Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 75, No 12, 2022

MATHEMATICS

Group theory

ON WEAKLY ss-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

Guo Zhong, Shixun Lin*,#, Yan Han*, Guoqiong Zhou*

Received on September 16, 2022
Presented by V. Drensky, Member of BAS, on September 27, 2022

Abstract

Let H be a subgroup of a finite group G. We say that H is weakly ss-supplemented in G if there exists a subgroup T of G and an s-semipermutable subgroup H_{ss} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ss}$. In this paper, we get some new characterizations of supersolvability and nilpotentcy of G by assuming some minimal subgroups of G are weakly ss-supplemented. Some recent results are extended and generalized.

Key words: weakly ss-supplemented subgroups, nilpotentcy, supersolvability, saturated formation

2020 Mathematics Subject Classification: 20D10, 20D20

1. Introduction. In this article, we suppose all groups are always finite, and both notions and notations have the same meaning as in Robinson [14]. As usual, $\pi(G)$ denotes the set of all prime divisors of the order of G, and H char G is the meaning that H is a characteristic subgroup of G. $\mathcal{U}$, $\mathcal{N}$ and $G^{\mathcal{F}}$ denote the classes of all supersolvable groups, nilpotent groups and the $\mathcal{F}$-residual (a formation $\mathcal{F}$), respectively. $Z_{\mathcal{F}}(G)$ is, then, the $\mathcal{F}$-hypercentre of G. $Z_{\infty}(G)$ is the hypercentre of G.

[#]Corresponding author.

The research is partially supported by Special Basic Cooperative Research Programmes of Yunnan Provincial Undergraduate Universities' Association (Grant No 202101BA070001-045) and Scientific Research Fund Projects of Yunnan Provincial Education Department (Grant No 2022J0979, No 2021J0480).

DOI:10.7546/CRABS.2022.12.02

We say a class of groups $\mathcal{F}$ is a *formation*, if $\mathcal{F}$ satisfies the following conditions: (1) $G/H \in \mathcal{F}$, if $G \in \mathcal{F}$; (2) $G/M \cap N \in \mathcal{F}$, if $G/M, G/N \in \mathcal{F}$, where H, M, N are normal subgroups of G. A formation $\mathcal{F}$ is called *saturated*, if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. At the same time, a formation $\mathcal{F}$ is one in which all subgroups of G are in $\mathcal{F}$ whenever $G \in \mathcal{F}$, then we call it *s-closed*. $\mathcal{U}$ is clearly an s-closed saturated formation. A subgroup H is called *$\mathcal{F}$-supplemented* in G, if G possesses a subgroup $T \in \mathcal{F}$ with $G = HT$, and T an $\mathcal{F}$-supplement of H in G.

Let H, K be subgroups of G, and then H, K are *permutable* when $HK = KH$. We say that H is *S-quasinormal* in G, which was introduced in [11], if $PH = HP$ for each Sylow subgroup P of G. Chen [6] extended the notion of S-quasinormal subgroups to s-semipermutable subgroups. Hence, the subgroup H is *s-semipermutable* such that $PH = HP$, if H is permutable with each Sylow p-subgroup P of G with $(p, |H|) = 1$.

It has recently become popular to use supplementation properties of subgroups to characterize the structure of a group. Wang [16], for example, presented c-normality and proposed some new criteria for group solvability and supersolvability. Skiba [15] presented the following notion as a generalization of c-normal subgroups: a subgroup H of G is *weakly s-permutable*, if there exists a subgroup T of G satisfying $G = HT$ and $H \cap T \le H_sG$, where H_sG is generated by all subgroups (S-quasinormal in G) of H. Wang [17] expanded the research of c-normal subgroups to c-supplemented subgroups in this way. A subgroup H of G is called *c-supplemented* in G, if a subgroup K of G exists satisfying $G = HK$ and $H \cap K \le H_G$, where $H_G = \cap_{g \in G} H^g$ is the largest normal subgroup of G that is contained in H.

As a continuation of that work, we combine all of the notions stated above with the conception of weakly ss-supplemented subgroups, as shown below.

Definition 1.1. A subgroup H of G is *weakly ss-supplemented* in G, if a subgroup T of G exists, and an s-semipermutable subgroup H_{ss} of G is contained in H satisfying $G = HT, H \cap T \le H_{ss}$.

