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NULLITY CONDITION ON TRANS-SASAKIAN 3-MANIFOLDS

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Abstract

In this paper, we are concerned with the κ-nullity condition on trans-Sasakian manifolds of dimension three. Such manifolds are classified under an additional assumption that the scalar curvature is invariant along the Reeb flow or a topology restriction.

Key words: trans-Sasakian manifold, κ-nullity condition, Sasakian manifold

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1. Introduction In 1988, Tanno [1] introduced the so called κ-nullity distribution $\mathcal{N}(\kappa)$ which is defined on a Riemannian manifold (M,g) by

$$\mathcal{N}(\kappa)\colon p\in M\to\mathcal{N}_p(\kappa)$$

and $\mathcal{N}_p(\kappa)$ is defined by

$$\mathcal{N}_p(\kappa)=\{Z\in T_pM\colon R(X,Y)Z=\kappa(g(Y,Z)X-g(X,Z)Y)\} \tag{1.1}$$

for any vector fields X, Y on M, where κ is a real constant. It is easily seen that M is of constant sectional curvature κ if and only if every vector on M belongs to $\mathcal{N}_p(\kappa)$ for every point $p\in M$.

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Such a notion plays a distinguished role in geometry of almost contact metric manifolds [2]. An almost contact metric manifold M^{2n+1} of dimension $2n+1$, $n \geq 1$, together with its almost contact metric structure (ϕ, ξ, η, g), is said to be a trans-Sasakian manifold (see [3,4]) if it satisfies

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.2}$$

for all vector fields X, Y, where α and β are two smooth functions. In view of (1.2), a trans-Sasakian manifold is often written as of type (α, β). In 1992, Marrero [5] proved that any trans-Sasakian manifold of dimension greater than three satisfies either $\alpha = 0$ or $\beta = 0$. But this property is not necessarily true for general trans-Sasakian manifolds of dimension three. In the past decade, to determine on what geometric conditions a compact or complete trans-Sasakian 3-manifold satisfies the above property has been proposed by Deshmukh [6] and later considered by many authors (see some recent literature by De [7–9], Deshmukh [6,10–15], W. Wang [16,17] and Y. Wang [18,19]).

As a trans-Sasakian 3-manifold of constant curvature is well understood, then one is interested in the local structure of it under a certain weaker condition. In this paper, we study trans-Sasakian 3-manifolds whose Reeb vector fields satisfy the κ-nullity condition. Without strong topology restrictions (such as compact and simply connected), we classify these manifolds under an additional assumption that the scalar curvature is invariant along the Reeb flow and construct a concrete example to verify that the previous assumption is essential. Another kind of nullity condition is also introduced.

2. Trans-Sasakian manifolds A smooth Riemannian manifold (M^{2n+1}, g) of dimension $2n+1$ is called an almost contact metric manifold if on M^{2n+1} there exist a $(1,1)$-type, $(1,0)$-type and $(0,1)$-type tensor fields ϕ, ξ and η, respectively, satisfying

$$\begin{gathered} \phi^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \end{gathered} \tag{2.1}$$

for any vector fields X and Y. ξ is said to be the Reeb or structure vector field. An almost contact metric manifold is said to be normal if $[\phi, \phi] = -2\mathrm{d}\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. An almost contact metric manifold is said to be trans-Sasakian if it satisfies equality (1.1). A 3-dimensional almost contact metric manifold is trans-Sasakian if and only if it is normal. This is not necessarily true for higher dimension.

A normal almost contact metric manifold is said to be an α-Sasakian manifold if $\mathrm{d}\eta = \alpha\Phi$ and $\mathrm{d}\Phi = 0$, where α is a nonzero constant. An α-Sasakian manifold reduces to a Sasakian manifold when $\alpha = 1$. A normal almost contact metric manifold is called a β-Kenmotsu manifold if it satisfies $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\beta\eta \wedge \Phi$, where β is a nonzero constant. A β-Kenmotsu manifold becomes a Kenmotsu manifold when $\beta = 1$. A normal almost contact metric manifold is said to be a cosymplectic manifold if it satisfies $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 0$. Obviously, the set of all α-Sasakian manifolds (resp. β-Kenmotsu) is a proper subset of that of all trans-Sasakian manifolds of type $(\alpha, 0)$ (resp. $(0, \beta)$).

