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A VERSION OF THE HADAMARD–LÉVY THEOREM FOR FRÉCHET SPACES

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Abstract

We prove a version of the Hadamard–Lévy theorem for Keller C_c^1 mappings between Fréchet spaces.

Key words: Hadamard–Lévy theorem, Fréchet spaces, global diffeomorphism

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This note serves to announce the continuation of the author's works [1,2] on the problem of finding sufficient conditions which imply that Keller C_c^1 local diffeomorphisms between Fréchet spaces are global ones.

A Fréchet space F is a complete metrizable locally convex space. Hence the topology of F can be given by an increasing sequence $(\|\cdot\|_{F,n})_{n\in\mathbb{N}}$ of seminorms. Let E be another Fréchet space whose topology is defined by an increasing sequence $(\|\cdot\|_{E,n})_{n\in\mathbb{N}}$ of seminorms. We denote the vector space of all continuous linear mappings from E into F by $CL(E,F)$.

In what follows we consider only Fréchet spaces over the field $\mathbb{R}$ of real numbers.

A bornology $\mathcal{B}_F$ on F is a covering of F satisfying the following axioms:

1. $\mathcal{B}_F$ is stable under finite unions;

2. if $A \in \mathcal{B}_F$ and $B \subseteq A$, then $B \in \mathcal{B}_F$.

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The compact bornology on F is the family $\mathcal{B}_F^c$ of relatively compact subsets of F having the set of all compact subsets of F as a base, in the sense that every $B \in \mathcal{B}_F^c$ is contained in some compact set (cf. [3]).

Let $\mathcal{B}_E^c$ be the compact bornology on E. We endow the vector space $CL(E, F)$ with the $\mathcal{B}_E^c$-topology which is the topology of uniform convergence on all compact subsets of E. This is a Hausdorff locally convex topology which can be defined by the family of all seminorms obtained as follows:

$$\|L\|_{B,n} \coloneqq \sup\{\|L(e)\|_{F,n} : e \in B\},$$

where $B \in \mathcal{B}_E^c$ and $n \in \mathbb{N}$. Likewise, we endow the space of continuous linear mappings from F into E, $CL(F, E)$, with the $\mathcal{B}_F^c$-topology.

Due to its many advantages, the notion of Keller C_c^1 mappings (which is equivalent to the notion of C^k mappings in the Michal–Bastiani sense) plays a significant role in the theory of differential calculus in locally convex spaces.

Let U be an open subset of E, and $\varphi \colon U \to F$ a mapping. The directional derivative of φ at $x \in U$ in the direction $h \in E$ is defined as

$$\mathtt{D}\varphi(x)h = \lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t}$$

whenever it exists. We say that φ is a Keller C_c^1 mapping (or differentiable of class Keller C_c^1) if the directional derivative $\mathtt{D}\varphi(x)h$ exists for each $h \in E$, and the induced mapping (the derivative) $\mathtt{D}\varphi \colon U \to CL(E, F)$ is continuous. In this definition the space $CL(E, F)$ is endowed with the $\mathcal{B}_E^c$-topology. We refer to KELLER's monograph [4] for further details.

The Hadamard–Lévy theorem was proved by Hadamard for finite dimensional spaces. Then, it was generalized by LÉVY [5] to Hilbert spaces. Later on, PLASTOCK [6] extended the theorem to the case of C^1 mappings between Banach spaces. In this article we extend the Hadamard–Lévy theorem to the case of Keller C_c^1 mappings between Fréchet spaces. Our approach relies on the path lifting property and the local surjectivity result.

Let $\varphi \colon U \subset E \to F$ be a Keller C_c^1 mapping. First, we give a sufficient condition for the local surjectivity of φ. That is, the existence of a solution to the equation $\varphi(e) = f$ for f close enough to $\varphi(u_1)$ for a point $u_1 \in U$. To solve this equation it is enough to assume that the derivative $\mathtt{D}\varphi(u_1)$ is surjective. The standard proof, inspired by the Picard method, is to construct a sequence of elements which converges to a solution.

