod_{k\in\mathbb{Z}_d}\prod_{l\in\mathbb{Z}_t} \frac{\sigma\left(v - e^{\frac{2\pi i k}{d}}v_0 - lB''_t + e^{\frac{2\pi i k}{d}}\mu_2\right)}{\sigma\left(v - e^{\frac{2\pi i k}{d}}v_0 - lB''_t\right)}$ *for* $\mu_2 \in \mathbb{C}\setminus\Lambda$ *have pole divisors* $(\psi_j)_\infty^{E\times E} = \pi^{-1}(C_j)$ *and* $\psi_j^d \in \mathcal{L}_{E\times E}(dB, dB^{\mathrm{sing}})^{\varphi_d(G_{d,t})}$;

(ii) if $\psi_1^{m_1}, \psi_2^{m_2}, \psi_0 \in \mathcal{L}_{E\times E}(dB, dB^{\mathrm{sing}})^{\varphi_d(G_{d,t})}$, $m_j \in \mathbb{N}$ *are such that* $\Psi :=$

$(\psi_1^{m_1}, \psi_2^{m_2}, \psi_0) : E \times E \dashrightarrow \Psi(E \times E)$ *is of* $\deg(\Psi) = dt$, *then* $V(dn) := l_{\mathbb{C}}\left(\theta_{dn}(\pi^*)^{-1}(S_o), \omega_{1[d]}^n, \ldots, \omega_{h[d]}^n\right)$ *with* $S_o := \{1, \psi_1^{m_1}, \psi_2^{m_2}, \psi_0\}$ *provides a regular embedding* $\Phi_{V(dn)} : Z \to \mathbb{P}(V(dn))$, $\forall n \in \mathbb{N}$.

As a result, $\Phi_{[\Gamma,dn]} : Z \to \mathbb{P}([\Gamma, dn]) = \mathbb{P}^{\frac{3e(X)dn(dn-1)}{2}+h-1}(\mathbb{C})$ *is a regular embedding for* $\forall n \in \mathbb{N}$.

Proof. (i) In $\psi_1(u)$, $\psi_2(v)$ one multiplies the arguments of σ by $e^{-\frac{2\pi ik}{d}}$, in order to conform with the assumptions of Lemma 3 and to derive the $(\Lambda,+)$-periodicity of these functions. Let us fix liftings $\widetilde{g_d} := \mathcal{T}(A_d, 0)\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{d}} \end{pmatrix}$, $\widetilde{\tau_t} := \mathcal{T}(B_t', B_t'') \in (\mathcal{T}(\mathbb{C}) \rtimes \mathrm{Aut}_{\mathrm{gr}}E)^2$ of g_d, $\tau_t \in (\mathrm{Aut}E)^2 = (\mathcal{T}(E) \rtimes \mathrm{Aut}_{\mathrm{gr}}E)^2$. Straightforwardly, $\frac{\widetilde{g_d}^*(\psi_j)}{\psi_j}$, $\frac{\widetilde{\tau_t}^*(\psi_j)}{\psi_j}$ have no zeroes and poles by $\left(e^{\frac{2\pi i}{d}} - 1\right) B_t'' \in \Lambda$ from the classification of $Y_{d,t}$. Thus, there are $c_{d,j}, \zeta_{t,j} \in \mathbb{C}$ with $\widetilde{g_d}^*(\psi_j) = c_{d,j}\psi_j$, $\widetilde{\tau_t}^*(\psi_j) = \zeta_{t,j}\psi_j$. By an induction on $r \in \mathbb{N}$, $(\widetilde{g_d}^r)^*(\psi_j) = c_{d,j}^r\psi_j$ and $(\widetilde{\tau_t}^r)^*(\psi_j) = \zeta_{t,j}^r\psi_j$. Since $\widetilde{g_d}^d, \widetilde{\tau_t}^t \in \mathcal{T}(\Lambda \times \Lambda)$ preserve ψ_j, one has $\psi_j = \left(\widetilde{g_d}^d\right)^*(\psi_j) = c_{d,j}^d\psi_j$, $\psi_j = \left(\widetilde{\tau_t}^t\right)^*(\psi_j) = \zeta_{t,j}^t\psi_j$, whereas $c_{d,j}^d = \zeta_{t,j}^t = 1$. Thus, $\widetilde{g_d}^*\left(\psi_j^d\right) = [\widetilde{g_d}^*(\psi_j)]^d = \psi_j^d$, $\widetilde{\tau_t}^*\left(\psi_j^t\right) = [\widetilde{\tau_t}^*(\psi_j)]^t = \psi_j^t$ and $\widetilde{\tau_t}^*\left(\psi_j^d\right) = \psi_j^d$, since t divides d. That proves the $\varphi_d(G_{d,t})$-invariance of ψ_j^d and concludes (i).