Putting $Y = \xi$ into (1.2) and using (2.1) we have

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{2.2}$$

for any vector field X. On a trans-Sasakian manifold we have

$$\xi(\alpha) = -2\alpha\beta. \tag{2.3}$$

In this paper, all manifolds are assumed to be connected.

3. κ-nullity condition Now suppose that the Reeb vector field of a trans-Sasakian 3-manifold satisfies the κ-nullity condition. Taking the covariant derivative of (2.2) gives

$$\begin{aligned} \nabla_Y \nabla_X \xi = &- Y(\alpha)\phi X - \alpha \nabla_Y \phi X + Y(\beta)(X - \eta(X)\xi) \\ &+ \beta(\nabla_Y X - (Y\eta(X))\xi - \eta(X)\nabla_Y \xi) \end{aligned} \tag{3.1}$$

for any vector fields X, Y. Using this and (2.2) in the curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

we obtain

$$\begin{aligned} R(X, Y)\xi = &- X(\alpha)\phi Y + Y(\alpha)\phi X - \alpha(\nabla_X \phi)Y + \alpha(\nabla_Y \phi)X \\ &+ X(\beta)(Y - \eta(Y)\xi) - Y(\beta)(X - \eta(X)\xi) - \beta g(\nabla_X \xi, Y)\xi \\ &+ \beta g(\nabla_Y \xi, X)\xi - \beta\eta(Y)\nabla_X \xi + \beta\eta(X)\nabla_Y \xi \end{aligned} \tag{3.2}$$

for any vector fields X, Y. Substituting (1.2) and (2.2) into (3.2) we get

$$\begin{aligned} R(X, Y)\xi = &(\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ &+ Y(\alpha)\phi X - X(\alpha)\phi Y - Y(\beta)(X - \eta(X)\xi) + X(\beta)(Y - \eta(Y)\xi) \end{aligned} \tag{3.3}$$

for any vector fields X, Y. Contracting X in (3.3) we have

$$Q\xi = (2\alpha^2 - 2\beta^2 - \xi(\beta))\xi + \phi\nabla\alpha - \nabla\beta, \tag{3.4}$$

where Q is the Ricci operator and ∇f denotes the gradient of a smooth function f. As the Reeb vector field satisfies the κ-nullity condition, from (1.1) we have

$$Q\xi = 2\kappa\xi.$$

Comparing this with (3.4) we have

$$(2\alpha^2 - 2\beta^2 - 2\kappa - \xi(\beta))\xi + \phi\nabla\alpha - \nabla\beta = 0.$$

Taking the inner product of this equality with ξ gives

$$\alpha^2 - \beta^2 - \kappa = \xi(\beta). \tag{3.5}$$

Applying (3.5) back in the previous relation we have

$$\xi(\beta)\xi + \phi\nabla\alpha - \nabla\beta = 0. \tag{3.6}$$

Theorem 3.1. *The Reeb vector field of trans-Sasakian* 3*-manifolds satisfies the* κ*-nullity condition if and only if* (3.5) *and* (3.6) *are valid for a constant* κ*.*

Proof. The proof of "only if" part has been discussed and here we only show the "if" part. Suppose that (3.5) and (3.6) are valid for a constant κ, with the help of (2.3), from (3.3) we have

$$R(X, \xi)\xi = \kappa(X - \eta(X)\xi)$$

for any vector field X. Let e be an arbitrary vector field orthogonal to the Reeb vector field, with the help of (3.6), from (3.3) we have

$$R(e, \phi e)\xi = 0.$$

Recall the dimension of the manifold is three, then above two equalities ensure that the Reeb vector field belongs to the κ-nullity distribution. □

On any 3-dimension Riemannian manifold, the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y) \end{aligned} \tag{3.7}$$

for any vector fields X, Y, Z, where r denotes the scalar curvature. Replacing Z by ξ in (3.7) yields

$$R(X,Y)\xi = \eta(Y)QX - \eta(X)QY + \eta(QY)X - \eta(QX)Y - \frac{r}{2}\eta(Y)X + \frac{r}{2}\eta(X)Y.$$