Obviously, weakly ss-supplemented subgroups of G include normal subgroups, c-normal subgroups, c-supplemented subgroups, permutable subgroups, S-quasinormal subgroups, and weakly s-permutable subgroups. However, the instances below demonstrate that the opposite is not always true.

Example 1.2. C_5 is weakly ss-supplemented in A_4C_5, since $A_4C_5 = C_5A_4$ and $C_5 \cap A_4 = 1$. But C_5 is obviously not normal, c-normal, permutable, s-permutable, and not weakly s-permutable in A_4C_5.

Example 1.3. Setting G$=\langle a, b \mid a^4 = 1, a^2 = b^2$ and $b^{-1}ab = a^{-3}\rangle$, then G is a 2-group and $\Phi(G) = \langle a^2, b^2\rangle = \langle a^2\rangle \times \langle b^2\rangle$. It, thus, is easy to know that $\langle b^2\rangle$ is s-semipermutable and weakly ss-supplemented in G. Unfortunately, $\langle b^2\rangle$ is not c-supplemented in G. In fact, $\langle b^2\rangle$ has only a supplemented subgroup G in G, but $\langle b^2\rangle \cap G = \langle b^2\rangle$ is not normal.

Example 1.4. In S_4 (the symmetric group of degree 4), $\langle(34)\rangle$ is weakly ss-supplemented and not s-semipermutable.

Minimal subgroups of Sylow subgroups are significant in identifying the structure of a group. For instance, BUCKLEY [5] claimed that an odd-order group G is supersolvable when arbitrary minimal subgroup of each Sylow subgroup of G is normal. Extensions were discovered after [5] in [2]. Further expansions have been discovered in [15, 17, 18] by introducing a saturated formation and the conditions of supplementation. The current article may be seen as a continuation of this path. More precisely, we will employ the weakly ss-supplemented subgroups to characterize the structures of some finite groups. Several previously published findings are consolidated and generalized.

2. Preliminaries. For the purpose of simplicity, we will refer to several well-known discoveries from the literature that will be useful in the next section.

Lemma 2.1 ([11]). *Let H and K be subgroups of a group G.*

(i) *If H is S-quasinormal in G, then H is subnormal in G.*

(ii) *Let $N \trianglelefteq G$. If H is S-quasinormal in G, then HN/N is S-quasinormal in G/N.*

Lemma 2.2 ([19]). *Suppose H is an s-semipermutable subgroup of G.*

(i) *If $H \leq K \leq G$, then H is s-semipermutable in K.*

(ii) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(iii) *If $H \leq O_p(G)$, then H is S-quasinormal in G.*

Lemma 2.3 ([13], Lemma 2.2). *If P is an S-quasinormal p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.4 ([12], Lemma 2.8). *Suppose G is a group and P is a normal p-subgroup of G contained in $Z_\infty(G)$, then $C_G(P) \geq O^p(G)$.*

We often need the following lemma in our proofs.

Lemma 2.5. *Let G be a group and p a prime. If subgroups H, N are weakly ss-supplemented, normal in G, respectively, then*

(i) *H is weakly ss-supplemented in M, if $H \leq M \leq G$.*

(ii) *If H is a p-group, then H/N is weakly ss-supplemented in G/N, where $N \trianglelefteq H$.*

(iii) *$(HN)/N$ is weakly ss-supplemented in G/N, when H is a p-group and N is a p'-subgroup.*

Proof. It is a routine check of Lemma 2.2. □

Lemma 2.6 ([10], III, Satz 5.2). *Suppose G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

(i) *G has a normal Sylow p-subgroup P and $G = PQ$, where Q is a nonnormal cyclic Sylow q-subgroup for some prime $q \neq p$.*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(iii) *The exponent of P is p or 4.*

Lemma 2.7 ([15], Lemma 2.16). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

Lemma 2.8 ([9], Lemma 2.7). *Let $\mathcal{F}$ be a formation, H is $\mathcal{F}$-supplemented in G, then*

(i) *If $N \trianglelefteq G$, then HN/N is $\mathcal{F}$-supplemented in G/N.*

(ii) *If $H \leq K \leq G$ and F is s-closed, then H is $\mathcal{F}$-supplemented in K.*

