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Complex analysis

CLASSES AND BOUNDARY PROPERTIES OF FUNCTIONS IN THE OPEN UNIT DISK

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Abstract

Let ψ be a Blaschke product and $d\theta(\operatorname{supp}\psi) = 0$. In this paper we prove that the functions of Bourgain algebra $(\psi H^\infty(D), L^\infty(D))_b$ have essential non-tangential limit at almost every point of $T = \{z :| z |= 1\}$.

Key words: bounded analytic functions, Bourgain algebras, boundary behaviour

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1. Introduction. Let Y be a Banach algebra and X be a linear subspace of Y. The Bourgain algebra $X_b = (X, Y)$ of X relative to X is defined to the set of all $f \in Y$ such that: if $f_n \to 0$ weakly in X, then $\operatorname{dist}(f.f_n, X) \to 0$.

The distance, $\operatorname{dist}(f.f_n, X)$ between $f.f_n$ and X is the quotient norm of the coset $f.f_n + X$ in the space Y/X. Cima and Timoney proved in [1] that: X_b is a closed subalgebra of Y and contains the constant functions; if X is an algebra, then $X \subset X_b$. The concept of Bourgain algebras was introduced by Cima and Timoney [1] in conection with Bourgain's work on the Dunford–Pettis property [2]. Subsequently, several authors have studied Bourgain algebras [3–7].

Let D be the open unit disk and $H^\infty(D)$ denote the Banach algebra of all bounded analytic functions on D with the supremum norm. Let $L^\infty = L^\infty(T)$ be the algebra of essentially bounded, measurable functions with respect to Lebesgue

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measure $d\theta/2\pi$ on $T = \{z :| z |= 1\}$. By considering boundary functions on T, we can consider that $H^\infty = H^\infty(T)$ is an essentially supremum-norm closed subalgebra of L^∞. Let $C(\overline{D})$ be the space of all continuous functions on the closed unit disk $\overline{D}$ and let $A(\overline{D}) = H^\infty(D) \cap C(\overline{D})$ denote the disk algebra, i.e. the algebra of all continuous functions on $\overline{D}$ which are analytic on D.

In [4] and [7] Cima, Stroethoff and Yale characterized Bourgain algebras $(H^\infty(D), L^\infty(D))_b$ and $(A(\overline{D}), L^\infty(D))_b$ of H^∞ and $A(\overline{D})$ with respect to algebra $L^\infty(D)$ of Lebesgue measurable, essentially bounded functions on the unit disk D. Some results for Bourgain algebras of subalgebras are proven in [8–10].

For $\xi \in T$ and $R \in (0,1)$ the open non-tangential cone $\Gamma_R(\xi)$ at ξ is the interior of the convex hull of ξ and the disk $\{z \in \mathbb{C} : |z| \leq R\}$. We say $f \in L^\infty(D)$ that has essential non-tangential limit at ξ if

$$\operatorname{esssup}\{|\ f(z) - L\ | : z \in \Gamma_R(\xi), |z| > 1 - \delta\} \to 0 \quad \delta \to 0^+$$

for all $R \in (0,1)$. In this case we will write $f^*(\xi)$ for L. Define BV to be the set of all $f \in L^\infty(D)$ such that an essential non-tangential limit $f^*(\xi)$ exists for almost every $\xi \in T$.

In [4] Cima, Stroethoff and Yale point out classes of functions that have non-tangential limits almost everywhere on the circle. They prove that if X is a linear subspace of $L^\infty(D)$ which satisfies $A(\overline{D}) \subseteq X \subseteq BV$, then $(X, L^\infty(D))_b \subseteq BV$.

For a sequence $\{z_n\}_n$ in $\mathbb{D}$ with $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$, the function:

$$B(z) = \prod_{n=1}^{\infty} \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z}, z \in D,$$

is called a Blaschke product with zeros $\{z_n\}_n$. A sequence $\{z_n\}_n$ in D is called interpolating if for every bounded sequence $\{a_n\}_n$ of complex numbers there is a function $f \in H^\infty$ such that $f(z_n) = a_n$ for all n. If $\{z_n\}_n$ is an interpolating sequence, then $B(z)$ is also called interpolating. The study of interpolating sequences is useful in many areas of function theory and operator theory [11,12].