As the Reeb vector field belongs to the κ-nullity distribution, from (1.1) and the above equality we get

$$\eta(Y)QX - \eta(X)QY + \eta(QY)X - \eta(QX)Y = \left(\frac{r}{2} + \kappa\right)(\eta(Y)X - \eta(X)Y)$$

for any vector fields X, Y. Replacing Y by ξ in the above equality, with the aid of $Q\xi = 2\kappa\xi$, we get

$$QX = \left(\frac{r}{2} - \kappa\right)X - 3\left(\frac{r}{6} - \kappa\right)\eta(X)\xi$$

for any vector field X. Taking the covariant derivative of the above equality we get

$$(3.8)\quad (\nabla_Y Q)X = \frac{1}{2}Y(r)X - \frac{1}{2}Y(r)\eta(X)\xi - 3\left(\frac{r}{6}-\kappa\right)g(\nabla_Y\xi, X)\xi - 3\left(\frac{r}{6}-\kappa\right)\eta(X)\nabla_Y\xi$$

for any vector fields X, Y. Recall that on a Riemannian manifold we have the following formula

$$\operatorname{div} Q = \frac{1}{2}\nabla r.$$

With the aid of the above formula, from (3.8) we have

$$\frac{1}{2}\xi(r) + 3\left(\frac{r}{6}-\kappa\right)\operatorname{div}\xi + 3\left(\frac{r}{6}-\kappa\right)\eta(\nabla_\xi\xi) = 0,$$

which is simplified by means of (2.2) yielding

$$\xi(r) = 2\beta(6\kappa - r).$$

Recall that the manifold is assumed to be connected. We consider the following two cases.

Case 1. $\beta = 0$. Using this in (3.5) we see that α is a constant. Using this in (2.2) we know that the Reeb vector field ξ is a Killing vector field of constant length one. Recall that if on a Riemannian manifold M there exists a Killing vector field ζ of constant length satisfying

$$k^2(\nabla_X\nabla_Y\zeta - \nabla_{\nabla_X Y}\zeta) = g(Y,\zeta)X - g(X,Y)\zeta$$

for a non-zero constant k and any vector fields X, Y, then M is homothetic to a Sasakian manifold (see [20]). By (2.2) one can check that the above equality is valid and hence in this case the manifold is homothetic to a Sasakian 3-manifold when α is a nonzero constant. When $\alpha = \beta = 0$, the manifold is a cosymplectic 3-manifold.

Case 2. $6\kappa = r$. Using this we have $Q = 2\kappa\,\mathrm{Id}$ and hence from (3.7) we see that the manifold is of constant sectional curvature κ.

According to the above analyses we have

Theorem 3.2. *If the Reeb vector field of a trans-Sasakian 3-manifold satisfies the κ-nullity condition and the scalar curvature is invariant along the Reeb flow, then either the manifold is of constant sectional curvature κ or it is homothetic to a Sasakian manifold or a cosymplectic manifold.*

Under some topology conditions, the invariance of the scalar curvature along the Reeb flow is redundant. In fact, if the Reeb vector field of a trans-Sasakian 3-manifold satisfies the κ-nullity condition, then ξ is an eigenvector field of the Ricci operator. Then the following theorem follows from Theorem 3.1 and Lemma 5.1 in [18].

Theorem 3.3. *If the Reeb vector field of a compact trans-Sasakian* 3*-manifold satisfies the* κ*-nullity condition, then either the manifold is cosymplecitc or it is homothetic to a Sasakian manifold.*

In Theorem 3.2, the invariance of the scalar curvature along the Reeb flow is essential. Next we present a concrete example to verify such an assertion.

Example 3.4. Let (x, y, z) be the standard Cartesian coordinates of $\mathbb{R}^3$. On $\mathbb{R}^3$ we define a Riemannian metric g as the following:

$$g = \frac{1}{e^{2f(z)}} \mathrm{d}x \otimes \mathrm{d}x + \frac{1}{e^{2f(z)}} \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z,$$

where $f(z)$ is a non-constant smooth function on $\mathbb{R}^3$ and satisfies the following ODE

$$f''(z) - \big(f'(z)\big)^2 - \kappa = 0 \tag{3.9}$$

for a constant κ. For example, we may select

$$f(z) = \begin{cases} -\ln|\cos\sqrt{\kappa}z| & \text{if } \kappa > 0, \\ \displaystyle\int_0^{\sqrt{-\kappa}z} (1 + e^{2x})(1 - e^{2x})^{-1}\,\mathrm{d}x & \text{if } \kappa < 0, \\ -\ln|z| & \text{if } \kappa = 0. \end{cases}$$

From the above metric, an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ is given by

$$e_1 = e^{f(z)}\frac{\partial}{\partial x}, \quad e_2 = e^{f(z)}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

The Lie bracket on the tangent space is given by

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -f'e_2, \quad [e_3, e_1] = f'e_1.$$

