Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences
Tome 75, No 9, 2022

MATHEMATICS
Graph theory

FINITE GROUPS WHOSE CO-PRIME ORDER GRAPHS HAVE POSITIVE GENUS

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Received on June 7, 2022
Presented by V. Drensky, Member of BAS, on June 21, 2022

Abstract

Given a finite group G, the co-prime order graph of G is the simple undirected graph whose vertex set is G, and two distinct vertices x, y are adjacent if $\gcd(o(x), o(y)) = 1$ or p, where p is a prime, and $o(x)$ and $o(y)$ are the orders of x and y, respectively. In this paper, we prove that, for a fixed positive integer k, there are finitely many finite groups whose co-prime order graphs have (non)orientable genus k. As applications, we classify all finite groups whose co-prime order graphs have (non)orientable genus one and two.

Key words: co-prime order graph, genus, finite group
2020 Mathematics Subject Classification: 05C25, 05C40

1. Introduction and results. Throughout this paper, all graphs are finite, undirected, with no loops and no multiple edges. Let Γ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of Γ, respectively. The *join* $\Gamma + \Delta$ of two graphs Γ and Δ is a graph having vertex set $V(\Gamma) \cup V(\Delta)$, and its edge set consists of $E(\Gamma) \cup E(\Delta)$ and all edges joining a vertex of Γ and a vertex of Δ. Denote by K_n the *complete graph* of order n. A *complete bipartite graph* is a bipartite graph in which each vertex in the first set is joined to every single vertex in the second set. The complete bipartite graph is denoted by $K_{m,n}$, where

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This research was supported by the National Natural Science Foundation of China (Grant No. 11801441), and the Natural Science Basic Research Program of Shaanxi (Program No. 2020JQ-761).
DOI:10.7546/CRABS.2022.09.03

this graph contains m vertices in the first set and n vertices in the second set. A subset of $V(\Gamma)$ is a *clique* if any two distinct vertices in this subset are adjacent in Γ.

An *embedding* of a graph into a surface is a drawing of the graph on the surface in such a way that its edges may intersect only at their endpoints. A graph is called *planar* if it can be embedded in the plane. A non-planar graph can be embedded in some surface obtained from the sphere by attaching some handles or crosscaps. We denote by $\mathbb{S}_k$ a sphere with k handles and by $\mathbb{N}_k$ a sphere with k crosscaps. Remark that both $\mathbb{S}_0$ and $\mathbb{N}_0$ are the sphere itself, and $\mathbb{S}_1$ and $\mathbb{N}_1$ are a torus and a projective plane, respectively. The smallest non-negative integer k such that a graph Γ can be embedded on $\mathbb{S}_k$ is called the *orientable genus* or *genus* of Γ, and is denoted by $\gamma(\Gamma)$. The *nonorientable genus* of Γ, denoted $\overline{\gamma}(\Gamma)$, is the smallest integer k such that Γ can be embedded on $\mathbb{N}_k$. A graph Γ is called *toroidal* if $\gamma(\Gamma) = 1$ and Γ is called *projective-planar* if $\overline{\gamma}(\Gamma) = 1$.

All groups considered in this paper are finite. Let G be a group. An element of order 2 is called an *involution* of G. For $g \in G$, we use $o(g)$ to denote the order of g. The cyclic group of order n is denoted by $\mathbb{Z}_n$. A *maximal cyclic subgroup* of G is a cyclic subgroup, which is not a proper subgroup of some cyclic subgroup of G. Denote by $\mathcal{M}_G$ the set of all maximal cyclic subgroups of G.

Graphs associated with groups and other algebraic structures have been actively investigated, because they have valuable applications (cf. [1]) and are related to automata theory (cf. [2,3]). The *power graph* of a group G has the vertex set G and two distinct elements are adjacent if one is a power of the other (cf. [4]). The *order supergraph* [5] of the power graph of G is a graph with vertex set G, and two distinct vertices x, y are adjacent if either $o(x)|o(y)$ or $o(y)|o(x)$. The *order prime graph* (also called *coprime graph*) of G has the vertex set G and two distinct elements x, y are adjacent if $\gcd(o(x), o(y)) = 1$, which was first introduced by Sattanathan and Kala [6].