Let ψ be an inner function, i.e. $\psi \in H^\infty(D)$ such that $\left|\psi(e^{i\theta})\right| = 1$ almost everywhere on T. The support of ψ is the set of points $\lambda \in T$ for which there is a sequence $\{z_n\}_n \subset D$ of points such that $z_n \to \lambda$ and $\psi(z_n) \to 0$. Then $\operatorname{supp} \psi$ is a closed subset of T. Note that if ψ is a Blaschke product, then $\operatorname{supp} \psi$ coincides with the cluster set of $\{a_n\}_n$. If $\operatorname{supp} \psi$ is a finite set, then ψ is called a Blaschke product with a finite number of singularities.

Blaschke products are analytic in a much larger region than the open unit disk.

Theorem 1.1 ([13]). *The Blaschke product $B(z)$ whose zeros are $a_1, a_2, \ldots, a_n, \ldots$; $0 < |a_n| < 1$ converges at all points z in the complex plane except those in the compact set K consisting of*

(1) *the points* $z = \dfrac{1}{\overline{a}_n}$ *;*

(2) *the points* z *on the unit circle which are accumulation points of the sequence* $\{a_n\}$.

The convergence is uniform on any closed set in the plane which is disjoint from K, and the product $B(z)$ is thus analytic off K.

Note that if $B(z)$ is a Blaschke product, then $|B(z)| = 1$ in $T \setminus \operatorname{supp} B$.

Let ψ be a Blaschke product, with a finite number of singularities. In [14] it is shown that $(\psi A(\overline{D}), L^\infty(D))_b \subset BV$.

In this paper we point out classes of functions that have non-tangential limits almost everywhere on the circle. Let ψ be a Blaschke product and $d\theta(\operatorname{supp}\psi) = 0$. Actually we prove that $(\psi H^\infty(D), L^\infty(D))_b \subset BV$. The idea for the proof of Theorem 3.1 is coming from [4]. Note that $A(\overline{D})$ is not contained in $\psi H^\infty(D)$.

2. Preliminaries. For $\xi \in T$ and $0 < \delta < 1$ we will write $E(\xi, \delta) = \{\omega \in D : |\omega - \xi| < \delta\}$. For $f \in L^\infty(D)$ and a nonempty set $E \subset D$ define the essential oscillation of f over E by

$$\omega(f, E) = \operatorname{esssup}\{|f(z) - f(w)| : z, \omega \in E\}$$

and the essential non-tangential oscillation of f at $\xi \in T$ by

$$\omega_R(f, \xi) = \lim_{\delta \to 0^+} \omega(f, E(\xi, \delta) \cap \Gamma_R(\xi)).$$

Then, as proved in [4] the following equations apply:

(1) $BV = \{f \in L^\infty(D) : \omega_R(f, \xi) = 0$ for $R \in (0, 1)$ and for almost every $\xi \in T\}$;

(2) $\omega_R(f.g, \xi) = |f^*(\xi)|\, \omega_R(g, \xi)$ if $g \in L^\infty(D)$ and f has essential limit $f^*(\xi)$;

(3) $\omega_R(f, \xi) = \omega_R(f - g, \xi)$ when $\omega_R(g, \xi) = 0$ and $f, g \in L^\infty(D)$.

Let E be a Lebesgue measurable subset of $\mathbb{R}$ and we denote by $|E|$ the measure of E and by $U(x, \delta)$ – the neighbourhood of x with radius $\delta > 0$. The quantity

$$D(E, x) = \lim_{\delta \to 0^+} \frac{|E \cup U(x, \delta)|}{2\delta}$$

if it exists, is the metric density of E at the point x. If $D(E, X) = 1$, the point x is called a point of density of E. It is well known that $D(E, x) = 1$ for almost all points x in E, i.e. if $D(E) = \{x \in E : D(E, x) = 1\}$, then $|D(E)| = |E|$. Note that if x is a point of density of E, then the set $F = E \cup U(x, \delta)$ has positive measure. If $y \in (x - \delta, x + \delta)$ we identify y with e^{iy}, when working on the circle T.