Lemma 2.9 ([20], Lemma 2.8). *Let $\mathcal{F}$ be a saturated formation. Assume G is a group such that G does not belong to $\mathcal{F}$ and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G. Then*

(i) *$G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G.*

(ii) *$G^{\mathcal{F}}$ is a p-subgroup for some prime p.*

(iii) *$G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.*

(iv) *$G^{\mathcal{F}}$ is either elementary abelian or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

Lemma 2.10 ([15], Lemma 2.16). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

3. Main results. We obtain some main results in this section.

Theorem 3.1. *Assume N is a normal subgroup of G, and G/N is supersolvable. If each cyclic subgroup $\langle x\rangle$ of every noncyclic Sylow subgroup of N with prime order or order 4 (if the Sylow 2-subgroup of N is nonabelian) without a supersolvable supplement in G is weakly ss-supplemented in G, then G is supersolvable.*

Proof. Suppose the theorem is not true, and take a counterexample (G, N), where $|N| + |G|$ is minimal. We proceed the proof step by step.

(1) Supersolvability exists for all proper subgroups of G.

Assume H is a proper subgroup of G and that $\langle x\rangle$ is a prime order cyclic subgroup of arbitrary noncyclic Sylow subgroup of $H \cap N$. $\langle x\rangle$ is obviously a prime order cyclic subgroup of a noncyclic Sylow subgroup of N. According to the hypothesis of the theorem, $\langle x\rangle$ possesses a supersolvable supplement in G or is weakly ss-supplemented. Lemma 2.8 says that $\langle x\rangle$ does have a supersolvable supplement $H \cap T$ in H, if $\langle x\rangle$ possesses a supersolvable supplement T in G. Lemma 2.5 states that if $\langle x\rangle$ is weakly ss-supplemented in G, it is also weakly ss-supplemented in H.

Let $\langle y\rangle$ be a 4-order cyclic subgroup of $H \cap N$, if the Sylow 2-subgroups of $H \cap N$ are nonabelian. The Sylow 2-subgroups of N are obviously nonabelian, and $\langle y\rangle$ is a 4-order cyclic subgroup of N. The hypothesis thus suggests that $\langle y\rangle$ is either weakly ss-supplemented or possesses a supersolvable supplement in G. By applying the preceding reasoning once again, we may conclude that $\langle y\rangle$ is either weakly ss-supplemented or possesses a supersolvable supplement in H. As a result, the hypothesis is correct for $(H, H \cap N)$. H must be supersolvable, since G is minimal. So, although G itself is not supersolvable, all of its proper subgroups are. According to the well-known DOERK's result in [7], there is a normal Sylow p-subgroup P of G satisfying $G = PM$, where subgroup $M \leq G$ is supersolvable maximal, and subgroup $P/\Phi(P) \trianglelefteq G/\Phi(P)$ is minimal. Furthermore, the exponent of P is p for $p > 2$, while it is at most 4 for $p = 2$.

(2) $P = N$ is noncyclic.

Meanwhile, since G/P is a homomorphic image of M, it hence is supersolvable. Since G/N is supersolvable according to the hypothesis, $G/(N \cap P)$ is also supersolvable. Clearly, we have that $(G, N \cap P)$ fulfills the hypothesis. If $N \cap P < N$, then G must be supersolvable by (G, N). Therefore, $N \cap P = N$, implying that N is a p-group. As $N \trianglelefteq G$, and subgroup $P/\Phi(P) \trianglelefteq G/\Phi(P)$ is minimal, it means $\Phi(P) = N\Phi(P)$ or $P = N\Phi(P)$. In the first situation, $N \leq \Phi(P) \leq \Phi(G)$, thus $G/\Phi(G)$ implying that G is also supersolvable, which is incongruous. As a consequence, $P = N\Phi(P)$, and thus $P = N$. Since G/P is supersolvable, and if P is cyclic, then G is supersolvable. It is incongruous.

(3) $\langle x\rangle$ is S-quasinormal in G for arbitrary x in P.

