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MAXIMUM MODULUS OF THE DERIVATIVES OF A POLYNOMIAL WITH ZEROS OUTSIDE A CIRCLE

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Abstract

In this paper, we present higher-order derivative extensions of a result established by Aziz and Mohammad [Proc. Amer. Math. Soc., 81 (1981), 549–553], thereby encompassing several known results as particular cases. In addition, a detailed numerical example along with graphical analysis is provided to demonstrate the accuracy and sharpness of the proposed results.

Key words: polynomial, zero, maximum modulus, s-th derivative, Gauss–Lucas theorem

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1. Introduction. Polynomial approximation is a fundamental concept in mathematics and applied sciences, offering a means to represent complex functions through simpler polynomial forms. This approach facilitates efficient analysis and computation, making it essential in areas such as numerical analysis, signal processing, computer-aided design, physics, and engineering.

A variety of methods have been developed to address such approximation problems, each tailored to specific settings and objectives. Among these, a particularly influential approach involves Bernstein's inequality, especially its trigonometric form, which is central to establishing inverse theorems in approximation theory (see Borwein and Erdélyi [1, p. 241]). This result provides estimates on how well a polynomial of fixed degree approximates a continuous function, in

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terms of its derivatives and Lipschitz constants. This line of work culminates in the classical Bernstein inequality [2], which asserts that if t is a real trigonometric polynomial of degree n, then

$$\max_{0\le\theta<2\pi} |t^{(m)}(\theta)| \le n^m \max_{0\le\theta<2\pi} |t(\theta)|.$$

This inequality also holds for all complex trigonometric polynomials t of degree n, which in turn yields, as a special case, the following algebraic polynomial form of Bernstein's inequality on the unit disk.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{1.1}$$

Equality holds in (1.1) *if and only if $p(z)$ has all its zeros at the origin.*

If we restrict to the class of polynomials having no zero in $|z| < 1$, then inequality (1.1) can be improved as

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

Inequality (1.2) was conjectured by Erdős and later verified by Lax [3].

Here, we use $\mathcal{P}_{n,k}$ to denote the class of polynomials $p(z)$ of degree n having no zero in $|z| < k$, $k \ge 1$. As a partial generalization of (1.2), Malik [4] proved that if $p \in \mathcal{P}_{n,k}$, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Notably, Govil and Rahman [5, Theorem 4] extended inequality (1.3) to the s-th derivative of $p(z)$, and proved under the same hypothesis with $1 \le s < n$ that

$$\max_{|z|=1} |p^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \max_{|z|=1} |p(z)|. \tag{1.4}$$

The notion that inequality (1.4) holds true for $0 \le s < n$ was proposed by Jain in [6, Lemma 5].

The refinement and broadening of inequalities related to complex polynomials have been the focus of substantial research. For a more comprehensive exploration of these inequalities and their diverse applications, valuable perspectives can be found in the monographs by Milovanović et al. [7], Marden [8], and Rahman and Schmeisser [9]. Additionally, Gardner et al. [10, Chap. 6] offer a modern treatment, presenting techniques grounded in geometric function theory to derive various polynomial inequalities.

The investigation of inequalities describing the relative growth of a polynomial $p(z)$ on two circles $|z| = 1$ and $|z| = R \geq 1$, particularly those relating the sup-norm of $|p(z)|$ on the unit circle to that of $|p(Rz)|$, is a classical subject in complex analysis. Over time, these inequalities have been extended to various domains and examined under different norm settings. By a simple deduction from the maximum modulus principle, the following result concerning the growth of a polynomial of degree n was obtained by Pólya and Szegő [11] for $R \geq 1$ as

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.5}$$

Ankeny and Rivlin [12] examined $p \in \mathcal{P}_{n,1}$, and achieved a refinement of (1.5) by establishing that for $R \geq 1$,

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|. \tag{1.6}$$

Aziz and Mohammad [13] generalized inequality (1.6) by proving that if $p \in \mathcal{P}_{n,k}$, then for $R \geq 1$,

$$\max_{|z|=R} |p(z)| \leq \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n} \max_{|z|=1} |p(z)|. \tag{1.7}$$

The study of maximum modulus inequalities for polynomials has been widely explored (see [14–17]). However, the progress in the case of higher-order derivatives has been quite limited. To address this gap, we develop generalized versions of inequality (1.7) applicable to higher-order derivatives.

