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INTUITIONISTIC LOGICS OF CONFIRMATION AND DISCOURAGEMENT

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Abstract

We analyse intuitionistic logics of confirmation and discouragement. This means that our acceptance of some formula φ in a possible world w gets an additional support or disapprovement. We use pre-ordered neighbourhood frames to model this situation. Our systems have certain connection with so-called logics of false belief and unknown truths.

Key words: intuitionistic modal logic, non-normal modal logic, neighbourhood semantics

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1. Introduction. This article may be considered an appendix to our earlier work [1] which was concentrated on the logics of *false belief*. They describe the following situation: we do not believe in certain formula φ in a given world w but we are *encouraged* to accept this formula. Another interpretation says that we believe (wrongly) that φ is false while actually it is true.

In the present paper we study a different approach. However, we still use pre-ordered neighbourhood frames. We show four interpretations of the simplest intuitionistic mono-modal logic with axiom `T`. They refer to the additional *confirmation* or *denial* of a given formula in a world w. This aspect is modelled by means of neighbourhoods. Moreover, contrary to the systems of false belief, now we accept φ in w.

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Our framework is intuitionistic. Thus, it relies on the pre-order which allows us to speak about certain hierarchy of possible worlds. For example, if $v \geq w$, then we may think about temporal interpretation: that v is a set of circumstances which appears *after* the state of matters connected with w.

This is an outline of an ongoing project. Thus, these results should be treated as an initial point for further investigation. We shall sketch some of its possible ways.

As for the general study of non-normal intuitionistic modal logics, the reader can check [2] (an overview with many new results).

2. Language, frames and models. Let us define some basic notions.

Definition 2.1. **iE**-alphabet consists of:

(1) PV which is a fixed denumerable set of propositional variables $p, q, r, s, \ldots$

(2) Logical connectives which are $\wedge$, $\vee$, $\rightarrow$ and $\neg$.

Now we define pre-ordered neighbourhood frame (**pn**-frame) which is a kind of scaffolding for our future structures.

Definition 2.2. **pn**-frame is a triple $F = \langle W, \mathcal{N}, \leq \rangle$, where $\leq$ is a partial order on $W \neq \varnothing$ and $\mathcal{N}$ is a function from W into $P(P(W))$.

Definition 2.3. **pn**-model is a quadruple $M = \langle W, \mathcal{N}, \leq, V \rangle$, where $\langle W, \mathcal{N}, \leq \rangle$ is **pn**-frame and V is a function from PV into $P(W)$ such that: if $w \in V(q)$ and $w \leq v$, then $v \in V(q)$. In this case we write: $w \Vdash q$.

Forcing of complex formulas is defined in a standard manner. Note that our models are intuitionistic, hence:

$$w \Vdash \varphi \rightarrow \psi \;\Leftrightarrow\; \text{ for any } \; v \geq w, w \nVdash \varphi \;\text{ or }\; w \Vdash \psi.$$

Now we shall introduce four modal operators and corresponding models. Due to the limited space, we shall work with all of them simultaneously. Some definitions and proofs will be shortened to avoid repetition of obvious things. For example, we will use $V(\varphi)$ as a shortcut for $\{v \in W; v \Vdash \varphi\}$.

We assume that the reader is familiar with some basic notions (like prime theory, strong completeness, etc.) or symbols (like syntactic consequence $\vdash$), and he can recognize when we rely on this knowledge.

Definition 2.4. $\Box$**Log** (resp. $\Diamond$**Log**, $\blacksquare$**Log**, $\bullet$**Log**)-alphabet in its non-modal part is identical with **iE**-alphabet but it contains the following modal operator:

i) $\Box$ for $\Box$**Log**.

ii) $\Diamond$ for $\Diamond$**Log**.

iii) $\blacksquare$ for $\blacksquare$**Log**.

iv) $\bullet$ for $\bullet$**Log**.