Let $x \neq 1$ be an arbitrary element of P. By the assertion (1), $\langle x\rangle$ is therefore a prime order or 4-order cyclic group. Suppose T is arbitrary supplement of $\langle x\rangle$ in G, as a result, $G = \langle x\rangle T$ suggests that $\langle x\rangle(P \cap T) = P \cap \langle x\rangle T = P \cap G = P$. As $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, thus $(P \cap T)\Phi(P) \trianglelefteq G$. We obtain $P = T \cap P$ or $T \cap P \leq \Phi(P)$, since $P/\Phi(P)$ is a chief factor of G. Hence, $P = \langle x\rangle$ is cyclic, if $T \cap P \leq \Phi(P)$ for some supplement T. It runs counter to assertion (2). Assume that $P = T \cap P$ for each supplement T. The supplement of $\langle x\rangle$ in G is thus $T = G$ which is unique. Since G is not supersolvable, and $\langle x\rangle$ is weakly ss-supplemented in G according to the hypothesis. As a consequence, $\langle x\rangle_{ss} = T \cap \langle x\rangle = \langle x\rangle$ is s-semipermutable in G. At this point, we see that $\langle x\rangle \leq P \leq O_p(G)$, hence by Lemma 2.2, $\langle x\rangle$ is S-quasinormal in G.

(4) The contradiction.

Assume $T/\Phi(P)$ is arbitrary nontrivial cyclic subgroup of $P/\Phi(P)$, and $|P/\Phi(P)| \neq p$. Let x be an element of $T/\Phi(P)$ with the property $\langle x\rangle\Phi(P) = T$. As $\langle x\rangle$ is S-quasinormal in G according to (3), so is $T/\Phi(P)$ in $G/\Phi(P)$ from Lemma 2.1. As a result of ([15], Lemma 2.11), $P/\Phi(P)$ has a maximal subgroup that is normal in $G/\Phi(P)$. It, however, does not hold since $P/\Phi(P)$ is a chief factor of G. Consequently, P is cyclic, and $p = |P/\Phi(P)|$, which are opposed to the choice of G. □

The goal of Theorem 3.2 is to unify and enhance some findings, such as ([2], Theorem 2) and ([4], Theorem 4.1).

Theorem 3.2. *Assume $\mathcal{F}$ is a saturated formation that contains $\mathcal{U}$, and $N \trianglelefteq G$ satisfies that $G/N \in \mathcal{F}$. If arbitrary cyclic subgroup $\langle x\rangle$, which does not have a supersolvable supplement in G, of each noncyclic Sylow subgroup of N (if the Sylow 2-subgroup of N is nonabelian) is weakly ss-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Suppose this theorem is false, and consider G to be a minimal order counterexample. The proof is made up of multiple steps:

(1) Supersolvability exists for the subgroup N.

According to Lemma 2.5, the subgroup $\langle x\rangle$ in the theorem is weakly *ss*-supplemented in N. Based on Theorem 3.1, we conclude that N is supersolvable.

(2) $G^{\mathcal{F}}$ is a p-group for some prime p, and $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. $G^{\mathcal{F}}$ has an exponent p for $p > 2$ and an exponent of no more than 4 for $p = 2$.

Let $p = \max \pi(N)$ and P be a Sylow p-subgroup of N. It is easy to know that P char $N \trianglelefteq G$ since N is supersolvable, thus $P \trianglelefteq G$. Lemma 2.6 shows that the hypothesis is true for $(G/P, N/P)$, and the minimality of G means that the factor group G/P belongs to $\mathcal{F}$. $G^{\mathcal{F}} \leq P$ is therefore a p-group. Since $G \notin \mathcal{F}$ and $\mathcal{F}$ is a saturated formation, we have $G^{\mathcal{F}} \nleq \Phi(G)$. Suppose the subgroup $M \leq G$ is maximal with $G^{\mathcal{F}} \nleq M$, then $G = MG^{\mathcal{F}} = MF(G)$. Because $M/(M \cap N) \cong MN/N = G/N \in \mathcal{F}$, it follows that the hypothesis is true for $(M, M \cap N)$. The minimality of G results in $M \in \mathcal{F}$. Furthermore, using Lemma 2.9, we obtain that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, while $G^{\mathcal{F}}$ has an exponent p for $p > 2$ and an exponent of no more than 4 for $p = 2$.

(3) For each element $x \in G^{\mathcal{F}}$, $\langle x\rangle$ is S-quasinormal in G.