2. Main results. In this paper, we extend inequality (1.7) to higher derivatives of the polynomial $p(z)$. More precisely, we prove the following theorem.

Theorem 2.1. *If $p \in \mathcal{P}_{n,k}$, then for $R \geq 1$ and $0 \leq s < n$,*

$$\max_{|z|=R} |p^{(s)}(z)| \leq \left\{ \frac{d^s}{dR^s}(R^n + 1) \right\} \frac{(R + k)^n}{(R + k)^n + (1 + Rk)^n} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Remark 2.2. Taking $k = 1$ in Theorem 2.1, we get a result due to Singha and Chanam [18, Corollary 2.6], which gives the s-th derivative generalization of inequality (1.6) due to Ankeny and Rivlin [12].

Moreover, we extend inequality (2.1) by incorporating the minimum modulus of the polynomial on $|z| = k$, which serves as an enhanced generalization of inequality (1.7) by Aziz and Mohammad [13].

Theorem 2.3. *If $p \in \mathcal{P}_{n,k}$, then for some fixed complex number γ with $|\gamma| < 1$, for $R \geq 1$, and $0 \leq s < n$,*

$$\begin{aligned} \max_{|z|=R} \left| \frac{d^s}{dz^s}\{p(z) - \gamma m\} \right| \\ \leq \left\{ \frac{d^s}{dR^s}(R^n + 1) \right\} \frac{(R + k)^n}{(R + k)^n + (1 + Rk)^n} \left(\max_{|z|=1} |p(z)| - |\gamma| m \right), \end{aligned} \tag{2.2}$$

where $m = \min_{|z|=k} |p(z)|$.

Remark 2.4. Further, taking $k = 1$ in Theorem 2.3, we get the result due to Singha and Chanam [18, Corollary 2.14], which gives an enhanced version for the s-th derivative of inequality (1.6) due to Ankeny and Rivlin [12].

Remark 2.5. Putting $s = 0$ in inequality (2.2) of Theorem 2.3, we get

$$\max_{|z|=R} |p(z) - \gamma m| \leq \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n} \left(\max_{|z|=1} |p(z)| - |\gamma| m \right). \tag{2.3}$$

Suppose z_0 on $|z| = R$ be such that $\max_{|z|=R} |p(z)| = |p(z_0)|$, then

$$\max_{|z|=R} |p(z) - \gamma m| \geq |p(z_0) - \gamma m| \geq |p(z_0)| - |\gamma| m. \tag{2.4}$$

Combining (2.4) and (2.3), we have

$$\max_{|z|=R} |p(z)| - |\gamma| m \leq \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n} \left(\max_{|z|=1} |p(z)| - |\gamma| m \right),$$

and upon simplification, this leads to the following intriguing result, which serves as an improvement of inequality (1.7) by Aziz and Mohammad [13].

Corollary 2.6. *If $p \in \mathcal{P}_{n,k}$, then for some fixed complex number γ with $|\gamma| < 1$, and $R \geq 1$,*

(2.5)

$$\max_{|z|=R} |p(z)| \leq \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n} \max_{|z|=1} |p(z)| - \frac{R^n (R + k)^n - (1 + Rk)^n}{(R + k)^n + (1 + Rk)^n} |\gamma| m,$$

where $m = \min_{|z|=k} |p(z)|$.

Remark 2.7. Since $R \geq 1$, we have

$$\frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n} \geq 1,$$

which is equivalent to

$$R^n (R + k)^n - (1 + Rk)^n \geq 0.$$

In view of this fact, it follows that inequality (2.5) improves inequality (1.7).

Remark 2.8. Further, taking $k = 1$ in Corollary 2.6, and letting $|\gamma| \to 1$, we get a result due to Aziz and Dawood [19, Theorem 3], which gives the improvement of inequality (1.6) due to Ankeny and Rivlin [12].