Definition 2.5. $\Box$**pn** (resp. $\Diamond$**pn**, $\blacksquare$**pn**, $\bullet$**pn**) frame is a **pn**-frame with the following additional restriction:

(for $\Box$**pn**-frames)

(1) $$[w \leq v, v \in X \subseteq W, X \in \mathcal{N}_w] \Rightarrow X \in \mathcal{N}_v.$$

(for $\Diamond$**pn**-frames)

(2) $$[w \leq v, v \in X \subseteq W, -X \notin \mathcal{N}_w] \Rightarrow -X \notin \mathcal{N}_v.$$

(for $\blacksquare$**pn**-frames)

(3) $$[w \leq v, v \in X \subseteq W, -X \in \mathcal{N}_w] \Rightarrow -X \in \mathcal{N}_v.$$

(for $\bullet$**pn**-frames)

(4) $$[w \leq v, v \in X \subseteq W, X \notin \mathcal{N}_w] \Rightarrow X \notin \mathcal{N}_v.$$

Definition 2.6. $\Box$**pn** (resp. $\Diamond$**pn**, $\blacksquare$**pn**, $\bullet$**pn**)-model is a **pn**-model with valuation and forcing of modal operator defined as below:
i) $w \Vdash \Box\varphi \Leftrightarrow w \Vdash \varphi$ and $V(\varphi) \in \mathcal{N}_v$ (in $\Box$**pn**-model).
ii) $w \Vdash \Diamond\varphi \Leftrightarrow w \Vdash \varphi$ and $-V(\varphi) \notin \mathcal{N}_w$ (in $\Diamond$**pn**-model).
iii) $w \Vdash \blacksquare\varphi \Leftrightarrow w \Vdash \varphi$ and $-V(\varphi) \in \mathcal{N}_w$ (in $\blacksquare$**pn**-model).
iv) $w \Vdash \bullet\varphi \Leftrightarrow w \Vdash \varphi$ and $V(\varphi) \notin \mathcal{N}_w$ (in $\bullet$**pn**-model).

Let us discuss our interpretation of those operators. First, we have $\Box$ which is often used to model *necessity*. Our understanding of $\Box$ is: we accept φ in w and, additionally, our decision is supported or confirmed by the fact that $V(\varphi) \in \mathcal{N}_w$.

Second, we have $\Diamond$. This symbol is often used to speak about *possibility*. However, here we assume that forcing of $\Diamond\varphi$ implies (in particular) forcing of φ. We have the following situation: we accept φ and we *cannot* say that we are dissuaded from this decision: the set $-V(\varphi)$ is not among w-neighbourhoods.

Assume for a moment that our language contains both $\Box$ and $\Diamond$ and our model satisfies both needful conditions of monotonicity (i.e. it is $\Box\Diamond$**pn**-model). We show that $\Box\varphi \to \neg\Diamond\neg\varphi$ is a tautology. Assume that some w denies this formula. Then there is $v \geq w$ such that $v \Vdash \Box\varphi$ and $v \nVdash \neg\Diamond\neg\varphi$. Hence, $v \Vdash \varphi$ and $V(\varphi) \in \mathcal{N}_v$. However, there is $s \geq v$ such that $s \Vdash \Diamond\neg\varphi$. Then $s \Vdash \neg\varphi$. But this is impossible because $s \geq v$ and we have persistence of truth.

Now consider $W = \{w, v\}$, where $\mathcal{N}_w = \mathcal{N}_v = W$, $w \leq v$ and $V(\varphi) = \{v\}$. This model satisfies both required monotonicity conditions (namely, for $\Box$ and $\Diamond$). We see that $w \Vdash \neg\Diamond\neg\varphi$, because for any $x \geq w$ we have $x \nVdash \neg\varphi$. But $w \nVdash \Box\varphi$ because $w \nVdash \varphi$. Hence, $\neg\Diamond\neg\varphi \to \Box\varphi$ is not a tautology in the class of $\Box\Diamond$**pn**-models.

Note that $\Box\varphi \to \Diamond\varphi$ is not a tautology in this class. Clearly, it will become true if we assume that $X \in \mathcal{N}_w \Rightarrow -X \notin \mathcal{N}_w$ (for any $w \in W$).