Using reasoning similar to those applied in the proof of Theorem 3.1.

(4) The contradiction.

Suppose $T/\Phi(G^{\mathcal{F}})$ is an arbitrary nontrivial cyclic subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$, and an element x belongs to $T/\Phi(G^{\mathcal{F}})$, then $T = \langle x\rangle\Phi(G^{\mathcal{F}})$. By Lemma 2.1, we know that $T/\Phi(G^{\mathcal{F}})$ is S-quasinormal in $G/\Phi(G^{\mathcal{F}})$, since $\langle x\rangle$ is S-quasinormal in G from the assertion (3). According to ([15], Lemma 2.11), $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup that is normal in $G/\Phi(G^{\mathcal{F}})$. We may deduce that $G^{\mathcal{F}}$ is cyclic and $|G^{F}/\Phi(G^{\mathcal{F}})| = p$, as $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. So, by Lemma 2.10, we get $G \in \mathcal{F}$, which is a contradiction. □

Theorem 3.3. *Assume a subgroup H of G is normal, and G/H is nilpotent. If $Z_\infty(G)$ contains all prime order cyclic subgroups of H, and arbitrary 4-order cyclic subgroup of H, which does not have arbitrary supersolvable supplement in G, is weakly ss-supplemented in G, then G is nilpotent.*

Proof. Assume the theorem is not true and G is a minimal order counterexample. We prove this theorem by the steps below.

(1) All proper subgroups of G are nilpotent.

Suppose G has a proper subgroup K. Since G/H is nilpotent, so is $K/(K \cap H) \cong KH/H \leq G/H$. Therefore, $Z_\infty(G) \cap K \leq Z_\infty(K)$ contains all prime order subgroups of $K \cap H$.

If a 4-order cyclic subgroup L of $K \cap H$ has no supersolvable supplement in K, then it also has no supersolvable supplement in G. Hence, from the hypothesis, L is weakly ss-supplemented in G, and so is it in K by Lemma 2.5. Consequently, $(K, K \cap H)$ meets the hypothesis, and the minimality of G reveals that K is nilpotent. It follows that the group G is not nilpotent, but its all proper subgroups are nilpotent. In the light of Lemma 2.6, $G = PQ$, where P, Q are a normal Sylow p-subgroup, a nonnormal cyclic Sylow q-subgroup of G, respectively, for some prime $p \neq q$. And then, the subgroup $P/\Phi(P)$ of $G/\Phi(P)$ is minimal normal, and $\exp(P) = p$ for $p > 2$, while no more than 4 for $p = 2$.

(2) $P \leq H, p = 2$ and $\exp(P) = 4$.

G/P and G/H are both nilpotent, and so is $G/(P \cap H) \lesssim G/P \times G/H$. If $P \nleq H$, then $P \cap H < P$ and $Q(P \cap H) < G$. Hence, by assertion (1), $Q(P \cap H)$ is nilpotent. We, then, obtain that Q char $Q(P \cap H)$ and $Q \times (P \cap H) = Q(P \cap H)$. Moreover, $G/(P \cap H) = (P/(P \cap H))(Q(P \cap H)/(P \cap H))$ makes that $Q(P \cap H)/(P \cap H) \trianglelefteq G/(P \cap H)$ and $Q(P \cap H) \trianglelefteq G$. We have a contradiction $Q \trianglelefteq G$, thus $P \leq H$. If $\exp(P) = p$, then $P = P \cap H \leq Z_\infty(G)$. According to Lemma 2.4, there exists a contradiction $G = P \times Q$. As a result, $p = 2$, $\exp(P) = 4$.

(3) $o(x) = 4$, for each $x \in P/\Phi(P)$.

Assume an element x exists with $o(x) = 2$, which belongs to $P/\Phi(P)$, and set $M = \langle x \rangle^G$. We have $M \leq P$ and $M\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Finally, $M = M\Phi(P) = P \leq Z_\infty(G)$ because the subgroup $P/\Phi(P)$ of $G/\Phi(P)$ is minimal normal, which is a contradiction.

(4) The contradiction.