Example 2.9. Consider the polynomial $p(z) = z^4 + 3^4$ and $|\gamma| = 1$ (for γ with $|\gamma| = 1$, the above results follow by continuity). Then $p(z)$ is a polynomial of degree 4 having all its zeros on $|z| = 3$, and we have $\max_{|z|=1} |p(z)| = 82$ and $\min_{|z|=2} |p(z)| = 65$. Then by inequality (1.7) of Aziz and Mohammad [13], we have $\max_{|z|=3} |p(z)| \leq 1388.797$, while by inequality (2.5) of Corollary 2.6, $\max_{|z|=3} |p(z)| \leq 352.921$, resulting in an improvement of 74.59% over the bound obtained from (1.7).

3. Lemmas. For the proofs of the theorems, we require the following lemmas. The first lemma is due to Jain [6].

Lemma 3.1. *If $p(z)$ is a polynomial of degree at most n, and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for $0 \leq s < n$,*

$$|p^{(s)}(z)| + |q^{(s)}(z)| \leq \left\{ \left| \frac{d^s}{dz^s}(1) \right| + \left| \frac{d^s}{dz^s}(z^n) \right| \right\} \max_{|z|=1} |p(z)| \quad \textit{for} \quad |z| \geq 1.$$

Next, we state and prove the following lemma of which, the logic behind its proof is essentially contained in the proof of Theorem 1 of Aziz and Mohammad [13]. However, to ensure completeness, we provide a brief outline of the proof here.

Lemma 3.2. *If $p \in \mathcal{P}_{n,k}$, and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for $R \geq 1$, $0 \leq s < n$, and for all z on $|z| = 1$,*

$$|p^{(s)}(Rz)| \leq \left(\frac{R+k}{1+Rk} \right)^n |q^{(s)}(Rz)|.$$

Proof. Since $p(z)$ has all its zeros in $|z| \geq k \geq 1$, we write

$$p(z) = \prod_{j=1}^{n} (z - r_j e^{i\theta_j}), \quad \text{where} \quad r_j \geq k, \ j = 1, 2, \ldots, n.$$

Then $q(z) = z^n\overline{p(\frac{1}{\bar{z}})} = \prod_{j=1}^{n} (1 - z r_j e^{-i\theta_j})$ has all its zeros in $|z| \leq \frac{1}{k}$, $\frac{1}{k} \leq 1$, so that clearly we have for $0 \leq \theta < 2\pi$ and for $R \geq 1$,

$$\begin{aligned} \left| \frac{p(Re^{i\theta})}{q(Re^{i\theta})} \right| &= \prod_{j=1}^{n} \left| \frac{Re^{i\theta} - r_j e^{i\theta_j}}{1 - Rr_j e^{i(\theta - \theta_j)}} \right| = \prod_{j=1}^{n} \left| \frac{Re^{i(\theta-\theta_j)} - r_j}{1 - Rr_j e^{i(\theta-\theta_j)}} \right| \\ &\leq \prod_{j=1}^{n} \left(\frac{R + r_j}{1 + Rr_j} \right) \leq \prod_{j=1}^{n} \left(\frac{R+k}{1+Rk} \right) = \left(\frac{R+k}{1+Rk} \right)^n. \end{aligned}$$

Thus for all z on $|z| = 1$, we have

$$|p(Rz)| \leq \left(\frac{R+k}{1+Rk} \right)^n |q(Rz)|. \tag{3.1}$$

From (3.1), we also have for any complex number λ with $|\lambda| > 1$,

$$|p(Rz)| < \left| \lambda \left(\frac{R+k}{1+Rk} \right)^n q(Rz) \right|.$$

Since $q(Rz)$ has all its zeros in $|z| \leq \frac{1}{Rk}$, $\frac{1}{Rk} \leq 1$, by Rouche's theorem, it follows that the polynomial

$$p(Rz) - \lambda \left(\frac{R+k}{1+Rk} \right)^n q(Rz)$$

has all its zeros in $|z| \leq \frac{1}{Rk}$, $\frac{1}{Rk} \leq 1$.

Gauss–Lucas theorem will then imply that the polynomial

$$R^s p^{(s)}(Rz) - \lambda \left(\frac{R+k}{1+Rk}\right)^n R^s q^{(s)}(Rz), \quad 1 \le s < n,$$

has all its zeros in $|z| \le \frac{1}{Rk}$, $\frac{1}{Rk} \le 1$.