(ii) By Lemma 2, $\Phi_{V(dn)}$ is regular. Due to $\theta_{dn}(1)(Z \setminus M) \subseteq \mathbb{C}^*$ and $\theta_{dn}(1)|_M \equiv 0$, $\Phi_{V(dn)}(Y_{d,t} \setminus C) \cap \Phi_{V(dn)}(M) = \varnothing$. The rational map $\overline{\Psi} := (\pi^*)^{-1}(\Psi) : Y_{d,t} \dashrightarrow \overline{\Psi}(Y_{d,t})$ is of $\deg(\overline{\Psi}) = \frac{\deg(\Psi)}{dt} = 1$. By assumption, $|M_i \cap (\partial_\Gamma \mathbb{B}/\Gamma)| \geq 2$, so that all the fibres of $\Phi_{V(dn)}|_{M_i}$, $\Phi_{V(dn)}|_M$ are finite and $\Phi_{V(dn)}$, $\Phi_{[\Gamma,dn]}$ are injective by $V(dn) \subseteq [\Gamma, dn]$. □

Theorem 5 (Di Cerbo and Stover [5])**.** *Let* $E = \mathbb{C}/\Lambda$ *for* $\Lambda = \mathcal{O}_{\sqrt{-1}}$, $\lambda_0 := 0$, $\lambda_1 := \frac{1}{2}$, $\lambda_2 := \frac{\sqrt{-1}}{2}$, $\lambda_3 := \frac{1+\sqrt{-1}}{2}$, $T_j := \lambda_j + \Lambda \in E^{2-\mathrm{tor}}$ *for* $0 \leq j \leq 3$. *Consider the elliptic curves* $B_j := E\left(1, \sqrt{-1}^j\right)$, $1 \leq j \leq 4$, $B_{4+k} := T_k \times E$, $B_{6+k} := E \times T_k$, $1 \leq k \leq 2$ *and put* $B := \sum_{j=1}^{8} B_j$, $\mathfrak{g}_2 := \mathcal{T}(T_3, T_3)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in (\mathrm{Aut}E)^2$, $t_{ij} := \pi(T_i, T_j) \in Y_2 := E \times E/\langle \mathfrak{g}_2 \rangle$. *Then* $C_j := \pi(B_j)$,$1 \leq j \leq 2$ *are singular,* $C_k := \pi(B_{2k-1})$, $3 \leq k \leq 4$ *are intersecting smooth components of* $C := \sum_{j=1}^{4} C_j$, *and the blow up* $X = (\mathbb{B}/\Gamma)'$ *of* Y_2 *at* $C^{\mathrm{sing}} = \{t_{00}, t_{11}, t_{12}\}$ *is a neat toroidal compactification with* $e(X) = 3$ *and* $D := X \setminus (\mathbb{B}/\Gamma) = \sum_{j=1}^{4} D_j$.

From now on the notations from Theorem 5 are valid.