Now we define an almost contact structure (ϕ, ξ, η, g) on $\mathbb{R}^3$ as the following:

$$\xi = e_3, \quad \eta = g(e_3, \cdot), \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the orthonormal ϕ-basis $\{e_1, e_2, e_3\}$. From a direct calculation, one finds that $(\mathbb{R}^3, g)$ is a trans-Sasakian 3-manifold of type $(0, -f'(z))$ (see [18]). Since $\alpha = 0$, $(\mathbb{R}^3, g)$ is not homothetic to a Sasakian manifold. Taking into account Theorem 3.1 and (3.9) it is easily seen that the Reeb vector field satisfies the κ-nullity condition. The Levi-Civita connection of the metric g is given by

$$\nabla_{e_i} e_j = \begin{pmatrix} f'e_3 & 0 & -f'e_1 \\ 0 & f'e_3 & -f'e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying these, the curvature tensor of the manifold is given by

$$
\begin{aligned}
R(e_1,e_2)e_1 &= (f')^2 e_2,\\
R(e_1,e_2)e_2 &= -(f')^2 e_1,\\
R(e_1,e_2)e_3 &= 0,\\
R(e_1,e_3)e_1 &= \left((f')^2 - f''\right) e_3,\\
R(e_1,e_3)e_2 &= 0,\\
R(e_1,e_3)e_3 &= \left(f'' - (f')^2\right) e_1,\\
R(e_2,e_3)e_1 &= 0,\\
R(e_2,e_3)e_2 &= \left((f')^2 - f''\right) e_3,\\
R(e_2,e_3)e_3 &= \left(f'' - (f')^2\right) e_2.
\end{aligned} \tag{3.10}
$$

With the aid of (3.9) and (3.10), the scalar curvature r is given by

$$r = 4\kappa - 2(f')^2.$$

Next we consider two cases for f'.

- $f'(z) = \text{const}$. Then from (3.9) it follows that $f'(z) = \pm\sqrt{-\kappa}$ with $\kappa < 0$, or equivalently, $f(z) = \pm\sqrt{-\kappa}z + C$ with C being an integral constant. In this case, $r = 6\kappa = \text{const}$ and hence the scalar curvature r is invariant along the Reeb flow. By using (3.10) we check that $(\mathbb{R}^3, g)$ is of constant sectional curvature κ, which confirms the result in Theorem 3.2.

- $f'(z) \neq \text{const}$. Then the scalar curvature r is not invariant along the Reeb vector field. From (3.10) it is obvious that $(\mathbb{R}^3, g)$ is not of constant sectional curvature. In this case, the manifold is homothetic to neither a Sasakian manifold nor a cosymplectic manifold. This shows that the invariance of the scalar curvature along the Reeb flow is essential in Theorem 3.2.

On an almost contact metric manifold (M, ϕ, ξ, η, g), the operator $2h := \mathcal{L}_\xi\phi$ plays an important role in studies of the almost contact manifolds. By means of this operator, many kinds of nullity distributions were introduced. For example, the Reeb vector field is said to belong to the (κ, μ, ν)-nullity distribution [2] if

$$R(X,Y)\xi = (\kappa\mathrm{Id} + \mu h + \nu h')(\eta(Y)X - \eta(X)Y)$$

for any vector fields X, Y, Z, where $h' = h \circ \phi$ and κ, μ and ν are constant. Such a nullity notion includes the (κ, μ)-nullity and $(\kappa, \mu)'$-nullity conditions that were extensively studied in contact geometry.

However, on a trans-Sasakian manifold, h vanishes identically as the manifold is normal (this is direct to verify by means of (1.2) and (2.2)). Therefore, on a trans-Sasakian manifold we introduce the following nullity condition which is different from (1.1). The Reeb vector field of an almost contact metric manifold is said to belong to the pseudo κ-nullity distribution if

$$R(X,Y)\xi = \kappa(\eta(Y)\phi X - \eta(X)\phi Y)$$

for any vector fields X, Y, where κ is a constant.