We know from assertion (3) that all elements of $P/\Phi(P)$ are 4-order. Assume T is a supplement of $\langle x \rangle$ $(x \in P/\Phi(P))$ in G, and then $P = P \cap \langle x \rangle T = \langle x \rangle (P \cap T)$. At the same time, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ is carried out from that $P/\Phi(P)$ is abelian, thus we have $(P \cap T)\Phi(P) \trianglelefteq G$. There is $T \cap P = P$ or $T \cap P \leq \Phi(P)$, as $P/\Phi(P)$ is a chief factor of G. We claim that $P = \langle x \rangle$ is cyclic since $P \cap T \leq \Phi(P)$ for some supplement T, and therefore G is nilpotent by ([10], IV, Satz 2.8), which is a contradiction. For arbitrary supplement T, we suppose that $P = T \cap P$. The unique supplement of $\langle x \rangle$ in G is thus $T = G$. If G is supersolvable, then we get that $Q \trianglelefteq G$ as $q > p = 2$. As a consequence, $G = P \times Q$ is nilpotent, resulting in a contradiction. Thus, from the hypothesis of the theorem, $\langle x \rangle$ is weakly ss-supplemented in G. Thereby, $\langle x \rangle_{ss} = T \cap \langle x \rangle = \langle x \rangle$ is s-semipermutable in G. Lemma 2.2 states that $\langle x \rangle \leq P \leq O_p(G)$, and $\langle x \rangle$ is S-quasinormal in G.

As a consequence, we get that $\langle x \rangle Q \leq G$, and assume $\langle x \rangle Q$ is a proper subgroup of G by ([10], IV, Satz 2.8). So, $\langle x \rangle Q = \langle x \rangle \times Q$, and $\langle x \rangle Q$ is nilpotent. It concludes that x is an element of $N_G(Q)$. Furthermore, there exist $G = P \times Q$, $P \leq N_G(Q)$. □

Theorem 3.4. *Assume $\mathcal{F}$ is a saturated formation containing $\mathcal{N}$. If each 4-order cyclic subgroup of $G^{\mathcal{F}}$ is weakly ss-supplemented in G, then $G \in \mathcal{F}$ iff all prime order cyclic subgroups of $G^{\mathcal{F}}$ are in the $\mathcal{F}$-hypercentre $Z_{\mathcal{F}}(G)$ of G.*

Proof. We just show the "only if" part. Suppose the theorem is false, and let G be a minimal order counter-example.

Assume $\langle x \rangle$ is a prime order subgroup of $G^{\mathcal{F}}$. Then $\langle x \rangle \leq G^{\mathcal{F}} \cap Z_{\mathcal{F}}(G)$, by ([8], IV, Theorem 6.10), which indicates that $Z(G^{\mathcal{F}})$ contains $\langle x \rangle$. Each 4-order cyclic subgroup of $G^{\mathcal{F}}$ is weakly ss-supplemented in $G^{\mathcal{F}}$, according to Lemma 2.5. Theorem 3.3 shows that $G^{\mathcal{F}}$ is nilpotent and so solvable. If $G^{\mathcal{F}} \leq \Phi(G)$, then $G/\Phi(G) \in \mathcal{F}$, resulting in $G \in \mathcal{F}$, a contradiction. As a consequence, G has a maximal subgroup M satisfying $MF(G) = MG^{\mathcal{F}} = G$. We may deduce from ([1], Theorem 3.5) that M is an $\mathcal{F}$-critical maximal subgroup, and G/M_G does not belong to $\mathcal{F}$. Since $M/(M \cap G^{\mathcal{F}}) \cong G/G^{\mathcal{F}} \in \mathcal{F}$, we obtain $M^{\mathcal{F}} \leq M \cap G^{\mathcal{F}}$, and hence $M^{\mathcal{F}} \leq G^{\mathcal{F}}$.

Let $1 = N_0 \leq N_1 \leq \cdots \leq N_t = Z_{\mathcal{F}}(G) \leq \cdots \leq G$ be a chief series of G through $Z_{\mathcal{F}}(G)$. Therefore, we obtain that $1 = M \cap N_0 \leq M \cap N_1 \leq \cdots \leq M \cap N_t = M \cap Z_{\mathcal{F}}(G) \leq \cdots \leq M$ is a normal series of M through $M \cap Z_{\mathcal{F}}(G)$.