Theorem 6. *Note that*

$$e_{157} := \frac{\sigma(iu - v + \lambda_3)\sigma(-iu + i\lambda_1 + \lambda_3)\sigma(v - \lambda_1 + \lambda_3)}{\sigma(iu - v)\sigma(-iu + i\lambda_1)\sigma(v - \lambda_1)},$$

$$e_{467} := \frac{\sigma(u - v + \lambda_3)\sigma(-u + \lambda_2 + \lambda_3)\sigma(v - \lambda_1 + \lambda_3)}{\sigma(u - v)\sigma(-u + \lambda_2)\sigma(v - \lambda_1)},$$

$e_{56} := \dfrac{\sigma(u)\sigma(-u + \lambda_3)}{\sigma(u - \lambda_1)\sigma(-u + \lambda_2)} v$ *are from* $\mathcal{L}_{E\times E}\left(B, B^{\text{sing}}\right)$*. If* $\widetilde{\mathfrak{g}_2} := \mathcal{T}(\lambda_3, \lambda_3)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in (\mathcal{T}(\mathbb{C}) \rtimes \text{Aut}_{\text{gr}}E)^2$*, then* $f_{134} := (\pi^*)^{-1}(e_{157} - \widetilde{\mathfrak{g}_2}^*(e_{157}))$*,* $f_{234} := (\pi^*)^{-1}\left(e_{467} - \widetilde{\mathfrak{g}_2}^*(e_{467})\right)$*,* $f_3 := (\pi^*)^{-1}(e_{56}) \in \mathcal{L}_{Y_2}\left(C, C^{\text{sing}}\right)$*,* $\Upsilon_{134} := \theta_1(f_{134})$*,* $\Upsilon_{234} := \theta_1(f_{234})$*,* $\Upsilon_3 := \theta_1(f_3)$ *is a* $\mathbb{C}$*-basis of* $[\Gamma, 1]$ *and* $\Phi_{[\Gamma,1]} = [\Upsilon_{134} : \Upsilon_{234} : \Upsilon_3] : Z \to \mathbb{P}([\Gamma, 1]) = \mathbb{P}^2(\mathbb{C})$ *is a non-regular, finite dominant rational map.*

Proof. Note that $(e_{i_1\dots i_s})_\infty^{E\times E} = B_{i_1} + \dots + B_{i_s}$. For $f \in \mathcal{L}_{Y_2}(C, C^{\text{sing}})$, $C_j \subset (f)_\infty^{Y_2}$ if and only if $\theta_1(f)(\kappa_j) \neq 0$. Let $c_1\Upsilon_{134} + c_2\Upsilon_{234} + c_3\Upsilon_3 = 0$, $c_i \in \mathbb{C}$. The evaluation at κ_i, $1 \leq i \leq 3$ yields a homogeneous linear system on c_i with invertible coefficient matrix $\begin{pmatrix} \Upsilon_{134}(\kappa_1) & 0 & 0 \\ 0 & \Upsilon_{234}(\kappa_2) & 0 \\ \Upsilon_{134}(\kappa_3) & \Upsilon_{234}(\kappa_3) & \Upsilon_3(\kappa_3) \end{pmatrix}$ and proves the linear independence of Υ_{134}, Υ_{234}, Υ_3. If $V_i := \{\omega \in [\Gamma, n] | \omega(\kappa_j) = 0, \forall 1 \leq j \leq i\}$, $\Upsilon \in [\Gamma, 1]$, then $\varphi_1 := \Upsilon_{134}(\kappa_1)\Upsilon - \Upsilon(\kappa_1)\Upsilon_{134} \in V_1$, $\varphi_{12} := \Upsilon_{234}(\kappa_2)\varphi_1 - \varphi_1(\kappa_2)\Upsilon_{234} \in V_2$, $\varphi_0 := \Upsilon_3(\kappa_3)\varphi_{12} - \varphi_{12}(\kappa_3)\Upsilon_3 \in V_3$. When $\varphi_o \not\equiv 0$, $e := \pi^*\theta_1^{-1}(\varphi_o)$ has $(e)_\infty^{E\times E} = \pi^{-1}(C_4) = B_7 + B_8$ by $[\Gamma, 1]_o = 0$ and $(e)_\infty^{E\times(v_o+\Lambda)} = \varnothing$, $\forall v_o + \Lambda \in E \setminus \{T_1, T_2\}$, so that $e = e(v) = \dfrac{\sigma(v - \alpha_1)\sigma(v - \lambda_3 + \alpha_1)}{\sigma(v - \lambda_1)\sigma(v - \lambda_2)} = \widetilde{\mathfrak{g}_2}^*(e) \notin \mathcal{L}_{E\times E}(B, B^{\text{sing}})^{\varphi_1(\langle \mathfrak{g}_2 \rangle)}$ is an absurd. Thus, $\varphi_0 \equiv 0$, $\varphi_{12} \in \text{Span}_{\mathbb{C}}(\Upsilon_3)$, $\varphi_1 \in \text{Span}_{\mathbb{C}}(\Upsilon_{234}, \Upsilon_3)$, $\Upsilon \in \text{Span}_{\mathbb{C}}(\Upsilon_{134}, \Upsilon_{234}, \Upsilon_3)$ and Υ_{134}, Υ_{234}, Υ_3 is a $\mathbb{C}$-basis of $[\Gamma, 1]$.