Suppose that the Reeb vector field of a trans-Sasakian 3-manifold belongs to the pseudo κ-nullity distribution. By definition, from (3.3) we get

$$(3.11)\quad (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y)+(2\alpha\beta-\kappa)(\eta(Y)\phi X-\eta(X)\phi Y) \\ +Y(\alpha)\phi X - X(\alpha)\phi Y - Y(\beta)(X-\eta(X)\xi)+X(\beta)(Y-\eta(Y)\xi)=0$$

for any vector fields X, Y. Replacing Y by ξ the the above equality, with the aid of (2.3), we have

$$\kappa\phi X = (\alpha^2-\beta^2-\xi(\beta))(X-\eta(X)\xi)$$

for any vector field X. It follows that $\kappa=0$ and $\alpha^2-\beta^2-\xi(\beta)=0$. Using this in (3.11), with the help of (2.3), we obtain again

$$\nabla\alpha + \phi\nabla\beta + 2\alpha\beta\xi = 0.$$

The curvature of a plane section spanned by the Reeb vector field and a vector field orthogonal to ξ is called Reeb sectional curvature. From the above analyses and Theorem 3.1 we have

Theorem 3.5. *If the Reeb vector field of a trans-Sasakian* 3*-manifold belongs to the pseudo κ-nullity distribution, then Reeb sectional curvatures vanish identically, and the Reeb vector field belongs to the* 0*-nullity distribution.*

The study of the pseudo κ-nullity distribution on almost contact metric manifolds is interesting to consider.

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REFERENCES

[1] Tanno S. (1985) Ricci curvatures of contact Riemannian manifolds, Tohoku Math. J., **40**(3), 441–448.

[2] Blair D. E. (2010) Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics, **203**, New York, Birkhäuser.

[3] Chinea D., C. Gonzalez (1990) A classification of almost contact metric manifolds, Ann. Mat. Pura Appl., **156**(1), 15–36.

[4] Oubiña A. (1985) New classes of almost contact metric structures, Publ. Math. Debrecen, **32**(3–4), 187–193.

[5] Marrero J. C. (1992) The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl., **162**, 77–86.
[6] Deshmukh S. (2014) Geometry of 3-dimensional trans-Sasakaian manifolds, An. Stiint. Univ. Al. I. Cuza Iasi Mat. (N. S.), **63**(1), 183–192.
[7] De U. C., A. Mondal (2009) On 3-dimensional normal almost contact metric manifolds satisfying certain curvature conditions, Commun. Korean Math. Soc., **24**(2), 265–275.
[8] De U. C., A. K. Mondal (2013) The structure of some classes of 3-dimensional normal almost contact metric manifolds, Bull. Malays. Math. Sci. Soc., **36**(2), 501–509.
[9] De U. C., M. M. Tripathi (2003) Ricci tensor in 3-dimensional trans-Sasakian manifolds, Kyungpook Math. J., **43**(2), 247–255.
[10] Deshmukh S. Trans-Sasakian manifolds homothetic to Sasakian manifolds, Mediterr. J. Math., **f13**(5), 2951–2958.
[11] Deshmukh S., F. Al-Solamy (2016) A note on compact trans-Sasakian manifolds, Mediterr. J. Math., **13**(4), 2099–2104.
[12] Deshmukh S., U. C. De (2018) A note on trans-Sasakian manifolds, Publ. Math. Debrecen, **92**(1–2), 159–169.
[13] Deshmukh S., U. C. De, F. Al-Solamy (2016) Trans-Sasakian manifolds homothetic to Sasakian manifolds, Publ. Math. Debrecen, **88**(3–4), 439–448.
[14] Deshmukh S., A. Ishan, O. Belova, S. B. Al-Shaikh (2021) Some conditions on trans-Sasakian manifolds to be homothetic to Sasakian manifolds, Math., **9**(16), 1887.
[15] Deshmukh S., M. M. Tripathi (2013) A note on compact trans-Sasakian manifolds, Math. Slovaca, **63**(6), 1361–1370.
[16] Wang W. (2021) Trans-Sasakian 3-manifolds with Einstein-like Ricci operators, Quaest. Math., **45**, doi: 10.2989/16073606.2021.1950861.
[17] Wang W., X. Liu (2018) Ricci tensors on trans-Sasakian 3-manifolds, Filomat, **32**(12), 4365–4374.
[18] Wang Y. (2018) Minimal and harmonic Reeb vector fields on trans-Sasakian 3-manifolds, J. Korean Math. Soc., **55**(6), 1321–1336.
[19] Wang Y., W. Wang (2019) A remark on trans-Sasakian 3-manifolds, Rev. Union Mat. Argentina, **60**(1), 257–264.
[20] Okumura K. (1964) Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures, Tohoku Math. J., **16**(3), 270–284.

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