

CHARACTERIZABLE AND V-CHARACTERIZABLE GROUPS

Jinshan Zhang, Xiu Liu^{*,#}, Dandan Liu, Yanhua Huang

Received on September 17, 2022

Presented by V. Drensky, Member of BAS, on September 27, 2022

Abstract

Let G be a finite group. The spectrum $\pi_e(G)$ is the set of all element orders of G . A vanishing element of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G . Denote by $\text{Vo}(G)$ the set of the orders of vanishing elements of G . For a set Ω of positive integers, let $h(\Omega)$ ($v(\Omega)$) be the number of isomorphism classes of finite group G such that $\pi_e(G) = \Omega$ ($\text{Vo}(G) = \Omega$, respectively). A group G is called characterizable (V-characterizable) if $h(\pi_e(G)) = 1$ ($v(\text{Vo}(G)) = 1$, respectively). In this note, we discuss the relation between characterizable and V-recognizable. Moreover, by an application of the relation, we prove that the group M_{22} is V-characterizable.

Key words: finite groups, characters, vanishing elements

2020 Mathematics Subject Classification: 20C15

1. Introduction. Given a finite group G , a vanishing element of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G . We will denote by $\text{Van}(G)$ the set of vanishing elements of G . Denote by $\pi(G)$ the set of prime divisors of the orders of G and by $\pi_e(G)$ the set of the element

[#]Corresponding author.

This work is supported by the NNSF of China (12071181) and also supported by the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (E10510071, 2022QYJ04); the joint special fund project for basic research in local undergraduate universities in Yunnan Province (202101BA070001-045); the teaching reform research project of Zhaotong University (Ztjx2022014).

DOI:10.7546/CRABS.2022.11.01

orders of G . Our aim in this paper is to analyze a particular subset of $\pi_e(G)$, the set $\text{Vo}(G)$ of the orders of elements in $\text{Van}(G)$. We know that $\text{Vo}(G)$ can encode some information about the structure of G (see [1–3]).

For a set Ω of positive integers, let $h(\Omega)$ be the number of isomorphism classes of finite group G such that $\pi_e(G) = \Omega$. A group G is called characterizable if $h(\pi_e(G)) = 1$. We define a similar concept for the set of orders of vanishing elements.

Definition 1.1. For a set Ω of positive integers, let $v(\Omega)$ be the number of isomorphism classes of finite group G such that $\text{Vo}(G) = \Omega$. A non-abelian group G is called V-characterizable if $v(\text{Vo}(G)) = 1$.

Given a finite set of positive integers X , the *prime graph* $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes p such that there exists an element of X divisible by p , and two distinct vertices p and q are adjacent if and only if there exists an element of X divisible by p and q . For a finite group G , the graph $\Pi(\pi_e(G))$, which we denote by $GK(G)$, is also known as the Gruenberg–Kegel graph of G . We focus our attention on a particular subgraph of $GK(G)$. The prime graph $\Pi(\text{Vo}(G))$ which in this paper we denote by $\Gamma(G)$, is called the vanishing prime graph of G . The vanishing prime graph was introduced in [4, 5].

The relation between $h(\pi_e(G))$ and $v(\text{Vo}(G))$, we give in the following result.

Theorem 1.2. *Let G be a finite group and let H be a simple group of Lie type or a sporadic simple group with $\text{Vo}(G) = \text{Vo}(H)$. Assume that $\Gamma(H)$ is disconnected, and that N is a normal subgroup of G and that $G/N \cong H$. If $N > 1$, then G has a normal series*

$$1 \leq V < N \leq G,$$

where N/V is an elementary abelian p -group, for some prime p and

$$\pi_e(G/V) = \pi_e(H).$$

In particular, if H is characterizable, then H is V-characterizable.