If $\Phi_{[\Gamma,1]} := [\Upsilon_{134} : \Upsilon_{234} : \Upsilon_3]$, then $\Phi_{[\Gamma,1]}|_{Y_2\setminus C} = [f_{134} : f_{234} : f_3]$ is not defined at t_{03}. Let $f(u, v) := \dfrac{\pi^*(f_{234})(u, v)}{\pi^*(f_3)(u)}$, whereas $f(iu, v) = e^{\frac{\eta(i)}{2}} \dfrac{\pi^*(f_{134})(u, v)}{\pi^*(f_3)(u)}$ and we need $\Psi := (f(iu, v), f(u, v)) : A_o := E \times E \setminus B \dashrightarrow \mathbb{C}^2$ to be of $\text{rk}(\Psi) = 2$. For any $\phi : A_o \dashrightarrow \mathbb{C}$ denote $\text{grad}(\phi) := \left(\dfrac{\partial \phi}{\partial u}, \dfrac{\partial \phi}{\partial v}\right)$ to have $T_p^{1,0}\Psi^{-1}(x_1, x_2) = l_{\mathbb{C}}\left(\text{grad}(f(iu, v)), \text{grad}(f(u, v))\right)^\perp$. If $\text{rk}(\Psi) < 2$, then generically $\dim \Psi^{-1}(x_1, x_2) = 1$ and $l_{\mathbb{C}}\left(\text{grad}(f(u, v))\right) = l_{\mathbb{C}}\left(\text{grad}(f(iu, v))\right) = l_{\mathbb{C}}\left(\text{grad}(f(-u, v)\right)$. Thus, $\Psi_o := (f(-u, v), f(u, v)) : A_o \dashrightarrow \mathbb{C}^2$ is of $\text{rk}(\Psi_o) = 1$, $\zeta(u, v) := \dfrac{f(-u, v)}{f(u, v)} : A_o \dashrightarrow \mathbb{C}$ takes finitely many values by $f(u, v)(A_o) = \mathbb{C}$ and $\zeta(u, v) \equiv C_o \in \mathbb{C}$. If $f_\varepsilon(u, v) := \dfrac{1}{2}[f(u, v) + \varepsilon f(-u, v)]$, $\varepsilon = \pm 1$, then $(1 - C_o)f_1(u, v) = (1 + C_o)f_{-1}(u, v)$ requires $C_o = \pm 1$. Due to $\pi^*(f_3)(-u) = -\pi^*(f_3)(u)$, $\pi^*(f_{234})(-u, v) = -C_o\pi^*(f_{234})(u, v)$ and $e_{2467}(u, v) = e_{2458}(u, v)$ for $e_{2467}(u, v) := e_{467}(-u, v) + C_o e_{467}(u, v)$, $e_{2458}(u, v) := \widetilde{\mathfrak{g}_2}^*(e_{2467})(u, v)$. If

$v+\Lambda \neq T_1, T_2$, then $\infty = e_{2467}(\lambda_2, v) = \widetilde{\mathfrak{g}_2}^*(e_{2467})(\lambda_2, v) = e_{2467}(\lambda_1, v) = 0$ is an absurd, showing $\mathrm{rk}(\Psi) = 2$. $\square$