Lemma 2.1 ([4]). *Let E and Z be measurable subsets of the circle T with $|E| > 0$ and $|Z| = 0$. Then each $\xi \in D(E)$ is the limit of a sequence $\{\xi_n\}_n$ of distinct points in $D(E)\backslash Z$.*

We will need some interpolation statements.

Lemma 2.2 ([15]). *If $\{z_n\}_n \subset D$ is interpolating sequence, then there exist functions $\{f_n\}_n \subset H^\infty(D)$ and positive number M such that $f_n(z_n) = 1$ for all n, $f_n(z_k) = 0$, for $n \neq k$ and $\sum_{n=1}^{\infty} |f_n(z)| \leq M$ for $z \in D$.*

Lemma 2.3 ([15]). *Suppose that $\{f_n\}_n$ is a sequence in $H^\infty(D)$ such that $\sum_{n=1}^{\infty} |f_n(z)| \leq M$ for $z \in D$. Then $f_n \to 0$ weakly in $H^\infty(D)$.*

Lemma 2.4 ([15]). *Let $\{\xi_n\}_n$ be a sequence of distinct points on the circle T which converges to a point ξ which is not equal to any of the ξ_n. Then there exists a sequence $\{f_n\}_n \subset A(\overline{D})$ such that $f_n(\zeta_n) = 1$ for all n,$f_n(\zeta_k) = 0$ for $n \neq k$ and $\sum_{n=1}^{\infty} |f_n(z)| \leq 2$ for all $z \in D$.*

3. The main result.

Theorem 3.1. *Let ψ be a Blaschke product and $d\theta(\operatorname{supp}\psi) = 0$. The Bourgain algebra of $\psi H^\infty(D)$ relative to $L^\infty(D)$ is contained in BV, i.e. $(\psi H^\infty(D), L^\infty(D))_b \subset BV$.*

Proof. Assume that $f \in (\psi H^\infty(D), L^\infty(D))_b \subset BV$, but $f \notin BV$. Then there exists a set $B \subset T$, $d\theta(B) > 0$ such that for every $\zeta \in B$ there is no essential non-tangential limit $f^*(\zeta)$. Obviously the same is true for the set $B' = B\backslash supp\psi$: $B' \subset T, d\theta(B') > 0$ and for every $\zeta \in B'$ there is no essential non-tangential limit $f^*(\zeta)$. Since

$$BV = \{f \in L^\infty(D) : \omega_R(f, \zeta) = 0 \text{ for all } R \in (0,1) \text{ and for almost every } \zeta \in T\}$$

we obtain that for every $\zeta \in B'$ there exists $R_\zeta \in (0,1)$ such that $\omega_R(f,\zeta) \geq \delta_\zeta > 0$. As in [4] we can consider that $\omega_R(f,\zeta) \geq \delta > 0$ for all $\zeta \in B'$, where $R \in (0,1)$.

Let ζ_0 be a fixed point in $D(B') = \{z \in B' : D(B', z) = 1\}$. By Lemma 2.1 there exists a sequence $\{\zeta_n\}_n \subset D(B') \setminus \{\zeta_0\}$ from different points, which converges to ζ_0. Since $\operatorname{supp}\psi \cap B' = \varnothing$ then $\operatorname{supp}\psi \cap \{\zeta_n\}_n = \varnothing$ for every n. By Lemma 2.4 there is a sequence $\{\zeta_n\}_n \subset A(\overline{D})$ such that $f_n(\zeta_n) = 1$ for all n, $f_n(\zeta_k) = 0$, for $n \neq k$ and $\sum_{n=1}^{\infty} |f_n(z)| \leq 2$ for all $z \in D$. Then for functions $g_n(z) = \psi(z).f_n(z), n \in \mathbb{N}$, we obtain:

$$g_n \in \psi A(\overline{D}) \subset \psi H^\infty(D); g_n(\zeta_n) = \psi(\zeta_n) \text{ for all } n;$$

$$g_n(\zeta_k) = 0 \text{ for } n \neq k \text{ and } \sum_{n=1}^{\infty} |g_n(z)| \leq \sum_{n=1}^{\infty} |f_n(z)| \leq 2 \text{ for all } z \in D.$$