Motivated by the the works on the power graphs, the order prime graphs and the order supergraphs of power graphs, Banerjee [7] introduced the *co-prime order graph* of a group G, which is denoted by $\Theta(G)$ and is the graph whose vertex set is G, and two distinct vertices x, y are adjacent if $\gcd(o(x), o(y))$ is either 1 or a prime. It is easy to see that the order prime graph of G is a spanning subgraph of $\Theta(G)$. In [8], the authors called the co-prime order graph *prime coprime graph*. Recently, Sehgal et al. [9] investigated the Laplacian spectrum of the co-prime order graph of G when G is a finite abelian p-group and a dihedral group. Hao et al. [10] characterized all finite groups whose co-prime order graphs are complete and classified finite groups whose co-prime order graphs are planar.

Clearly, the problem of finding the graph genus is NP-hard. Many research articles have appeared on the genus of graphs constructed from some algebraic structures. For example, Wickham [11] proved that, for a fixed positive integer k, there are finitely many isomorphism classes of rings whose zero-divisor graph

has genus k. Also, SELVAKUMAR and SUBAJINI [12] classified the finite groups with toroidal coprime graphs.

In this paper, we study the (non)orientable genus of a co-prime order graph. To be specific, we prove that for a fixed positive integer k, there are finitely many finite groups whose co-prime order graphs have (non)orientable genus k. As applications, we classify all finite groups whose co-prime order graphs are toroidal and projective-planar. We also classify the finite groups whose co-prime order graphs have (non)orientable genus two.

The main results of the paper are the following three theorems.

Theorem 1.1. *Given a positive integer k, there are finitely many finite groups whose co-prime order graphs have (non)orientable genus k. In particular, if G is a finite group satisfying $\gamma(\Theta(G)) = k$ (resp. $\overline{\gamma}(\Theta(G)) = k$), then $|G| \leq 4k + 5$ (resp. $|G| \leq 2k + 5$).*

Theorem 1.2. (i) *$\Theta(G)$ is toroidal if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_5, \quad \mathbb{Z}_6, \quad S_3, \quad \mathbb{Z}_7, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_9. \tag{1}$$

(ii) *$\Theta(G)$ is projective-planar if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_5, \quad \mathbb{Z}_6, \quad S_3. \tag{2}$$

Theorem 1.3. (i) *$\gamma(\Theta(G)) = 2$ if and only if G is isomorphic to either $\mathbb{Z}_2^3$ or D_8.*
(ii) *$\overline{\gamma}(\Theta(G)) = 2$ if and only if G is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_9$.*

2. Preliminaries. In this section, we state some results before giving the proofs of our main theorems.

A group is called a $\mathcal{P}$-group [13] if every non-trivial element of the group has prime order. For example, every elementary abelian p-group $\mathbb{Z}_p^n$ (namely, the n-fold direct product of $\mathbb{Z}_p$) is a $\mathcal{P}$-group, and the symmetric group S_3 on 3 letters is also a $\mathcal{P}$-group.

Theorem 2.1 ([10], Theorem 2.1)**.** *$\Theta(G)$ is complete if and only if G is a $\mathcal{P}$-group.*

In fact, CHENG et al. [13] have characterized all $\mathcal{P}$-groups. It was shown in [13] that the $\mathcal{P}$-groups are either p-groups of exponent p or Frobenius groups of particular type, or they are isomorphic to the alternating group on 5 letters.

Given a positive integer n at least 3, the dihedral group of order $2n$ is denoted by D_{2n}, which has a presentation as follows:

$$D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle.$$

By a routine method, it is easy to check that

$$D_{2n} = \langle a \rangle \cup \{ab, a^2b, \ldots, a^{n-1}b, b\}, \quad o(a^ib) = 2 \text{ for all } 1 \leq i \leq n-1. \tag{3}$$

Given a positive integer n at least 2, JOHNSON [14], pp. 44–45 introduced the generalized quaternion group Q_{4n}, where Q_{4n} has a presentation

$$Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = e, y^{-1}xy = x^{-1} \rangle. \tag{4}$$

It is routine to check that the order of Q_{4n} is equal to $4n$, and this group contains a unique involution x^n, where an involution is an element of order 2. Moreover, $o(x^i y) = 4$ for all $1 \leq i \leq 2n$,

$$Q_{4n} = \langle x \rangle \cup \{x^i y : 1 \leq i \leq 2n\}, \tag{5}$$

and

$$\mathcal{M}_{Q_{4n}} = \{\langle x \rangle, \langle xy \rangle, \langle x^2 y \rangle, \ldots, \langle x^n y \rangle\}, \quad x^n \in \bigcap_{M \in \mathcal{M}_{Q_{4n}}} M. \tag{6}$$

In [10], the planar co-prime order graphs were classified.