Corollary 7. *Note that*

$$\psi_1(u) := e_{56}(u), \ \psi_2(v) = h_{78}(v) := \frac{\sigma(v)\sigma(-v+\lambda_3)}{\sigma(v-\lambda_1)\sigma(-v+\lambda_2)}$$

belong to $\mathcal{L}_{E\times E}\left(B, B^{\mathrm{sing}}\right)$, $h_{467}(u,v) := \dfrac{\sigma(u-v+\lambda_1)\sigma(-u+\lambda_2+\lambda_1)\sigma(v)}{\sigma(u-v)\sigma(-u+\lambda_2)\sigma(v-\lambda_1)}$ *is from* $\mathcal{L}_{E\times E}\left(B, 2B^{\mathrm{sing}}\right)$ *and let* $\omega_{i[2]} \in [\Gamma, 2]$, $i \in \{1,2,4\}$ *satisfy (1). Then* $\psi_0 := h_{467} + \widetilde{\mathfrak{g}_2}^*(h_{467})$, $\psi_2 \in \pi^* \mathcal{L}_{Y_2}(C, 2C^{\mathrm{sing}})$, *the rational map* $\Psi := \left(\psi_1^2, \psi_2, \psi_0\right)$: $E\times E \dashrightarrow \Psi(E\times E)$ *is of* $\deg(\Psi) = 2$, $V(2n) := l_{\mathbb{C}}\left(\theta_{2n}\,(\pi^*)^{-1} S_o, \omega^n_{1[2]}, \ldots, \omega^n_{4[2]}\right)$ *with* $S_o := \left\{1, \psi_1^2, \psi_2, \psi_0\right\}$ *has* $\dim V(2) = 7$, $\dim V(2n) = 8$, $\forall n \geq 2$ *and the morphisms* $\Phi_{V(2n)} : Z \to \mathbb{P}(V(2n))$, $\Phi_{[\Gamma, 2n]} : Z \to \mathbb{P}^{9n(2n-1)+3}(\mathbb{C})$ *are regular embeddings for* $\forall n \in \mathbb{N}$.