As an application of Theorem 1.2, we obtain the following result:

Theorem 1.3. *Let G be a finite group. Then $G \cong M_{22}$ if and only if $\text{Vo}(G) = \text{Vo}(M_{22})$.*

Remark 1.4. There exists non-abelian simple groups which are not V-recognizable. For the simple linear group $L_2(9)$, let $G \cong NA$, where $A \cong \text{SL}_2(4)$ and N is an elementary abelian 2-group and a direct sum of natural $\text{SL}_2(4)$ -modules, we easily conclude that $\text{Vo}(G) = \text{Vo}(L_2(9))$. In particular, $v(\text{Vo}(L_2(9))) = \infty$. Also, for the simple linear group $L_3(5)$, $\text{Vo}(L_3(5)) = \text{Vo}(\text{Aut}(L_3(5)))$, but $L_3(5) \not\cong \text{Aut}(L_3(5))$ because $|L_3(5)| \neq |\text{Aut}(L_3(5))|$.

In fact, we also know that both $L_2(9)$ and $L_3(5)$ are not recognizable. Naturally, we may ask the following question:

Question 1.5. Let G be a finite group and let H be a finite non-abelian simple group with $\text{Vo}(G) = \text{Vo}(H)$. If H is recognizable, then $G \cong H$?

All further unexplained notation is standard, readers may refer to [6] and [7].

2. Preliminary results. The following lemma provides some properties of the vanishing prime graph of a finite group and its relationship with the Gruenberg–Kegel graph. In what follows, we shall denote by $V(\mathcal{G})$ the vertex set of a graph $\mathcal{G}$, and by $n(\mathcal{G})$ the number of connected components of $\mathcal{G}$.

Lemma 2.1 ([4,5]). Let G be a finite group. Then the following hold:

- (1) If G is solvable, then $\Gamma(G)$ has at most two connected components.
- (2) If G is non-solvable and $\Gamma(G)$ is disconnected, then G has a unique non-abelian composition factor S , and $n(\Gamma(G)) \leq n(\text{GK}(S))$ unless G is isomorphic to A_7 .

Lemma 2.2 ([2], Proposition 2.1). Let G be a non-abelian simple group and p a prime number. If G is of Lie type, or if $p \geq 5$, then there exists $\chi \in \text{Irr}(G)$ of p -defect zero.

In the following lemma, we collect some basic remarks relating the vanishing elements of a group G and the vanishing elements of the quotients of G . We shall freely use these results.

Lemma 2.3 ([4,8]). Let N be a normal subgroup of G .

(1) Any character of G/N can be viewed, by inflation, as a character of G . In particular, if $xN \in \text{Van}(G/N)$, then $xN \subseteq \text{Van}(G)$.

(2) If $p \in \pi(N)$ and N has an irreducible character of p -defect zero, then every element of N of order divisible by p is a vanishing element of G .

(3) If $m \in \text{Vo}(G/N)$, then there exists an integer n such that $mn \in \text{Vo}(G)$.

Remark 2.4. Let G be a simple group of Lie type. By Lemma 2.2, G has characters of p -defect zero for every prime p and hence by [7], Theorem 8.17 every non-identity element of G is a vanishing element. Hence $\text{Vo}(G) = \pi_e(G) - \{1\}$.

3. The relation between characterizable and V-recognizable.

Lemma 3.1 ([4], Proposition 4.2). Let G be a group. Assume that $V(\Gamma(G)) \neq \pi(G)$. Then $\Gamma(G)$ is connected. Moreover, if G is non-solvable, then G has a unique non-abelian composition factor S and $S \cong A_5$.

Next, we first give a case which forces that $\text{Vo}(G)$ and $\pi_e(G)$ are almost the same.

Lemma 3.2 ([3], Lemma 3.2). Let G be a finite group, and let N be a normal subgroup of G . Assume that every non-identity element of G/N is a vanishing element of G/N . If N is an elementary abelian p -group for some prime p , then

$$\pi_e(G) = \text{Vo}(G) \cup \{1, p\}.$$

Moreover, if $\Gamma(G)$ is disconnected, then $p \in V(\Gamma(G))$.

In the following proof of Theorem 1.2, we will use the fact: let G and H be two groups, if $\text{Vo}(G) = \pi_e(H) - \{1\}$, then, for any $m \in \text{Vo}(G)$ and any non-identity factor n of m , we have that $n \in \text{Vo}(G)$.