Theorem 2.2 ([10], Theorem 2.3)**.** *$\Theta(G)$ is planar if and only if G is isomorphic to one of the following groups:*

(a) $\mathbb{Z}_3$*;*

(b) $\mathbb{Z}_2 \times \mathbb{Z}_2$*;*

(c) $\mathbb{Z}_{2^n}$*, where $n \geq 1$ is an integer;*

(d) $Q_{4\times 2^n}$*, where $n \geq 1$ is an integer.*

Given a group G, we use $\mathbf{P}_G$ to denote the set of all elements of order 1 or a prime in G. Namely, $\mathbf{P}_G = \{x \in G : o(x) = 1 \text{ or } p, \text{ where } p \text{ is a prime}\}$.

Observation 2.3. *Let G be a group of order n and $x \in G$. If $o(x)$ is 1 or a prime, then the degree of (x) is $n-1$ in $\Theta(G)$. Particularly, $\mathbf{P}_G$ is a clique of $\Theta(G)$.*

Recall the following elementary result on p-groups.

Lemma 2.4 ([15], Theorem 5.4.10)**.** *Given a prime p, a p-group having a unique subgroup of order p is either a cyclic group or a generalized quaternion 2-group.*

Theorem 2.5. ([16]) *Let $n \geq 3$ and $m \geq 2$. Then*

(a) $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$*;*

(b) *For $n \neq 7$, $\overline{\gamma}(K_n) = \lceil \frac{1}{6}(n-3)(n-4) \rceil$; $\overline{\gamma}(K_7) = 3$;*

(c) $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$*;*

(d) $\overline{\gamma}(K_{m,n}) = \lceil \frac{1}{2}(m-2)(n-2) \rceil$.

Lemma 2.6 ([17], Theorem 3.3)**.** *Let $m, n \geq 0$ and $m \geq n-1$. Then*

$$\overline{\gamma}(\overline{K_m} + K_n) = \begin{cases} 0, & \text{if } n \leq 2; \\ \lceil \frac{1}{2}(m-2)(n-2) \rceil, & \text{if } n \geq 3 \text{ and } (m,n) \neq (4,5); \\ 4, & \text{if } (m,n) = (4,5). \end{cases}$$

Lemma 2.7 ([16])**.** *Suppose that Γ is a connected graph with n vertices and m edges, where $n \geq 3$. Then $\gamma(\Gamma) \geq \frac{m}{6} - \frac{n}{2} + 1$.*

Lemma 2.8. *Let $|G| = n$. If $\gamma(\Theta(G)) \geq 1$, then $\Theta(G)$ has a subgraph isomorphic to $K_{3,n-3}$.*

Proof. We first claim that G has at least two distinct elements of prime order. In fact, if G is not a 2-group, clearly, the claim is valid. Suppose now that G is a 2-group. Assume to the contrary that G has a unique involution. Then by Lemma 2.4, we have that G is either cyclic or generalized quaternion. Theorem 2.2 implies $\Theta(G)$ is planar, contrary to $\gamma(\Theta(G)) \geq 1$. We conclude that G has at least two distinct involutions, as desired. Thus, our claim is valid.

Let x and y be two distinct elements of prime order and let

$$U = \{x, y, e\}, \quad V = G \setminus U.$$

Then Observation 2.3 implies that every element in U is adjacent to every element in V. It follows that $\Theta(G)$ has a subgraph isomorphic to $K_{3,n-3}$, as desired. □

3. Proofs of main results. We first prove Theorem 1.1.

Proof of Theorem 1.1. We first show that there are finitely many finite groups whose co-prime order graphs have orientable genus k. Let G be a finite group satisfying $\gamma(\Theta(G)) = k$. By Theorem 2.2, we deduce $|G| \geq 5$ as $k \geq 1$. To prove the desired result, it suffices to prove that the order of G is bounded by a constant depending only on k. Notice that $k \geq 1$. It follows from Lemma 2.8 that $\gamma(\Theta(G))$ has a spanning subgraph isomorphic to $K_{3,|G|-3}$. From Theorem 2.5, it follows that

$$\gamma(\Theta(G)) = k \geq \gamma(K_{3,|G|-3}) = \left\lceil \frac{|G|-5}{4} \right\rceil,$$

which yields $|G| \leq 4k+5$, as desired.