Therefore for all z on $|z| = 1$,

$$|p^{(s)}(Rz)| \le \left(\frac{R+k}{1+Rk}\right)^n |q^{(s)}(Rz)|, \quad 1 \le s < n,$$

which, on being combined with (3.1), completes the proof of Lemma 3.2. □

4. Proofs of the main results. We begin by proving Theorem 2.3.

Proof of Theorem 2.3. By hypothesis, $p(z)$ has all its zeros in $|z| \ge k$, $k \ge 1$. In case $m = \min\limits_{|z|=k} |p(z)| \ne 0$, consider $P(z) = p(z) - \gamma m$, where γ is a complex number with $|\gamma| < 1$. Now on $|z| = k$,

$$|\gamma m| < m \le |p(z)|.$$

Then by Rouche's theorem, it follows that $P(z)$ has all its zeros in $|z| > k$, and in case $m = 0$, $P(z) = p(z)$. Thus, in any case, $P(z)$ has all its zeros in $|z| \ge k$.

Applying Lemma 3.2 to $P(z)$, we have for $0 \le \theta < 2\pi$, where $R \ge 1$ and $0 \le s < n$,

$$|P^{(s)}(Re^{i\theta})| \le \left(\frac{R+k}{1+Rk}\right)^n |Q^{(s)}(Re^{i\theta})|, \tag{4.1}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

By Lemma 3.1, we have for $R \ge 1$, $0 \le s < n$, and for all z on $|z| = 1$,

$$|P^{(s)}(Rz)| + |Q^{(s)}(Rz)| \le \left\{\frac{d^s}{dR^s}(R^n + 1)\right\} \max_{|z|=1} |P(z)|. \tag{4.2}$$

Using (4.1) in (4.2), we get

$$\left\{1 + \left(\frac{1+Rk}{R+k}\right)^n\right\} |P^{(s)}(Rz)| \le \left\{\frac{d^s}{dR^s}(R^n + 1)\right\} \max_{|z|=1} |P(z)|,$$

i.e.

$$\max_{|z|=R} \left|\frac{d^s}{dz^s}\{p(z) - \gamma m\}\right| \le \left\{\frac{d^s}{dR^s}(R^n + 1)\right\} \frac{(R+k)^n}{(R+k)^n + (1+Rk)^n} \max_{|z|=1} |p(z) - \gamma m|. \tag{4.3}$$

Now, let z_1 on $|z| = 1$ be such that

$$\max_{|z|=1} |p(z) - \gamma m| = |p(z_1) - \gamma m|. \tag{4.4}$$

For an appropriate choice of the argument of γ, we have

$$|p(z_1) - \gamma m| = |p(z_1)| - |\gamma| m. \tag{4.5}$$

From (4.4) and (4.5), and using the fact that $\max_{|z|=1} |p(z)| \geq |p(z_1)|$, we get

$$\max_{|z|=1} |p(z) - \gamma m| \leq \max_{|z|=1} |p(z)| - |\gamma| m. \tag{4.6}$$

Combining (4.6) and (4.3), we get

$$\max_{|z|=R} \left| \frac{d^s}{dz^s} \{p(z) - \gamma m\} \right| \leq \left\{ \frac{d^s}{dR^s}(R^n + 1) \right\} \frac{(R+k)^n}{(R+k)^n + (1+Rk)^n} \left(\max_{|z|=1} |p(z)| - |\gamma| m \right).$$

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.1. The proof of this theorem proceeds in a manner analogous to that of Theorem 2.3; however, instead of applying Lemmas 3.2 and 3.1 to $P(z)$ as given in (4.1) and (4.2), we apply them directly to $p(z)$. The details are therefore omitted. □

5. Numerical example and graphical representations. As an illustration of the theoretical results obtained, this section presents a specific example and compares the bounds derived from Theorems 2.1 and 2.3.