Proposition 1. *Let U be open in E, and $\varphi \colon U \subset E \to F$ a Keller C_c^1 mapping. If the derivative $\mathtt{D}\varphi(u_1)$ is surjective for some $u_1 \in U$, then φ is locally surjective. Furthermore, if $\mathtt{D}\varphi(u)$ is surjective for all $u \in U$, then φ is open.*

Proof. We want to find a solution $\mathbf{e}$ to the equation $f = \varphi(e)$ for f in an open neighbourhood of $\varphi(u_1)$ and $\mathbf{e}$ in an open neighbourhood of u_1 provided that

f is close enough to $\varphi(u_1)$. Let $E_0 = \ker \mathtt{D}\varphi(u_1)$. The quotient space $E_1 = E/E_0$ is a Fréchet space whose topology is defined by the family of all seminorms

$$|||\hat{e}|||_{E_1,i} = \inf_{x\in\hat{e}} \|x\|_{E,i}\,,$$

where $\hat{e}$ is the coset of $e \in E$ and $i \in \mathbb{N}$. By the open mapping theorem (cf. [7], Theorem 4.35) there exists a linear continuous map $\phi^{-1}\colon F \to E_1$, where $\phi\colon E_1 \to F$ is a topological isomorphism induced by $\mathtt{D}\varphi(u_1)$. Define a sequence of cosets $\hat{e}_i \in E/E_0$ and a sequence of elements $e_i \in \hat{e}_i \subset E$ inductively by $\hat{e}_0 := E_0$, $e_0 \in \hat{e}_0$ small, and

$$\hat{e}_i = \hat{e}_{i-1} + \phi^{-1}(f - \varphi(u_1 + e_{i-1})). \tag{1}$$

We choose $e_i \in \hat{e}_i$ such that

$$|||\hat{e}_i - \hat{e}_{i-1}|||_{E_1,i} \geqslant \frac{1}{2}\|e_i - e_{i-1}\|_{E,i}. \tag{2}$$

The latter is possible because

$$|||\hat{e}_i - \hat{e}_{i-1}|||_{E_1,i} = \inf_{x\in\hat{e}_i} \|x - e_{i-1}\|_{E,i}\,.$$

Since $e_{i-1} \in \hat{e}_{i-1}$, $\hat{e}_{i-1} = \phi^{-1}(\mathtt{D}\varphi(u_1)e_{i-1})$, it follows that

$$\hat{e}_i = \phi^{-1}(f - \varphi(u_1 + e_{i-1}) + \mathtt{D}\varphi(u_1)e_{i-1}).$$

By subtracting this from the expression for $\hat{e}_{i-1}$ we get

$$\hat{e}_i - \hat{e}_{i-1} = -\phi^{-1}\big(\varphi(u_1 + e_{i-1}) - \varphi(u_1 + e_{i-2}) - \mathtt{D}\varphi(u_1)(e_{i-1} - e_{i-2})\big).$$

Since φ is of class Keller C^1_c, we may find a convex neighbourhood $\mathcal{V}$ of u_1 such that for given $\delta > 0$ we have

$$\|\mathtt{D}\varphi(v) - \mathtt{D}\varphi(u_1)\|_{B,i} < \delta, \quad i \in \mathbb{N}, \quad B \in \mathcal{B}^c_E,$$

for $v \in \mathcal{V}$. Suppose inductively that $u_1 + e_{i-1} \in \mathcal{V}$ and $u_1 + e_{i-2} \in \mathcal{V}$. Therefore, for $0 \leqslant t \leqslant 1$ we obtain

$$u_1 + te_{i-1} + (1-t)e_{i-2} = (1-t)(u_1 + e_{i-2}) + t(u_1 + e_{i-1}) \in \mathcal{V}.$$

Thus, by the mean value inequality ([4], Corollary 1.1.4 (1)) we have

$$|||\hat{e}_i - \hat{e}_{i-1}|||_{E_1,i} \leqslant \delta(\|\phi^{-1}\|_{B,i})(\|e_{i-1} - e_{i-2}\|_{E,i}).$$

Therefore, in virtue of equation (2) we obtain

$$\|e_i - e_{i-1}\|_{E,i} \leqslant 2|||\hat{e}_i - \hat{e}_{i-1}|||_{E_1,i} \leqslant 2\delta(\|\phi^{-1}\|_{B,i})(\|e_{i-1} - e_{i-2}\|_{E,i}).$$