Similarly, if G is a finite group with $\overline{\gamma}(\Theta(G)) = k$, then we have $|G| \leq 2k+5$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. (i) Clearly, $\Theta(\mathbb{Z}_5) \cong K_5$ and $\Theta(\mathbb{Z}_7) \cong K_7$, and so $\gamma(\Theta(\mathbb{Z}_5)) = \gamma(\Theta(\mathbb{Z}_7)) = 1$ by Theorem 2.5. Theorem 2.1 implies that $\Theta(S_3) \cong K_6$, and so $\gamma(\Theta(S_3)) = 1$ by Theorem 2.5. In view of Theorem 2.2, we have

$0 < \gamma(\Theta(\mathbb{Z}_6)) \leq \gamma(\Theta(K_6)) = 1$, which yields $\gamma(\Theta(\mathbb{Z}_6)) = 1$. Also, it is easy to verify that $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \overline{K_4} + K_4$. According to [18], Theorem 1, we have $\gamma(\Theta(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 1$.

Let $\mathbb{Z}_9 = \{0, 1, \ldots, 8\}$. Then $U = \{0, 3, 6\}$ is a clique of $\Theta(\mathbb{Z}_9)$, $\mathbb{Z}_9 \setminus U$ is an independent set of $\Theta(\mathbb{Z}_9)$, and every vertex in U is adjacent to every vertex in $\mathbb{Z}_9 \setminus U$. We can embed $\Theta(\mathbb{Z}_9)$ into the tours as shown in Fig. 1. So, we have

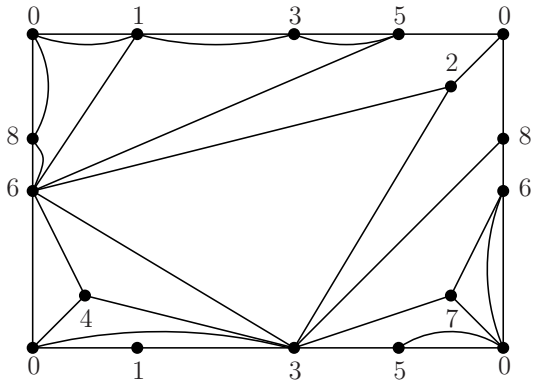


Fig. 1. An embedding of $\Theta(\mathbb{Z}_9)$ on a torus

$\gamma(\Theta(\mathbb{Z}_9)) \leq 1$. Since $\Theta(\mathbb{Z}_9)$ is not planar by Theorem 2.2, we have $\gamma(\Theta(\mathbb{Z}_9)) = 1$. We conclude that the co-prime order graph of any group in (1) is toroidal.

Suppose now that $\Theta(G)$ is toroidal. It suffices to prove that G is one of (1). By Theorems 1.1 and 2.2, we have

$$5 \leq |G| \leq 9.$$

Clearly, if $|G| \leq 7$, then G is one of $\mathbb{Z}_5, \mathbb{Z}_6, S_3$, and $\mathbb{Z}_7$, as desired. Notice that $\Theta(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong K_9$ by Theorem 2.1, and $\gamma(K_9) = 3$ by Theorem 2.5. Remark that there are 2 groups of order 9 up to isomorphism. As a result, if $|G| = 9$, then $G \cong \mathbb{Z}_9$, as desired. By Theorems 2.1 and 2.5 again, $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_8$ and $\gamma(K_8) = 2$. Also, it follows from Theorem 2.2 that $\gamma(\Theta(\mathbb{Z}_8)) = \gamma(\Theta(Q_8)) = 0$. Furthermore, it is easy to see that $\Theta(D_8) \cong \overline{K_2} + K_6$, which implies that $\Theta(D_8)$ has 27 edges. It follows from Lemma 2.7 that $\gamma(\Theta(D_8)) \geq 2$. Remark that there are, up to isomorphism, 5 groups of order 8. Consequently, we conclude that if $|G| = 8$, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, as desired.

(ii) Notice that $\overline{\gamma}(\Theta(\mathbb{Z}_5)) = \overline{\gamma}(\Theta(\mathbb{Z}_6)) = 1$ by Theorem 2.5. Also, since $1 < \overline{\gamma}(\Theta(S_6)) \leq \overline{\gamma}(K_6)$, we have $\overline{\gamma}(\Theta(S_6)) = 1$. Consequently, the co-prime order graph of any group in (1) is projective-planar. Suppose now that $\Theta(G)$ is projective-planar. It follows from Theorems 1.1 and 2.2 that $5 \leq |G| \leq 7$. As a result, G is isomorphic to one of $\mathbb{Z}_5, \mathbb{Z}_6, S_3$, and $\mathbb{Z}_7$. By Theorem 2.5, $\overline{\gamma}(\Theta(\mathbb{Z}_7)) = 3$ as $\Theta(\mathbb{Z}_7) \cong K_7$. Thus, G is a group in (1), as desired. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. (i) By Theorems 2.2, 1.1 and 1.2, it suffices to prove that if $8 \leq |G| \leq 13$ and $\gamma(\Theta(G)) \geq 2$, then $\gamma(\Theta(G)) = 2$ if and only if G is isomorphic to either $\mathbb{Z}_2^3$ or D_8.