Third, we have $\blacksquare$. Now φ is accepted by the subject but he is advised against this formula. This is expressed by the fact that $-V(\varphi) \in \mathcal{N}_w$. The set of worlds in which φ is *not* satisfied is one of our credible neighbourhoods.

The fourth case is $\bullet$. Now φ is still accepted in a given set of circumstances. However, the set of worlds which satisfy φ is not credible because it does not belong to the class of w-neighbourhoods. Another interpretation of $\bullet$ is that φ is true but unknown. This leads us to the so-called logics of *unknown truths*. We shall deal with this topic later.

The question is: how to combine all these operators in one propositional logic based on intuitionism?

It seems that any relationships between our operators should be rather *weak*. For example, if we are advised against φ (as in the case of $\blacksquare\varphi$), then it does not mean that we are encouraged to accept $\neg\varphi$. Not on the ground of intuitionistic logic. If we are *not* discouraged ($\Diamond\varphi$ applies to us), then it still does not mean that we are encouraged (by means of $\Box\varphi$). If φ is not suggested by our neighbourhoods ($\bullet\varphi$ case), then it does not mean that there is a neighbourhood which is against φ (as in the case of $\blacksquare\varphi$).

Note that we have three different things: **i)** lack of recommendation of some formula; **ii)** advising against this formula; **iii)** supporting the opposite formula, namely $\neg\varphi$. This last case was not analysed above. It would require a new operator, say **N**, defined in the following manner: $w \Vdash \mathbf{N}\varphi \Leftrightarrow w \Vdash \varphi$ and $V(\neg\varphi) \in \mathcal{N}_w$[1]. On the intuitionistic ground $V(\neg\varphi)$ may be different from $-V(\varphi)$. However, it is always true that $V(\neg\varphi) \subseteq -V(\varphi)$. As for the monotonicity condition for **N**, it can be:

$[w \leq v,\ v \in X$ and there is $Y \in \mathcal{N}_w$ such that $Y \subseteq -X] \Rightarrow Y \in \mathcal{N}_v$.

Suppose now that $w \in W$ and $w \Vdash \mathbf{N}\varphi$. Thus $w \Vdash \varphi$ and $V(\neg\varphi) \in \mathcal{N}_w$. Of course $v \Vdash \varphi$ and $V(\neg\varphi) \subseteq -V(\varphi)$. Hence, $V(\neg\varphi) \in \mathcal{N}_v$. Now $v \Vdash \mathbf{N}\varphi$.

Monotonicity and completeness. One can easily prove the following fact:

Theorem 2.7. *In every* $\Box$**pn** *(resp.* $\Diamond$**pn**, $\blacksquare$**pn**, $\bullet$**pn***)-model* $M = \langle W, \mathcal{N}, \leq, V\rangle$ *the following holds: if* $w \Vdash \gamma$ *and* $w \leq v$*, then* $v \Vdash \gamma$*.*

Proof. It is easy to see that the conditions imposed on models are coherent with monotonicity. Let us show the idea on the example of $\bullet$. Assume that we have $\bullet$**pn**-model, $w \in W$, $w \leq v$. Let $\gamma = \bullet\varphi$ and $w \Vdash \gamma$. Then $w \Vdash \varphi$ and $V(\varphi) \notin \mathcal{N}_w$. Then $v \Vdash \varphi$, i.e. $v \in V(\varphi) \notin \mathcal{N}_w$. From the monotonicity condition, $V(\varphi) \notin \mathcal{N}_v$. Thus $v \Vdash \bullet\varphi$. $\Box$

Now we shall present sound and complete axiomatization of our logics. What we are really doing is to present four different types of semantics for one propositional system with axiom `T`. In our future work we would like to combine all the operators in one system.

Definition 2.8. The $\Box$**Log** (resp. $\Diamond$**Log**, $\blacksquare$**Log**, $\bullet$**Log**)-logic is defined as the smallest set of formulas containing **IPC** and closed under the following set of rules: $\{\mathtt{T}x, \mathtt{MP}, \mathtt{RE}x\}$ where:

(1) **IPC** is the set of all intuitionistic axiom schemes and their modal instances.