Hence, if δ is small enough, then

$$\|e_i - e_{i-1}\|_{E,i} \leqslant \frac{1}{2}\|e_{i-1} - e_{i-2}\|_{E,i}.$$

Starting with e_0 such that $\sup_{i\in\mathbb{N}}\|e_0\|_{E,i}$ is small enough and $\|e_1 - e_0\|_{E,i} < \frac{1}{2}\|e_0\|_{E,i}$ for each $i \in \mathbb{N}$, we have

$$\begin{aligned}\|e_i\|_{E,i} &\leqslant \|e_0\|_{E,i} + \|e_1 - e_0\|_{E,i} + \|e_2 - e_1\|_{E,i} + \cdots + \|e_i - e_{i-1}\|_{E,i} \\ &\leqslant \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{i-1}}\right)\|e_0\|_{E,i} \leqslant 2\|e_0\|_{E,i}.\end{aligned}$$

Thus, the elements $u_1 + e_i$ stay inductively in $\mathcal{V}$. It follows that e_i is a Cauchy sequence, therefore e_i converges to some $a \in E$, because E is complete. As a consequence, $\hat{e}_i$ converges to $\hat{a}$ and $a \in \hat{a}$. Thus, in view of equation (1) we get $\phi^{-1}(f - \varphi(u_1 + a)) = 0$, and therefore by setting $\mathbf{e} = u_1 + a$ we obtain $f = \varphi(\mathbf{e})$. This proves the assertion of the theorem. □

Lemma 1. *Let $\varphi\colon E \to F$ be a Keller C_c^1 local diffeomorphism. Assume that for each $e \in E$ we have*

$$\|[\mathrm{D}\,\varphi(e)]^{-1}\|_{B,n} < \infty, \quad B \in \mathcal{B}_F^{c}, \quad n \in \mathbb{N}.$$

Then, there are open neighbourhoods $\mathcal{U}_e$ of e and $\mathcal{V}_{\varphi(e)}$ of $\varphi(e)$ such that $\varphi\colon \mathcal{U}_e \to \mathcal{V}_{\varphi(e)}$ is a Keller C_c^1 diffeomorphism and $\varphi(\overline{\mathcal{U}}_e) = \overline{\mathcal{V}}_{\varphi(e)}$.

Proof. Since φ is a local diffeomorphism, it follows that for each $e \in E$ there are open neighbourhoods $\mathcal{U}_e$ of e and $\mathcal{V}_{\varphi(e)}$ of $\varphi(e)$ such that $\varphi\colon \mathcal{U}_e \to \mathcal{V}_{\varphi(e)}$ is a diffeomorphism. Without loss of generality we may assume that $\mathcal{U}_e$ and $\mathcal{V}_{\varphi(e)}$ are convex. Let $f \in \overline{\mathcal{V}}_{\varphi(e)}$, so there is a sequence f_i in $\overline{\mathcal{V}}_{\varphi(e)}$ such that $f_i \to f$. Since φ^{-1} is of class Keller C_c^1 on $\overline{\mathcal{V}}_{\varphi(e)}$, for $e_i = \varphi^{-1}(f_i) \in \overline{\mathcal{U}}_e$ by the mean value inequality ([4], Corollary 1.1.4 (1)) we obtain

$$\|e_i - e_j\|_{E,n} \leqslant \sup_{h\in[f_i,f_j]} \left\|\mathtt{D}\varphi^{-1}(h)\right\|_{B,n} \|f_i - f_j\|_{F,n}, \quad n \in \mathbb{N}, \quad B \in \mathcal{B}_F,$$

where $[f_i, f_j]$ is the closed segment joining f_i to f_j.