Let $|G| = 8$. Since $\gamma(\Theta(G)) \geq 2$, we have that $G \cong \mathbb{Z}_2^3$ or D_8. Clearly, $\gamma(\Theta(\mathbb{Z}_2^3)) = \gamma(K_8) = 2$ by Theorems 2.1 and 2.5. Note that $\Theta(D_8) = K_6 + \overline{K_2}$. It follows from Lemma 2.7 that $\gamma(\Theta(D_8)) \geq \dfrac{3}{2}$. Moreover, since $\gamma(\Theta(D_8)) \leq \gamma(K_8) = 2$, we have $\gamma(\Theta(D_8)) = 2$.

Let $|G| = 9$. Since $\gamma(\Theta(G)) \geq 2$, we have $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Notice that $\gamma(\Theta(\mathbb{Z}_3 \times \mathbb{Z}_3)) = \gamma(K_9) = 3$.

Let $|G| = 10$. Then $G \cong \mathbb{Z}_{10}$ or D_{10}. Note that $\Theta(\mathbb{Z}_{10}) = K_6 + \overline{K_4}$. It follows from Lemma 2.7 that $\gamma(\Theta(\mathbb{Z}_{10})) = \gamma(K_6 + \overline{K_4}) \geq \dfrac{5}{2}$. Also, by (3), D_{10} is a $\mathcal{P}$-group. Thus, $\gamma(\Theta(D_{10})) = \gamma(K_{10}) = 4$ by Theorems 2.1 and 2.5.

Let $|G| = 12$. Then $G \cong \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, A_4, D_{12}, Q_{12}$. Note that $|E(\Theta(\mathbb{Z}_{12}))| = 44$, $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_6) = K_6 + \overline{K_6}$, and $\Theta(A_4)$ is complete. It follows from Lemma 2.7 that $\gamma(\Theta(\mathbb{Z}_{12})) \geq \dfrac{7}{3}$. Also, by (5), (6) and (3), both $\Theta(D_{12})$ and $\Theta(Q_{12})$ have a subgraph isomorphic to $K_{6,6}$. Thus, if $G \in \{\mathbb{Z}_2 \times \mathbb{Z}_6, A_4, D_{12}, Q_{12}\}$, then $\Theta(G)$ has a subgraph isomorphic to $K_{6,6}$, and so $\gamma(\Theta(G)) \geq \gamma(K_{6,6}) = 4$ by Theorem 2.5.

Finally, if $n = 11$ or 13, then $\gamma(\Theta(\mathbb{Z}_n)) = \gamma(\Theta(K_n)) \geq \gamma(K_{10}) = 4$, the proof is now complete.

(ii) By Theorems 2.2, 1.1 and 1.2, it suffices to prove that if $7 \leq |G| \leq 9$ and $\overline{\gamma}(\Theta(G)) \geq 2$, then $\overline{\gamma}(\Theta(G)) = 2$ if and only if G is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_9$.

Clearly, $\overline{\gamma}(\Theta(\mathbb{Z}_7)) = \overline{\gamma}(K_7) = 3$ by Theorems 2.1 and 2.5. Now let $|G| = 8$. Since $\overline{\gamma}(\Theta(G)) \geq 2$, we have that $G \cong D_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2^3$. Note that both $\Theta(D_8)$ and $\Theta(\mathbb{Z}_2^3)$ have a subgraph isomorphic to K_7. We deduce that $\overline{\gamma}(\Theta(D_8)) \geq \overline{\gamma}(K_7) = 3$ and $\overline{\gamma}(\Theta(\mathbb{Z}_2^3)) \geq \overline{\gamma}(K_7) = 3$. Now since $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \overline{K_4} + K_4$, it follows from Lemma 2.6 that $\overline{\gamma}(\Theta(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 2$, as desired.

Let $|G| = 9$. Then $G \cong \mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Note that $\overline{\gamma}(\Theta(\mathbb{Z}_3 \times \mathbb{Z}_3)) = \overline{\gamma}(K_9) = 5$ by Theorems 2.1 and 2.5. Now since $\Theta(\mathbb{Z}_9) \cong \overline{K_6} + K_3$, it follows from Lemma 2.6 that $\overline{\gamma}(\Theta(\mathbb{Z}_9)) = 2$, as desired. □

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