Proof of Theorem 1.2. Recall that $\text{Vo}(G) = \text{Vo}(H)$ and that $\Gamma(H)$ is disconnected, then it follows by the hypothesis and Lemma 2.1, that N is the solvable radical of G .

Assume that $N > 1$. Let $1 \leq V < N$ such that N/V is a chief factor of G . Since N is solvable, N/V is an elementary abelian p -group, for some prime p . Now, we consider the group $\tilde{G} := G/V$.

First, assume that H satisfies the condition (*): every non-identity element of H is a vanishing element of H . Then applying Lemma 3.2 in the group $\tilde{G}$, we get

$$\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, p\}.$$

Recall that every non-identity element of G/N is a vanishing element of G/N , thus $\text{Vo}(G/N) = \pi_e(G/N) - \{1\}$. Moreover, $\text{Vo}(G) = \text{Vo}(G/N)$, hence we conclude

$$\text{Vo}(G) = \pi_e(G/N) - \{1\} = \text{Vo}(G/N).$$

By the first equation above, we have that all of non-identity factors of each element of $\text{Vo}(G)$ are also included in $\text{Vo}(\tilde{G})$. Notice that any element in $\text{Vo}(\tilde{G})$ is a factor of some element in $\text{Vo}(G)$. Hence we obtain

$$\text{Vo}(\tilde{G}) \subseteq \text{Vo}(G) = \text{Vo}(G/N) = \text{Vo}(\tilde{G}/\tilde{N}).$$

Therefore, we conclude

$$\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, p\} \subseteq \text{Vo}(\tilde{G}/\tilde{N}) \cup \{1, p\} \subseteq \pi_e(\tilde{G}).$$

So, we have

$$\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}/\tilde{N}) \cup \{1, p\}.$$

Recall that $\text{Vo}(G/N) = \pi_e(G/N) - \{1\}$ and that $\text{Vo}(G/N) = \text{Vo}(\tilde{G}/\tilde{N})$, we obtain

$$\pi_e(\tilde{G}) = \pi_e(G/N) \cup \{p\}.$$

Since $\Gamma(G)$ is disconnected, it follows by Lemma 3.1 that $V(\Gamma(G)) = \pi(G)$, which implies that $p \in V(\Gamma(G))$. Notice that all of non-identity factors of each element of $\text{Vo}(G)$ are included in $\text{Vo}(G)$ and that $\text{Vo}(G) = \pi_e(G/N) - \{1\}$. Thus we have that $p \in \pi_e(G/N)$. Hence we get

$$\pi_e(\tilde{G}) = \pi_e(G/N) = \pi_e(H).$$

Let H be a simple group of Lie type. Then by Remark 2.4, H satisfies the condition (*). Let H be a sporadic simple group. A direct inspection of [6] shows that except for M_{22} and M_{24} , H satisfies the condition (*).

Now, suppose that $H \cong M_{22}$. Notice that any element in $\text{Vo}(\tilde{G})$ is a factor of some element in $\text{Vo}(G)$ and that $\text{Vo}(G) = \text{Vo}(M_{22})$. Hence get that

$$\text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{22}) \cup \{2\} = \{3, 4, 5, 6, 7, 8, 11\} \cup \{2\}.$$

Similarly, any element in $\text{Vo}(\tilde{G}/\tilde{N})$ is a factor of some element in $\text{Vo}(\tilde{G})$ and $\tilde{G}/\tilde{N} \cong M_{22}$. Then we conclude that

$$(1) \quad \{5, 6, 7, 8, 11\} \subseteq \text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{22}) \cup \{2\} = \pi_e(M_{22}) - \{1\}.$$

Note that $\pi_e(\tilde{G}) = \pi_e(\tilde{G} \setminus \tilde{N}) \cup \{1, p\}$. For any element x in $\tilde{G} \setminus \tilde{N}$, if x is a non-vanishing element of $\tilde{G}$, then $o(x\tilde{N}) = 2$, and so $o(x) = 2$, or $2p$. Therefore, we obtain that

$$(2) \quad \pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, 2, p, 2p\}.$$

Since $\Gamma(G)$ is disconnected, it follows by Lemma 3.1 that $V(\Gamma(G)) = \pi(G)$, which implies that $p \in V(\Gamma(G)) = \{2, 3, 5, 7, 11\}$.