Then, by the lemma's assumption

$$\sup_{h\in[f_i,f_j]} \left\|\mathtt{D}\varphi^{-1}(h)\right\|_{B,n} = \sup_{h\in[f_i,f_j]} \left\|[\mathtt{D}\varphi(\varphi^{-1}(h))]^{-1}\right\|_{B,n} < \infty, \quad n \in \mathbb{N}, \quad B \in \mathcal{B}_F.$$

Thus, e_i is a Cauchy sequence and since E is complete, it follows that $e_i \to \mathbf{e}$ for some $\mathbf{e} \in \overline{\mathcal{U}}_e$, and $f = \varphi(\mathbf{e}) \in \varphi(\overline{\mathcal{U}}_e)$. Thereby, $\varphi(\overline{\mathcal{U}}_e) \supseteq \overline{\mathcal{V}}_{\varphi(e)}$. Obviously $\varphi(\overline{\mathcal{U}}_e) \subseteq \overline{\mathcal{V}}_{\varphi(e)}$, so $\varphi(\overline{\mathcal{U}}_e) = \overline{\mathcal{V}}_{\varphi(e)}$. □

The following lemma establishes the path lifting property of mappings. The proof is similar to the case of Banach spaces ([8], Lemma 1.23).

Lemma 2. *Let* $\varphi\colon E \to F$ *be a Keller* C^1_c *local diffeomorphism. Assume that* $\gamma(s,t)\colon [0,1]\times[0,1] \to F$ *is a Keller* C^1_c *mapping in both variables. Also, assume that there exists a point* $f \in F$ *such that* $\gamma(s,0) = f$ *for all* $s \in [0,1]$*. Then, there exists a mapping* $\Phi(s,t)\colon [0,1]\times[0,1] \to E$ *of class Keller* C^1_c *in both variables such that* $\Phi(s,t) = \varphi^{-1}\circ\gamma(s,t)$ *for all* $s,t \in [0,1]$*, and* $\Phi(s,0) = e = \varphi^{-1}(f)$ *for all* $s \in [0,1]$.

Proof. Since φ is a local diffeomorphism, there are open neighbourhoods $\mathcal{U}_e$ of e and $\mathcal{V}_f$ of f such that $\varphi\colon \mathcal{U}_e \to \mathcal{V}_f$ is a diffeomorphism. There exists $\delta > 0$ such that $[0,1]\times[0,\delta) \subset \gamma^{-1}(\mathcal{U}_e)$, because $\gamma^{-1}(\mathcal{U}_e)$, which is an open set, contains the closed set $[0,1]\times\{0\}$. For small enough $\delta > 0$, define the mapping $\Phi(s,t) = \varphi^{-1}(\gamma(s,t))$ for $(s,t) \in [0,1]\times[0,\delta)$. Let $\hat{\delta}$ be the supremum of these numbers δ. We shall show that Φ is defined at $t = \hat{\delta}$ for each $s \in [0,1]$. As Φ is the composition of C^1_c mappings, then it is of class C^1_c in t and s ([4], Corollary 1.3.1). For each fixed $s \in [0,1]$, by [4], Corollary A.3.4 for each $t_i, t_j \in [0,1]$ we have

$$\|\Phi(s,t_i) - \Phi(s,t_j)\|_{E,n} \leqslant K|t_i - t_j|, \quad n \in \mathbb{N},$$

where K is a constant. Thus, $\Phi(s,t_i)$ is a Cauchy sequence in E, and so is convergent. The continuity of φ and γ yields

$$\varphi(\Phi(s,\hat{\delta})) = \lim_{t_i\to\hat{\delta}} \varphi(s,\Phi(s,t_i)) = \lim_{t_i\to\hat{\delta}} \gamma(s,t_i) = \gamma(s,\hat{\delta}),$$