[1] Strictly speaking, to make our definition inductive we should write $\{v \in W;$ for any $u \geq v, u \nVdash \varphi\}$ instead of $V(\neg\varphi)$.

(2) $\mathtt{T}_x$ is the axiom scheme $x\varphi \to \varphi$, where:

i) $x = \Box$, if logic is $\Box$**Log**.
ii) $x = \Diamond$, if logic is $\Diamond$**Log**.
iii) $x = \blacksquare$, if logic is $\blacksquare$**Log**.
iv) $x = \bullet$, if logic is $\bullet$**Log**.

(3) $\mathtt{MP}$ is *modus ponens*: $\varphi, \varphi \to \psi \Rightarrow \psi$.
(4) $\mathtt{RE}_x$ is *rule of extensionality*: $\varphi \leftrightarrow \psi \Rightarrow x\varphi \leftrightarrow x\psi$, where x is defined as in the case of $\mathtt{T}_x$.

The following theorem holds (we leave the proof to the reader):

Theorem 2.9. $\Box$**Log** *(resp.* $\Diamond$**Log***,* $\blacksquare$**Log***,* $\bullet$**Log***) is sound with respect to the class of all* $\Box$**pn** *(resp.* $\Diamond$**pn***,* $\blacksquare$**pn***,* $\bullet$**pn***)-frames.*

As for the canonical model, we define it simultaneously for each system.

Definition 2.10. $\Box$**can-pn**(resp. $\Diamond$**can-pn**, $\blacksquare$**can-pn**, $\bullet$**can-pn**)-model is a triple $\langle W, \leq, \mathcal{N}, \leq, V\rangle$ where:
(1) W is the set of all $\Box$**Log** (resp. $\Diamond$**Log**, $\blacksquare$**Log**, $\bullet$**Log**) prime theories[2].
(2) For every $w, v \in W$ we say that $w \leq v$ *iff* $w \subseteq v$.
(3) $\widehat{\varphi} = \{z \in W; \varphi \in z\}$.
(4) $\mathcal{N}$ is a function from W into $P(P(W))$ such that for every $w \in W$ and for each formula φ:
i) $\mathcal{N}_w = \{\widehat{\varphi}; \Box\varphi \in w\}$ (in $\Box$**can-pn**-model).
ii) $\mathcal{N}_w = \{X \subseteq W; X = W \setminus \widehat{\varphi}; \Diamond\varphi \notin w\}$ (in $\Diamond$**can-pn**-model).
iii) $\mathcal{N}_w = \{X \subseteq W; X = W \setminus \widehat{\varphi}; \blacksquare\varphi \in w\}$ (in $\blacksquare$**can-pn**-model).
iv) $\mathcal{N}_w = \{\widehat{\varphi}; \bullet\varphi \notin w\}$ (in $\bullet$**can-pn**-model).
(5) $V : PV \to P(W)$ is a function defined as follows: $w \in V(q) \Leftrightarrow q \in w$.

For convenience, let us agree on the following denotations:

CAN $= \{\Box$**can-pn**$, \Diamond$**can-pn**$, \blacksquare$**can-pn**$, \bullet$**can-pn**$\}$,

LOG $= \{\Box$**Log**$, \Diamond$**Log**$, \blacksquare$**Log**$, \bullet$**Log**$\}$.

We must check if our (canonical) neighbourhood functions are *well-defined.*

Lemma 2.11. *For each* $M \in$ **CAN** *and for each prime theory* w *based on the appropriate logic from* **LOG** *we have the following:*

- *In* $\Box$**can-pn***-model: if* $\widehat{\varphi} \in \mathcal{N}_w$ *and* $\widehat{\varphi} = \widehat{\psi}$*, then* $\Box\psi \in w$.
- *In* $\Diamond$**can-pn***-model: if* $W \setminus \widehat{\varphi} \in \mathcal{N}_w$ *and* $\widehat{\varphi} = \widehat{\psi}$*, then* $\Diamond\psi \notin w$.
- *In* $\blacksquare$**can-pn***-model: if* $\widehat{\varphi} \in \mathcal{N}_w$ *and* $\widehat{\varphi} = \widehat{\psi}$*, then* $\blacksquare\psi \notin w$.
- *In* $\bullet$**can-pn***-model: if* $W \setminus \widehat{\varphi} \in \mathcal{N}_w$ *and* $\widehat{\varphi} = \widehat{\psi}$*, then* $\bullet\psi \notin w$.