Assume that $p \in \{5, 7, 11\}$. Let S be a Sylow 3-subgroup of $\tilde{G}$. Then by (1) and (2), the group $\tilde{N}S$ is a Frobenius group with kernel $\tilde{N}$ and a complement S . Then S is cyclic. However, by [6], we obtain a contradiction, which implies that $p \in \{2, 3\}$. Recall that $\{5, 6, 7, 8, 11\} \subseteq \text{Vo}(\tilde{G}) \subseteq \{2, 3, 4, 5, 6, 7, 8, 11\}$, hence we conclude that

$$\pi_e(\tilde{G}) = \{1, 2, 3, 4, 5, 6, 7, 8, 11\} = \pi_e(M_{22}).$$

Next, suppose that $H \cong M_{24}$. Notice that any element in $\text{Vo}(\tilde{G})$ is a factor of some element in $\text{Vo}(G)$ and that $\text{Vo}(G) = \text{Vo}(M_{24}) = \pi_e(M_{24}) - \{1\}$. Hence we get that

$$\text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{24}).$$

Similarly, any element in $\text{Vo}(\tilde{G}/\tilde{N})$ is a factor of some element in $\text{Vo}(\tilde{G})$ and $\tilde{G}/\tilde{N} \cong M_{24}$. Then we conclude that

$$(3) \quad \{8, 10, 11, 12, 14, 15, 21, 23\} \subseteq \text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{24}) = \pi_e(M_{24}) - \{1\}.$$

Note that $\pi_e(\tilde{G}) = \pi_e(\tilde{G} \setminus \tilde{N}) \cup \{1, p\}$. For any element x in $\tilde{G} \setminus \tilde{N}$, if x is a non-vanishing element, then $o(x) = 2$, or $2p$. Therefore, we obtain that

$$(4) \quad \pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, 2, p, 2p\}.$$

Since $\Gamma(G)$ is disconnected, it follows by Lemma 3.1 that $V(\Gamma(G)) = \pi(G)$, which implies that $p \in V(\Gamma(G)) = \{2, 3, 5, 7, 11, 23\}$.

Assume that $p \in \{11, 23\}$. Let S be a Sylow 3-subgroup of $\tilde{G}$. Then by (3) and (4), the group $\tilde{N}S$ is a Frobenius group with kernel $\tilde{N}$ and a complement S . Then S is cyclic. Then by [6], we also obtain a contradiction, which implies that $p \in \{2, 3, 5, 7\}$. Recall that $\{8, 10, 11, 12, 14, 15, 21, 23\} \subseteq \text{Vo}(\tilde{G})$, hence we conclude that

$$\pi_e(\tilde{G}) = \pi_e(M_{24}).$$

The proof is completed. □

4. An application of Theorem 1.2. In this section, we are ready to prove Theorem 1.3. Denote by $s(\mathcal{H})$ the number of connected components of a graph $\mathcal{H}$. It follows from the classification of simple groups with disconnected vanishing prime graph that $s(\Gamma(H)) \leq 6$ for every finite group G (see [4]). First, using [6], we collect the sporadic simple groups with $s(\Gamma(H)) \geq 4$ in Table 1:

T a b l e 1
Sporadic simple groups with $s(\Gamma(H)) \geq 4$

H	$\text{Vo}(H)$	$V(\Gamma(H))$
$O'N$	$2 - 8, 10 - 12, 14 - 16,$ $19, 20, 28, 31$	$\{2, 3, 5, 7\},$ $\{11\}, \{19\}, \{31\}$
Ly	$2 - 12, 14, 15, 18,$ $20 - 22, 24, 25, 28, 30,$ $31, 33, 37, 40, 42, 67$	$\{2, 3, 5, 7, 11\},$ $\{31\}, \{37\}, \{67\}$
F'_{24}	$2 - 18, 20 - 24, 26 - 30, 33,$ $35, 36, 39, 42, 45, 60$	$\{2, 3, 5, 7, 11, 13\},$ $\{17\}, \{23\}, \{29\}$
M	$2 - 36, 38 - 42, 44 - 48,$ $50 - 52, 54 - 57, 59, 60,$ $62, 66, 68 - 71, 78, 84, 87,$ $88, 92 - 95, 104, 105, 110, 119$	$\{2, 3, 5, 7, 11, 13,$ $17, 19, 23,$ $29, 31, 47\},$ $\{41\}, \{59\}, \{71\}$
J_4	$2 - 8, 10 - 12, 14 - 16,$ $20 - 24, 28 - 31, 33,$ $35, 37, 40, 42 - 44, 66$	$\{2, 3, 5, 7, 11\},$ $\{23\}, \{29\},$ $\{31\}, \{37\}, \{43\}$
M_{22}	$3 - 8, 11$	$\{2, 3\}, \{5\}, \{7\}, \{11\}$
J_1	$2 - 3, 5 - 7, 10, 11, 15, 19$	$\{2, 3, 5\}, \{7\}, \{11\}, \{19\}$

The following result, which appears as in [9], Proposition 3.2, will turn out to be useful in the proof of Theorem 1.3.

Lemma 4.1. *Let N be normal in G . Suppose that G has a normal series*

$$1 \leq N < M \leq G,$$

where $M/N \cong \text{PSL}_3(4)$. If only 2 and 3 are connected in $\Gamma(G)$, then N is a 3-group.

Proof of Theorem 1.3. Assume that $\text{Vo}(G) = \text{Vo}(M_{22})$. Then by part (1) of Lemma 2.1, G is non-solvable. Let N be the solvable radical of G . Then by part (2) of Lemma 2.1, G has a normal series

$$1 \leq N < M \leq G,$$

where G/M is a solvable group, and M/N is a non-cyclic simple group. Now we consider the group $\overline{G} := G/N$. As N is the solvable radical of G , $\overline{G} \leq \text{Aut}(\overline{M})$ and $G/M \leq \text{Out}(\overline{M})$. Recall that $n(\Gamma(G)) = 4$; thus by part (2) of Lemma 2.1, $n(GK(\overline{M})) \geq 4$.

We show that $\overline{M}$ is a sporadic simple group. Let $\overline{M} \cong A_n$ for some $n \geq 12$. For odd n , set

$$a = (1, \dots, n-9)(n-8, n-7, n-6, n-5)(n-4, n-3, n-2, n-1, n).$$

For even n , set

$$a = (1, \dots, n-8)(n-7, n-6)(n-5, n-4, n-3, n-2, n-1).$$

By Lemma 2.2, there exists $\chi \in \text{Irr}(\overline{M})$ such that χ is of 5-defect zero. Then, by Lemma 2.3(2), $a \in \text{Van}(\overline{G})$. Observe that there exists an element $g \in G$ such that $40 \mid o(g)$, a contradiction. Hence the hypothesis yields that $n < 12$. Then by [6], $n(GK(\overline{M})) \leq 3$, and so we obtain a contradiction (note that $n(GK(\overline{M})) \geq 4$).

By the classification theorem of the finite simple groups, we can now suppose that $\overline{M}$ is a simple group of Lie type. Then by Lemma 2.2, for any prime divisor p of $|\overline{M}|$, there exists $\chi_p \in \text{Irr}(\overline{M})$ such that χ_p is of p -defect zero, and so every element of $\overline{M}$ of order divisible by p is a vanishing element of $\overline{G}$. Hence every non-identity element of $\overline{M}$ is a vanishing element of $\overline{G}$, and thus $M \setminus N \subseteq \text{Van}(G)$. Therefore get that

$$\pi_e(\overline{M}) - \{1\} = \text{Vo}(\overline{M}) \subseteq \text{Vo}(\overline{G}) \subseteq \text{Vo}(G) \cup \{2\} = \{2, 3, 4, 5, 6, 7, 8, 11\}.$$