Proof. At $\forall x_1 \in \mathbb{C}^*$, $(\psi_1^2)^{-1}(x_1) = \psi_1^{-1}(\sqrt{x_1}) \coprod \psi_1^{-1}(-\sqrt{x_1}) \subset E\times E$, $\psi_1^{-1}(\varepsilon\sqrt{x_1}) = \{(P(\varepsilon\sqrt{x_1}), Q), (T_3 - P(\varepsilon\sqrt{x_1}), Q) \,|\, Q \in E\}$, $\varepsilon = \pm 1$ by $(\psi_1)^E_\infty = \{T_1, T_2\}$. Since $\langle \widetilde{\mathfrak{g}_2}\rangle$ acts on $(\psi_1^2)^{-1}(x_1)$, if $P(\sqrt{x_1}) \notin E^{2-\mathrm{tor}}$, then $\psi_1^{-1}(-\sqrt{x_1}) = \{(T_3 + P(\sqrt{x_1}), Q), (-P(\sqrt{x_1}), Q) \,|\, Q \in E\}$. Also $\psi_2^{-1}(x_2) = \{(P, Q(x_2)), (P, T_3 - Q(x_2)) \,|\, P \in E\}$, $\forall x_2 \in \mathbb{C}$ by $(\psi_2)^E_\infty = \{T_1, T_2\}$. If $2P(\sqrt{x_1}), 2Q(x_2) \notin \langle T_3\rangle$, then $(\psi_1^2, \psi_2)^{-1}(x_1, x_2) = \coprod_{P\in S_{12}} \mathrm{Orb}_{\langle\widetilde{\mathfrak{g}_2}\rangle}(P, Q(x_2))$ for $S_{12} := \{\pm P(\sqrt{x_1}), T_3 \mp P(\sqrt{x_1})\}$, $|S_{12}| = 4$, $\sum_{P\in S_{12}} P = T_0$ and $(\psi_2, \psi_0)^{-1}(x_2, x_3) = \coprod_{P\in S_{20}} \mathrm{Orb}_{\langle\widetilde{\mathfrak{g}_2}\rangle}(P, Q(x_2))$ with $\sum_{P\in S_{20}} P = T_3$ by $(\psi_0(u, Q(x_2)))^E_\infty = \{\pm Q(x_2), T_1, T_2\}$, where $|S_{20}| = 4$ for a generic $x_3 \in \mathbb{C}$. That implies $\left|\Psi^{-1}(x) \cap [E\times Q(x_2)]\right| < 4$, $|\Psi(x)| < 8$ and $\deg\Psi \in \{2, 4\}$ as a divisor of $\deg(\psi_2, \psi_0) = 8$. If $\deg(\Psi) = 4$ and $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ is generic, then $\Psi^{-1}(x) = \mathrm{Orb}_{\langle\widetilde{\mathfrak{g}_2}\rangle}(P(\sqrt{x_1}), Q(x_2)) \coprod \mathrm{Orb}_{\langle\widetilde{\mathfrak{g}_2}\rangle}(P, Q(x_2))$ for $P \in S_{12} \setminus \{P(\sqrt{x_1})\}$. Thus, ψ_0 satisfies exactly one of the following 3 identities: (A) $\psi_0(u,v) = \psi_0(\lambda_3 - u, v)$, (B) $\psi_0(u,v) = \psi_0(\lambda_3 + u, v)$ or (C) $\psi_0(u,v) = \psi_0(-u, v)$. For $2(v_o + \Lambda) \notin \langle T_3\rangle$, note that $\psi_0(v_o, v_o) = \infty \neq \psi_0(\lambda_3 \pm v_o, v_o)$ rules out (A), (B). If (C), $h_{2467}(u,v) := h_{467}(u,v) - h_{467}(-u,v)$, $h_{2458}(u,v) := -\widetilde{\mathfrak{g}_2}^*(h_{2467})(u,v)$, then $h_{2467}(u,v) = h_{2458}(u,v)$ and $h_{2467}(\lambda_2, v_o) = \infty \neq -h_{2467}(\lambda_1, \lambda_3 - v_o) = h_{2458}(\lambda_2, v_o)$ for $v_o + \Lambda \in E \setminus E^{2-\mathrm{tor}}$. Thus, $\deg\Psi = 2$ and the regular map $\Phi_{V(2n)}$ is injective.

Note that $\dim\left(V(2n)/V(2n) \cap [\Gamma, 2n]_o\right) = 4$ by $\omega^n_{i[2]} \in V(2n)$. Due to 1, ψ_2, $\psi_0 \in \pi^*\mathcal{L}_{Y_2}(C, 2C^{\mathrm{sing}}) \subseteq \pi^*\theta_{2n}^{-1}([\Gamma, 2n]_o)$, $\forall n \in \mathbb{N}$ and $\psi_1^2 \in \pi^*\mathcal{L}_{Y_2}(2C, 2C^{\mathrm{sing}}) \subseteq \pi^*\theta_{2n}^{-1}([\Gamma, 2n]_o)$, $\forall n \geq 2$, one has $V(2) \cap [\Gamma, 2]_o = l_{\mathbb{C}}(\theta_2\,(\pi^*)^{-1}(S_o \setminus \{\psi_1^2\}))$ and $V(2n) \cap [\Gamma, 2n]_o = l_{\mathbb{C}}(\theta_2\,(\pi^*)^{-1}(S_o))$ for $\forall n \geq 2$. If $u_o + \Lambda \in E \setminus E^{2-\mathrm{tor}}$, then $(u_o + \Lambda, u_o + \Lambda) \in (\psi_0)^{E\times E}_\infty \setminus \left((\psi_1^2)^{E\times E}_\infty \cup (\psi_2)^{E\times E}_\infty\right)$, so that $\psi_0 \notin \mathrm{Span}_{\mathbb{C}}(1, \psi_1^2, \psi_2)$.

That is why, $\dim(V(2)\cap[\Gamma,2]_o)=3$ and $\dim\left(V(2n)\cap[\Gamma,2n]_o\right)=4$ for $\forall n\geq 2$. $\square$

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Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5 James Bouchier Blvd
1164 Sofia, Bulgaria
e-mail: kasparia@fmi.uni-sofia.bg