Proof. We shall not deal with all cases. Take, for example, $\blacksquare$**can-pn**-model. If $\widehat{\varphi} \in \mathcal{N}_w$, then $\blacksquare\varphi \notin w$. As we already know, $\varphi \leftrightarrow \psi \in \blacksquare$**Log**. Hence, $\blacksquare\varphi \leftrightarrow \blacksquare\psi \in \blacksquare$**Log** $\subseteq w$. Suppose that $\blacksquare\psi \in w$. Then, by means of $\mathtt{MP}$ and the fact that $\blacksquare\psi \to \blacksquare\varphi$ is a theorem, we have $\blacksquare\varphi \in w$. Contradiction. Hence, $\blacksquare\psi \notin w$. $\Box$

The next lemma deals with monotonicity.

[2] Clearly, each theory can be extended to the prime one.

Lemma 2.12. *Canonical models from* **CAN** *satisfy conditions (resp.)* **i**) - **iv**) *from Def. 2.5.*

Proof. We shall expose only one case here. Let M be a $\bullet$**can-pn**-model, $w \subseteq v \in X$ and $X \notin \mathcal{N}_w$. For any φ we have: $X \neq \widehat{\varphi}$ or $\bullet\varphi \in w$. In the first case, we are done, so we focus on the second case. As we know, $w \subseteq v$, hence $\bullet\varphi \in v$. Thus, X cannot be in $\mathcal{N}_v$. □

Finally, we obtain the crucial lemma:

Lemma 2.13. *In any model from* **CAN** *we have (for each γ and for each $w \in W$): $w \Vdash \gamma \Leftrightarrow \gamma \in w$.*

Proof. Consider $\bullet$**can-pn**-model, for example. Let $\gamma = \bullet\varphi$ and $w \in W$. Let $w \Vdash \gamma$, i.e. $w \Vdash \varphi$ and $V(\varphi) \notin \mathcal{N}_w$. By induction hypothesis, $\varphi \in w$ and $\widehat{\varphi} \notin \mathcal{N}_w$. By the definition of canonical neighbourhoods, $\bullet\varphi \in w$.

Now suppose that $\bullet\varphi \in w$. Then $\varphi \in w$. By induction, $w \Vdash \varphi$. Moreover, $\widehat{\varphi} \notin \mathcal{N}_w$. By induction, $V(\varphi) \notin \mathcal{N}_w$. □

Now we obtain our expected conclusion (the proof is standard):

Theorem 2.14. *Each system from* **LOG** *is strongly complete with respect to the appropriate class of models from* **CAN**.

Remark 2.15. Note that in Lemma 2.12 we did not use the fact that $v \in X$. In each case canonical model satisfies more restricted condition of monotonicity. For example, in case of $\Box$**can-pn**-model it would be: $[w \leq v, X \in \mathcal{N}_w] \Rightarrow X \in \mathcal{N}_v$. Hence, our systems are complete with respect to the narrower classes.

3. Some final considerations. As for the $\bullet$, it is typical for the logics of unknown truths, investigated by Gilbert and Venturi in [3] and by Fan in [4]. However, their systems were classical (not intuitionistic). Moreover, these authors analysed another operator, namely $\circ\varphi$, defined as $\neg\bullet\varphi$. In the classical framework $w \Vdash \circ\varphi \Leftrightarrow w \nVdash \varphi$ or $V(\varphi) \in \mathcal{N}_w$. Analogously, we may define $\bullet\varphi$ as $\neg \circ \varphi$.