Let f be the canonical definition of $\mathcal{F}$. We have $G/C_G(N_i/N_{i-1})$ is an element of $f(p)$ for arbitrary chief factor $N_i/N_{i-1} (1 \leq i \leq t)$ of G and arbitrary prime $p \big| |N_i/N_{i-1}|$. There exists $G = MC_G(N_i/N_{i-1})$, since $F(G) \leq C_G(N_i/N_{i-1})$ by ([10], III, Satz 4.3), and thus

$$M/C_M(N_i/N_{i-1}) = M/(M \cap C_G(N_i/N_{i-1})) \cong G/C_G(N_i/N_{i-1}) \in f(p).$$

We obtain that $M/C_M(N_i \cap M/N_{i-1} \cap M) \in f(p)$ since $C_M(N_i/N_{i-1}) \leq C_M(N_i \cap M/N_{i-1} \cap M)$, for arbitrary prime $p \big| |(N_i \cap M)/(N_{i-1} \cap M)|$.

Clearly, $Z_{\mathcal{F}}(G) \cap M \leq Z_{\mathcal{F}}(M)$ when the normal series of M above is refined to a chief series of M. As a result, $Z_{\mathcal{F}}(M)$ contains all prime order subgroups of $M^{\mathcal{F}}$, and, by Lemma 2.5, all cyclic 4-order subgroups of $M^{\mathcal{F}}$ are weakly ss-supplemented in M. Thus, M matches the hypothesis of this theorem.

The fact that G is minimal implies $M \in \mathcal{F}$. For some prime p, $G^{\mathcal{F}}$ is a p-group, according to Lemma 2.9, and the subgroup $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ of $G/\Phi(G^{\mathcal{F}})$ is minimal normal. Furthermore, $G^{\mathcal{F}}$ has an exponent p for $p > 2$ and an exponent of no more than 4 for $p = 2$. If $\exp(G^{\mathcal{F}}) = p$, then $G^{\mathcal{F}} = \Omega_1(G^{\mathcal{F}}) \leq Z_{\mathcal{F}}(G)$ based on the hypothesis. There exists an inconsistency $G \in \mathcal{F}$. Consequently, $p = 2$

and $\exp(G)^{\mathcal{F}} = 4$. We set $H = \langle x\rangle^G$, if there is an $x \in G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ and $o(x) = 2$. So, we obtain that $H \le \Omega_1(G^{\mathcal{F}}) \le Z_{\mathcal{F}}(G)$, and H is a normal subgroup of G.

Since the subgroup $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ of the group $G/\Phi(G^{\mathcal{F}})$ is minimal normal, $H = H\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}}$ is a contradiction. As a consequence, we obtain $o(x) = 4$ for all $x \in G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. Hence, based on the hypothesis, $\langle x\rangle$ is weakly ss-supplemented in G.

Assume T is arbitrary supplement of $\langle x\rangle$ in G. After that, $G^{\mathcal{F}} = G^{\mathcal{F}} \cap \langle x\rangle T = \langle x\rangle(G^{\mathcal{F}} \cap T)$. We have $(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \trianglelefteq G/\Phi(G^{\mathcal{F}})$ and $(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}}) \trianglelefteq G$, since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is commutative. Then $G^{\mathcal{F}} = T \cap G^{\mathcal{F}}$ or $T \cap G^{\mathcal{F}} \le \Phi(G^{\mathcal{F}})$, since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. For some supplement T, if $T \cap G^{\mathcal{F}} \le \Phi(G^{\mathcal{F}})$, then $\langle x\rangle = G^{\mathcal{F}} \trianglelefteq G$. Especially $\langle x\rangle = G^{\mathcal{F}}$ is S-quasinormal in G. If, for arbitrary supplement T, $G^{\mathcal{F}} \cap T = G^{\mathcal{F}}$, then the unique supplement of $\langle x\rangle$ in G is $G = T$. Thus, $\langle x\rangle_{ss} = T \cap \langle x\rangle = \langle x\rangle$ is s-semipermutable in G.