By Lemma 2.3 (with $\psi H^\infty(D)$ instead of $H^\infty(D)$) we have $g_n \to 0$ weakly in $\psi H^\infty(D)$. Since $f \in (\psi H^\infty(D), L^\infty(D))_b$ there is a sequence $\{\psi h_n\}_n \subset \psi H^\infty(D)$ such that $\|fg_n - \psi h_n\|_\infty = \|f.\psi f_n - \psi h_n\|_\infty = \varepsilon_n \to 0$ as $n \to \infty$. Then, because $\psi h_n \in H^\infty(D)$, there exists n_0 and for every $n > n_0$ the function $f.\psi f_n$ has essential non-tangential limits almost everywhere on unit circle.

Fix $n > n_0$. Since f_n is continuous in $\overline{D}$ and $f_n(\zeta_n) = 1$, then there exists $r > 0$ such that $|f_n(p)| > \dfrac{1}{2}$ for every $p \in U(\zeta_n, r) = \{z \in T : |z - \zeta_n| < r\}$. But ζ_n is a point of density of B' and we obtain $d\theta(B' \cap U(\zeta_n, r)) > 0$. Therefore we can find a point $\xi^{(n)} \in B'$ such that $\left|f_n(\xi^{(n)})\right| > \dfrac{1}{2}$. On the other hand, again from $d\theta(B' \cap U(\zeta_n, r)) > 0$, we can consider that $f.\psi f_n$ and ψh_n have essential non-tangential limit in $\xi^{(n)}$.

Then by (2) and (3) we have:

$$\frac{\delta}{2} < \left|f(\xi^{(n)})\right|.\delta = \left|\psi(\xi^{(n)})\right|\left|f(\xi^{(n)})\right|.\delta \le \left|(\psi f_n)(\xi^{(n)})\right|.\omega_R(f, \xi^{(n)})$$

$$= \omega_R(\psi f_n f, \xi^{(n)}) = \omega_R(\psi f_n f - \psi h_n, \xi^{(n)}) \le 2\|\psi f_n f - \psi h_n\| \le 2\varepsilon_n,$$

because $\left|\psi(\xi^{(n)})\right| = 1$ (theorem). This contradicts to the condition that $\varepsilon_n \to 0$.

Therefore $f \in BV$ and $(\psi H^\infty(D), L^\infty(D))_b \subset BV$. □

A closed subalgebra X between $A(\overline{D})$ and $H^\infty(D)$ is called an analytic subalgebra. Note that in the proof of Theorem 3.1 only two properties of $H^\infty(D)$ were used:

1. $H^\infty(D) \subset BV$. But if X is an analytic subalgebra, then $X \subset H^\infty(D) \subset BV$.

2. If $\{f_n\}_n \subset A(\overline{D}$ are the peak point functions in Lemma 2.4 and $g_n = \psi.f_n$, then $g_n \in \psi A(\overline{D} \subset \psi H^\infty(D)$ and $g_n \to 0$ weakly in $\psi H^\infty(D)$. But if X is an analytic subalgebra, then $g_n \in \psi A(\overline{D}) \subset \psi X$ and $g_n \to 0$ weakly in ψX because $A(\overline{D}) \subset X$.

Consequently, if X is an analytic subalgebra, then $(\psi X, L^\infty(D))_b \subset BV$.