hence $\Phi(s,t)$ is defined at $t = \hat{\delta}$. We shall show by contradiction that $\hat{\delta} = 1$. Suppose $\hat{\delta} < 1$. There are open neighbourhoods $\mathcal{U}_s$ of $\Phi(s,\hat{\delta})$ and $\mathcal{V}_s$ of $\gamma(s,\hat{\delta})$ such that φ is a diffeomorphism of $\mathcal{U}_s$ onto $\mathcal{V}_s$ for each $s \in [0,1]$. Since the curve $\Phi(s,\hat{\delta})$, $s \in [0,1]$, is compact, it has a finite sub-covering with neighbourhoods $\mathcal{U}_{s_i}$, $i = 1,\dots,n$. Therefore, the sets $\mathcal{V}_{s_i} = \varphi(\mathcal{U}_{s_i})$, $i = 1,\dots,n$, cover the curve $\gamma(s,\hat{\delta})$. Since each $\gamma^{-1}(V_{s_i})$ is open and contains $(\hat{\delta}, s_i)$, it follows that it contains an open rectangle $(s_i - \beta_i, s_i + \beta_i)\times(\hat{\delta} - \alpha_i, \hat{\delta} + \alpha_i)$, where $(s_i - \beta_i, s_i + \beta_i) \subset \gamma^{-1}(\cdot,\hat{\delta})(\mathcal{V}_{s_i})$ and these rectangles cover $[0,1]$. Now, we can define the extension of the function $\Phi(s,t)$ for all $s \in [0,1]$ and $0 \leqslant t < \hat{\delta} + (\alpha = \min_i \alpha_i)$, by setting $\Phi(s,t) = (\varphi|_{U_i})^{-1}\circ\gamma(s,t)$ if $(s,t) \in [\hat{\delta}, \hat{\delta} + \alpha)\times(s_i - \beta_i, s_i + \beta_i)$, for $i = 1,\dots,n$. This contradicts the definition of $\hat{\delta}$, and therefore $\hat{\delta} = 1$. Finally, $\Phi(s,0) = \varphi^{-1}(\gamma(s,0)) = \varphi^{-1}(f) = e$ for all $s \in [0,1]$. □

Theorem 1. *Let* $\varphi\colon E \to F$ *be a Keller* C^1_c *local diffeomorphism. If for each* $e \in E$ *we have*

$$\|[\mathrm{D}\,\varphi(e)]^{-1}\|_{B,n} < \infty, \quad B \in \mathcal{B}^c_F, \quad n \in \mathbb{N},$$

then φ *is a global diffeomorphism.*

Proof. It suffices to prove that φ is injective and surjective.

Injectivity. Suppose $\varphi(u_0) = \varphi(u_1) = \mathbf{v}$. Define the curves $\alpha(t) = (1-t)u_0 + tu_1$, $\beta(t) = \varphi(\alpha(t))$, and $\gamma(s,t) = (1-s)\mathbf{v} + s\beta(t)$ for $t,s \in [0,1]$. Since φ is a local diffeomorphism, $\gamma(s,t)$ is of class Keller C^1_c in both variables, and

$\gamma(s,0) = \varphi(u_0)$ for all $s \in [0,1]$, it follows by Lemma 2 that there exists a mapping $\Phi(s,t) = \varphi^{-1}(\gamma(s,t))\colon [0,1] \times [0,1] \to E$ of class Keller C^1_c in both variables. Then, by the chain rule ([4], Corollary 1.3.2)

$$0 = \frac{d}{ds}\gamma(s,1) = \frac{d}{ds}\varphi(\Phi(s,1)) = \mathtt{D}\varphi(\Phi(s,1))\frac{d}{ds}\Phi(s,1),$$

because $\gamma(s,1) = \mathbf{v}$ for all $s \in [0,1]$. Thus, the theorem's assumption implies that $\frac{d}{ds}\Phi(s,1) = 0$, and so $\Phi(0,1) = \Phi(1,1)$. Similarly, we get $\frac{d}{dt}\Phi(0,t) = 0$. Therefore, $\Phi(0,0) = \Phi(0,1)$ which implies

$$u_0 = \Phi(0,0) = \Phi(1,1) = u_1.$$

Surjectivity. By Proposition 1, $\varphi(E)$ is open. We shall prove that it is closed too. Since φ is a local diffeomorphism for each $e \in E$, by Lemma 1 there are open neighbourhoods $\mathcal{U}_e$ of e and $\mathcal{V}_{\varphi(e)}$ of $\varphi(e)$ such that $\varphi\colon \mathcal{U}_e \to \mathcal{V}_{\varphi(e)}$ is a diffeomorphism and $\varphi(\overline{\mathcal{U}}_e) = \overline{\mathcal{V}}_{\varphi(e)}$. Then

$$\overline{\varphi(E)} \subseteq \overline{\bigcup_{e\in E} \mathcal{V}_{\varphi(e)}} \subseteq \bigcup_{e\in E} \overline{\mathcal{V}}_{\varphi(e)} = \bigcup_{e\in E} \varphi(\overline{\mathcal{U}}_e) \subseteq \varphi(E).$$

That is, $\varphi(E)$ is closed. □

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