We may see that $\langle x\rangle \le G^{\mathcal{F}} \le O_p(G)$, and $\langle x\rangle$ is S-quasinormal in G according to Lemma 2.2. Consequently, $\langle x\rangle$ is normalized by each Sylow q-subgroup $Q \le M$ for each $q \in \pi(G)$ with $q \ne 2$. Thus Q acts on $\langle x\rangle$ through conjugation. However, the automorphism group of the 4-order cyclic group is the 2-order cyclic group, then we claim that Q acts on $\langle x\rangle$ trivially, and centralizes $\langle x\rangle$ as well. Thereby, we have that $O^2(M)$ centralizes $\langle x\rangle$ and it also centralizes $G^{\mathcal{F}}$. We obtain $O^2(M) \trianglelefteq G$ since $G = MG^{\mathcal{F}}$, and hence G/M_G is a 2-group. Finally, there is a contradiction $G/M_G \in \mathcal{F}$ since $\mathcal{N} \subseteq \mathcal{F}$, which is exactly what we wanted. □

Theorem 3.1 in [3] and Theorem 4.3 in [4] are obviously the corollaries of Theorem 3.4.

REFERENCES

[1] Ballester-Bolinches A. (1989) $\mathcal{H}$-Normalizers and local definitions of saturated formations of finite groups, Israel J. Math., **67**(3), 312–326.

[2] Ballester-Bolinches A., M. C. Pedraza-Aguilera (1996) On minimal subgroups of finite groups, Acta Math. Hungar., **73**, 335–342.

[3] Ballester-Bolinches A., Y. Wang (2000) C-Supplemented subgroups of finite groups, J. Pure Appl. Algebra, **153**(2), 121–127.

[4] Ballester-Bolinches A., Y. Wang, X. Guo (2000) Finite groups with some C-normal minimal subgroups, Glasgow Math. J., **42**(3), 383–389.

[5] Buckley J. (1970) Finite groups whose minimal subgroups are normal, Math. Zeitschr., **116**, 15–17.

[6] Chen Z. (1987) On a theorem of Srinivasan, J. Southwest China Normal Univ. (Nat. Sci. Ed.) (Chin. Ser.), **1**, 1–4.

[7] Doerk K. (1966) Minimal nicht überauflösbare, endliche gruppen, Math. Zeitschr., **91**, 198–205.

[8] Doerk K., T. Hawkes (1992) Finite Soluble Groups, Walter de Gruyter.

[9] Guo W., F. Xie, B. Li (2009) On some open questions in theory of generalized permutable subgroups, Sci. China Ser. A-Math., **52**(10), 2132–2144.

[10] Huppert B. (1968) Endliche Gruppen I, Springer-Verlag.

[11] Kegel O. H. (1962) Sylow-Gruppen und subnormalteiler endlicher gruppen, Math. Zeitschr., **78**, 205–221.
[12] Li Y., Y. Wang (2004) On π-quasinormally embedded subgroups of finite group, J. Algebra, **281**(1), 109–123.
[13] Li Y., Y. Wang, H. Wei (2003) The influence of π-quasinormality of maximal subgroups of Sylow subgroups of a finite group, Arch. Math., **81**, 245–252.
[14] Robinson D. (1982) A Course in the Theory of Groups, Springer-Verlag.
[15] Skiba A. N. (2007) On weakly s-permutable subgroups of finite groups, J. Algebra, **315**(1), 192–209.
[16] Wang Y. (1996) C-normality of groups and its properties, J. Algebra, **180**, 954–956.
[17] Wang Y. (2000) Finite groups with some subgroups of Sylow subgroups c-supplemented, J. Algebra, **224**, 467–478.
[18] Wei H. (2001) On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups, Comm. Algebra, **29**, 2193–2200.
[19] Zhang Q., L. Wang (2005) The influence of s-semipermutable subgroups on the structure of a finite group, Acta Math. Sinica (Chin. Ser.), **48**(1), 81–88.
[20] Zhao T., X. Li, Y. Xu (2011) Weakly s-supplementally embedded minimal subgroups of finite groups, Proc. Edinburgh Math. Soc., **54**(3), 799–807.

School of Information Science and Technology
Guangdong University of Foreign Studies
Guangzhou 510000, P. R. China
e-mail: `guozhong@um.edu.mo`

**School of Mathematics and Statistics*
Zhaotong University
Zhaotong 657000, P. R. China
e-mails: `shixunlin@163.com`
`hanyan702@126.com`
`guoqiongzhou@yeah.net`