Then $\max\{\pi_e(\overline{M})\} = 11$. So by [10], Table 1, $\overline{M} \cong \text{PSL}_3(4)$ (note that $s(\Pi(\overline{M})) \geq 4$). It follows by the hypothesis and Lemma 4.2 that N is a 3-group. Recall that $11 \in \pi(G)$. Thus by [6], we obtain a contradiction, which implies that $\overline{M}$ is a sporadic simple group. Recall that $s(\Pi(\overline{M})) \geq 4$, thus $\overline{M}$ is one of the groups in Table 1.

Assume that $\overline{M}$ is not isomorphic to M_{22} . Then by [6], for any prime divisor p of $|\overline{M}|$, there exists $\chi_p \in \text{Irr}(\overline{M})$ such that χ_p is of p -defect zero, and so every element of $\overline{M}$ of order divisible by p is a vanishing element of $\overline{G}$. Hence every non-identity element of $\overline{M}$ is a vanishing element of $\overline{G}$, and thus any non-identity element of $\overline{M}$ is a factor of some element in $\text{Vo}(G)$, then we obtain a contradiction, which implies that $\overline{M}$ is isomorphic to M_{22} .

Since $\overline{M} \cong M_{22}$, by [6], we conclude that $M = G$. By [11], the group M_{22} is recognizable, and then by Theorem 1.2, $G \cong M_{22}$, the proof is completed. $\square$

REFERENCES

- [1] BUBBOLONI D., S. DOLFI, P. SPIGA (2009) Finite groups whose irreducible characters vanish only on p -elements, *J. Pure Appl. Algebra*, **213**(3), 370–376.
- [2] DOLFI S., E. PACIFICI, L. SANUS, P. SPIGA (2009) On the orders of zeros of irreducible characters, *J. Algebra*, **321**(1), 345–352.

- [3] ZHANG J., C. SHAO, Z. SHEN (2017) A new characterization of Suzuki's simple groups, *J. Algebra Appl.*, **16**(11), 1750216, 6 pp.
- [4] DOLFI S., E. PACIFICI, L. SANUS, P. SPIGA (2010) On the vanishing prime graph of finite groups, *J. Lond. Math. Soc.* (2), **82**(1), 167–183.
- [5] DOLFI S., E. PACIFICI, L. SANUS, P. SPIGA (2010) On the vanishing prime graph of solvable groups, *J. Group Theory*, **13**(2), 189–206.
- [6] CONWAY J. H., R. T. CURTIS, S. P. NORTON, R. A. PARK, R. A. WILSON (1985) *Atlas of finite groups*, Clarendon Press.
- [7] ISAACS I. M. (1976) *Character theory of finite groups*. Pure and Applied Mathematics, No. **69**. New York-London, Academic Press [Harcourt Brace Jovanovich, Publishers].
- [8] GHASEMABADI M. F., A. IRANMANESH, F. MAVADATPOUR (2015) A new characterization of some finite simple groups, *Sibirsk. Mat. Zh.* **56**(1), 94–99 (in Russian); English translation in *Sib. Math. J.*, **56**(1) (2015), 78–82.
- [9] LIU D. D., Y. H. HUANG, J. S. ZHANG (2022) Recognition of some finite simple groups by the vanishing prime graph, preprint.
- [10] VASIL'EV A. V. (2005) On recognition of all finite nonabelian simple groups with orders having prime divisors at most 13, *Sib. Math. J.*, **46**(2), 246–253, <https://doi.org/10.1007/s11202-005-0024-z>.
- [11] SHI W. J. (1988) A characteristic property of the Mathieu groups, *Chinese Ann. Math. Ser. A*, **9**(5), 575–580, (in Chinese).

School of Science
Sichuan University of Science
and Engineering
Zigong 643000, P. R. China
 e-mail: zjscdut@163.com
 190085710@qq.com
 maggie434100@163.com

**School of Mathematics and Statistics*
Zhaotong University
Zhaotong 657000, P. R. China
 e-mail: 1105510794@qq.com