Example 5.1. Let $p(z) = z^4 + 3^4$ with all zeros $3e^{\frac{i\pi}{4}(1+2l)}$, $l = 0, 1, 2, 3$, on $|z| = 3$, so that Theorems 2.1 and 2.3 hold for $1 \leq k \leq 3$. Then on the circle $|z| = R$, we have

$$|p(Re^{i\theta})| = \sqrt{R^8 + 6561 + 162R^4 \cos 4\theta}.$$

Clearly, $M_R = \max_{|z|=R} |p(z)| = \max_{0 \leq \theta < 2\pi} |p(Re^{i\theta})| = R^4 + 3^4$, and $m = \min_{|z|=k} |p(z)| = p(k) = 3^4 - k^4$.

We intend to illustrate that the bounds of Theorem 2.3 improve most for $|\gamma| = 1$ by considering level graphs for $|\gamma|$, followed by comparison of the bounds of Theorem 2.1, and Theorem 2.3 for $|\gamma| = 1$.

We can consider the difference between the right and the left hand-sides in inequalities (2.1) of Theorem 2.1 and (2.2) of Theorem 2.3 as

$$\Delta(R,k) = \begin{cases} \dfrac{12R^2(R+k)^4}{(R+k)^4 + (1+Rk)^4} M_1 - M_R'', & \text{(inequality (2.1))}, \\ \dfrac{12R^4(R+k)^4}{(R+k)^4 + (1+Rk)^4} (M_1 - |\gamma| m) - M_R'', & \text{(inequality (2.2))}. \end{cases}$$

Graphics of the function $(R,k) \mapsto \Delta(R,k)$ for inequality (2.2) for $|\gamma| = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ are presented in Fig. 1 (left), indicated by blue, grey, orange, and green colours, respectively. The graphs of the function $(R,k) \mapsto \Delta(R,k)$ for the two inequalities are shown in the same figure (right) with the yellow and green regions corresponding to inequalities (2.1) from Theorem 2.1, and (2.2) from Theorem 2.3 for $|\gamma| = 1$, respectively. A smaller value of $\Delta(R,k)$ indicates a tighter bound. From the figure, it is evident that inequality (2.2) of Theorem 2.3 provides a stronger improvement compared to inequality (2.1) of Theorem 2.1, across the range of R and k values.

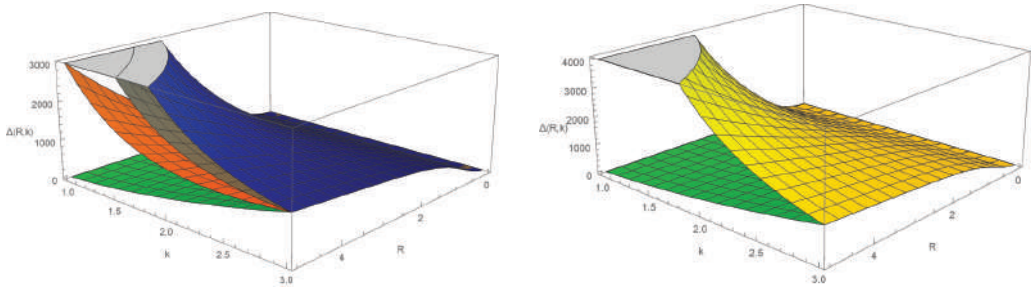


Fig. 1. (Left) The function $(R,k) \mapsto \Delta(R,k)$, where $0 < R \leq 5$ and $1 \leq k \leq 3$ for inequality (2.2) when $|\gamma| = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$; (right) comparison of the differences $(R,k) \mapsto \Delta(R,k)$ in the inequalities (2.1), and (2.2) for $|\gamma| = 1$

Inequality (2.1) from Theorem 2.1 for $R = 3$ and $k = 2$ yields the bound $\max_{|z|=3} |p''(z)| \leq 1829.147$. In contrast, inequality (2.2) from Theorem 2.3 provides a significantly tighter bound of $\max_{|z|=3} |p''(z)| \leq 379.213$. This represents an improvement of approximately 79.27% compared to the bound from (2.1).

6. Conclusion This work aims to extend classical polynomial inequalities, especially those of the Ankeny and Rivlin-type, to the setting of higher-order derivatives. Employing a novel approach, the paper derives these extensions and includes a comparative numerical example to estimate the obtained bounds, thereby illustrating the effectiveness of the proposed results.

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