For $f \in L^\infty(D), \xi \in T$, and $t > 0$ we put

$$\omega(f, \xi, t) = \operatorname{esssup}\{|f(z) - f(w)| : z, w \in E(\xi, t)\},$$

where $E(\xi, t) = z \in D : |z - \xi| < t$. Essential oscillation of f at the point $\xi \in T$ is defined as

$$\omega(f, \xi) = \lim_{\delta \to 0^+} \omega(f, \xi, t).$$

Let $W(D)$ be the set of functions in $L^\infty(D)$ whose essential oscillations are bounded away from zero at only finitely many points of the unit circle, i.e.

$$W(D) = \{f \in L^\infty : \text{ for every } \delta > 0 \text{ the set } \{\xi \in T : \omega(f, \xi) \ge \delta\} \text{ is finite}\}.$$

The following result can be proved as in Theorem 3.1.

Theorem 3.2. *Let ψ be a Blaschke product and $d\theta(supp\psi) = 0$. The Bourgain algebra of $\psi A(\overline{D})$ relative to $L^\infty(D)$ is contained in $W(D)$, i.e. $(\psi A(\overline{D}), L^\infty(D))_b \subset W(D)$.*

The algebra $\psi H^\infty(D) + \mathbb{C}$, where ψ is a Blaschke product and $\mathbb{C}$ denotes the span of the constant function in $H^\infty(D)$ has been studied by several authors. The papers [16], [17] and [18] considered Nevanlinna–Pick interpolation problems for these algebras. The case when the Blaschke product has a finite number of zeros was considered in [17] and the case of one zero at 0 with multiplicity 2 was discussed in [18]. In [19] the corona theorem for $\psi H^\infty(D) + \mathbb{C}$ has been proved in several different ways. In [9] it is shown that if ψ is a finite Blaschke product, then the Bourgain algebra of $\psi H^\infty + \mathbb{C}$ with respect to the algebra L^∞ coincides with the Sarason algebra $H^\infty + C$.

Obviously $\psi H^\infty(D) \subset \psi H^\infty(D) + \mathbb{C}$. It can be seen, from the proof of Theorem 3.1, that Bourgain's algebra of $\psi H^\infty(D) + \mathbb{C}$ with respect to the algebra $L^\infty(D)$ is contained in BV. The following theorem shows the relationship between the corresponding Bourgain algebras.

Theorem 3.3. *Let ψ be a Blaschke product. Then $(\psi H^\infty(D), L^\infty(D))_b$ is contained in $(\psi H^\infty(D) + \mathbb{C}, L^\infty(D))_b$.*

Proof. Let $f \in (\psi H^\infty(D), L^\infty(D))_b$ and the sequence $g_n = \psi h_n + c_n \to 0$ weakly in $\psi H^\infty(D) + \mathbb{C}$. Then $g_n \to 0$ weakly in $H^\infty(D)$. For the point evaluation $\varphi_{z_0} \in (H^\infty(D))^*$, $z_0 \in D$ such that $\psi(z_0) = 0$ we have

$$c_n = \psi(z_0)h_n(z_0) + c_n = \varphi_{z_0}(\psi h_n + c_n) \to 0 \quad \text{when} \quad n \to 0.$$

If $\varphi \in (\psi H^\infty(D))^*$ and Φ is the Hahn–Banach extension of φ on $H^\infty(D)$, then

$$\varphi(\psi.h_n) = \Phi|_{\psi H^\infty}(\psi.h_n) = \Phi|_{\psi H^\infty}(g_n - c_n) = \Phi(g_n) - c_n.\Phi(1) \to 0$$

and $\{\psi.h_n\}_n$ tends to zero weakly in $\psi H^\infty(D)$. Since $f \in (\psi H^\infty(D), L^\infty(D))_b$ there exists a sequence $\{p_n\}_n \subset \psi H^\infty(D) \subset \psi H^\infty(D) + \mathbb{C}$ such that $\|f.\psi h_n - p_n\|_D \to 0$. Then

$$\|f.g_n - p_n\|_D = \|f.\psi h_n + f.c_n - p_n\|_D \leq \|f.\psi h_n - p_n\|_D + \|f\|_D|c_n| \to 0$$

and hence $f \in (\psi H^\infty(D) + \mathbb{C}, L^\infty(D))_b$. The proof is complete. □

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