In the intuitionistic setting things are more complicated. First, if w is a possible world in an arbitrary $\bullet$**pn**-frame, then:

$w \Vdash \neg \circ \varphi \Leftrightarrow$ for any $v \geq w$, $v \nVdash \circ\varphi \Leftrightarrow$ for any $v \geq w$, $v \Vdash \varphi$ and $V(\varphi) \notin \mathcal{N}_w$ $\Leftrightarrow w \Vdash \bullet\varphi$.

As we can see, there is no problem with $\bullet\varphi \leftrightarrow \neg \circ \varphi$. Note that in the right-to-left part of the last equivalence we used the fact that forcing of $\bullet$ is monotonic. Now consider the second alleged tautology:

$w \Vdash \neg \bullet \varphi \Leftrightarrow$ for any $v \geq w$, $v \nVdash \bullet\varphi \Leftrightarrow$ for any $v \geq w$, $v \nVdash \varphi$ or $V(\varphi) \in \mathcal{N}_w$ $\Rightarrow w \Vdash \circ\varphi$.

The penultimate statement implies that $w \Vdash \circ\varphi$ (because $w \geq w$). But the converse need not to be true. This is because monotonicity of forcing fails for $\circ$ in $\bullet$**pn**-frames. Note that we assumed that if $w \Vdash \circ\varphi$, then $w \nVdash \varphi$. This does not mean, in general, that φ is discarded in each world placed above w. Besides, if $V(\varphi) \in \mathcal{N}_w$, then in $\bullet$**pn**-frames we do not have any guarantee that $V(\varphi) \in \mathcal{N}_v$ for every $v \geq w$.

Of course we can strenghten our assumptions. Hence, let us define: $w \Vdash \circ\varphi$

$\Leftrightarrow$ for any $v \geq w$, $v \nVdash \varphi$ or $V(\varphi) \in \mathcal{N}_w$. Moreover, consider these $\bullet$**pn**-frames which satisfy additional condition: if $X \in \mathcal{N}_w$ and $v \geq w$, then $X \in \mathcal{N}_v$[3].

In this framework forcing of $\circ\varphi$ becomes monotonic and $\neg \bullet \varphi \leftrightarrow \circ\varphi$ is a tautology. However, the duality between $\bullet\varphi$ and $\neg \circ \varphi$ fails:

$w \Vdash \neg \circ \varphi \Leftrightarrow$ for any $v \geq w$, $v \nVdash \circ\varphi \Leftrightarrow$ for any $v \geq w$ there is $u \geq v$ such that $u \Vdash \varphi$ and $V(\varphi) \notin \mathcal{N}_w$.

The last statement does not imply that $w \Vdash \bullet\varphi$. One can easily find a countermodel in which $w \nVdash \varphi$ but there is some u above w that supports the formula in question. On the other hand, if $\circ$ is interpreted in the new way, then the fact that $w \Vdash \bullet\varphi$ implies that $w \Vdash \neg \circ \varphi$.

All these issues should be studied in the context of intuitionism. Our initial research suggests that this may be interesting. For example, consider system $\mathbf{B}_k$, investigated in [3]. It contains axioms $\circ\top$, $\bullet\varphi \to \varphi$ and $(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$. It is closed under MP and the rule $\varphi \to \psi \Rightarrow (\circ\varphi \wedge \varphi) \to (\circ\psi \wedge \psi)$. In the proof of completeness of this system (with respect to the class of filters) Gilbert and Venturi used the fact that if $\varphi \notin w$ (where w is a maximal theory), then $\neg\varphi \in w$ and hence $\circ\varphi \in w$ (because from the axiom $\bullet\varphi \to \varphi$ we may infer that $\neg\varphi \to \neg \bullet \varphi \in w$, i.e. $\neg\varphi \to \circ\varphi \in w$). However, in intuitionistic prime theories we cannot say that if $\varphi \notin w$, then $\neg\varphi \in w$.

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[3]Note that Cond. 1 from Def. 2.5 would be too weak to enforce monotonicity of $\circ$.