

ON THE THEOREMS OF BIERNACKI

Radoš Bakić 

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Abstract

Let $p(z)$ be a complex polynomial of degree n . If we know the location of its zeros, then we can locate the zeros of its m -th integral. This result has been obtained by Biernacki, and his bound depends only on m , and not on n . We found an alternative bound that depends on both n and m , and is better than the previous one in infinitely many cases.

Key words: zeros of polynomial, composite polynomials

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Let $p(z)$ be a complex polynomial of degree n . For such a polynomial we define polynomial $F_m(z)$ in the following manner:

$$F_0(z) = p(z)$$
$$F_m(z) = \int_0^z F_{m-1}(t)dt.$$

It is a natural problem to locate the zeros of $F_m(z)$, once we know the location of the zeros of $p(z)$. The answer to this question is given in the well-known theorem due to Biernacki:

Theorem 1 (Biernacki). *If the zeros of $p(z)$ lie in the disc $|z| \leq 1$, then the zeros of $F_m(z)$ lie in the disc $|z| \leq m + 1$.*

The above estimation is sharp, as shown by the example $f(z) = 1 + z$.

As we see, the above estimation depends only on m , and not on n . The aim of this paper is to show that the above estimation can be significantly improved when n and m are sufficiently large.

BIERNACIKI derived his bound in [1] using the classical composition theorem due to Szegő [2]. He proved that the zeros of $F_m(z)$ are contained in the central disc that contains the zeros of the polynomial

$$q(z) = \binom{n+m}{m} + \binom{n+m}{m+1}z + \dots + \binom{n+m}{n+m}z^n.$$

After that, using Rouché's theorem, he proved that the zeros of $q(z)$ are contained in the region $|z| \leq m+1$, and therefore obtained his theorem.

We shall use a different approach in order to locate the zeros of $q(z)$. Since the case $m = 1$ is analysed in [1], we shall assume that $m > 1$.

Let w be a zero of $q(z)$ of the maximum modulus. Then by Vieta's rule it follows that

$$(1) \quad |w| \geq \sqrt[n]{\binom{n+m}{m}}.$$

Also $q(w) = 0$ implies that

$$0 = q(w) = w^m q(w) = (w+1)^{(n+m)} - \sum_{k=0}^{m-1} \binom{n+m}{k} w^k \quad \text{i.e.}$$

$$(w+1)^{n+m} = \sum_{k=0}^{m-1} \binom{n+m}{k} w^k.$$

From the above inequality we can obtain that

$$(2) \quad (|w| - 1)^{n+m} \leq \sum_{k=0}^{m-1} \binom{n+m}{k} |w|^k.$$

Now we are going to use the following elementary inequality that can be found in [1]

$$\sum_{k=0}^m \binom{n}{k} x^k \leq \binom{n}{m} (x+1)^m, \quad x \geq 0.$$

If we apply this inequality (with $x = |w|$) to (2), we have that

$$(|w| - 1)^{n+m} \leq \sum_{k=0}^{m-1} \binom{n+m}{k} |w|^k \leq \binom{n+m}{m-1} (|w| + 1)^{m-1}.$$

Now from the above inequality and (1) (for $|w| \neq 1$) we obtain that

$$\begin{aligned}
 (3) \quad (|w| - 1)^{n+1} &\leq \binom{n+m}{m-1} \left(\frac{|w|+1}{|w|-1} \right)^{m-1} \\
 &= \binom{n+m}{m-1} \left(1 + \frac{2}{|w|-1} \right)^{m-1} \\
 &\leq \binom{n+m}{m-1} \left(1 + \frac{2}{\sqrt[n]{\binom{n+m}{m}} - 1} \right)^{m-1} \quad (\text{using (1)}) \\
 &< \binom{n+m}{m-1} e^{\frac{2(m-1)}{\sqrt[n]{\binom{n+m}{m}} - 1}},
 \end{aligned}$$

where we used the inequality

$$(1 + x/k)^k < e^x, \quad k, x > 0.$$

Finally we obtain

$$(4) \quad |w| < 1 + \sqrt[n+1]{\binom{n+m}{n+1}} e^{\frac{2(m-1)}{(n+1)(\sqrt[n]{\binom{n+m}{n}} - 1)}}.$$

Suppose now that m is fixed and $n \rightarrow \infty$. In that case we will have that both terms $\sqrt[n+1]{\binom{n+m}{n+1}} = \sqrt[n+1]{\binom{n+m}{m-1}}$ and $\sqrt[n]{\binom{n+m}{n}} = \sqrt[n]{\binom{n+m}{m}}$ have 1 for their limit, since the involved binomial coefficients are polynomials in n of degrees $m-1$ and m , respectively. Also

$$\lim_{n \rightarrow \infty} (n+1) \left(\sqrt[n]{\binom{n+m}{n}} - 1 \right) = \infty,$$

which can be obtained by L'Hospital's rule applied on the quotient of the functions

$$f(x) = \left(\frac{m+x}{m} \right)^{\frac{1}{x}} - 1 \quad \text{and} \quad g(x) = \frac{1}{x+1}.$$

Hence we see that the limit of the right-hand side of (4) when $n \rightarrow \infty$ is equal to 2, and therefore does not depend on m .

It implies that for a fixed m and sufficiently large n , gives us better bound than $m+1$ given by Biernacki.

Now we want to further simplify the obtained bound (4). If we use the inequalities

$$\sqrt[n]{\binom{n+m}{n}} > \frac{n+m}{n} = 1 + \frac{m}{n},$$

and

$${}^{n+1}\sqrt{\binom{n+m}{n+1}} \leq \frac{(n+m)e}{n+1},$$

then (4) can be transformed in the following manner

$$\begin{aligned} |w| &< 1 + {}^{n+1}\sqrt{\binom{n+m}{n+1}} e^{\frac{2(m-1)}{(n+1)\left(\sqrt[n]{\binom{n+m}{n}}-1\right)}} \\ &< 1 + \frac{(n+m)e}{n+1} e^{\frac{2(m-1)n}{m(n+1)}} \\ &< 1 + \frac{(n+m)e^3}{n+1}. \end{aligned}$$

From the above inequality we see that when both n and m are sufficiently large, then (4) is also a better bound than $m+1$. We shall formulate our result as the following theorem.

Theorem 2. *Let $p(z)$ be a complex polynomial of degree n , and let us define a polynomial $F_m(z)$ in the following manner:*

$$F_0(z) = p(z), \quad F_m(z) = \int_0^z F_{m-1}(t) dt.$$

If the zeros of $p(z)$ lie in the disc $|z| \leq 1$, then the zeros of $F_m(z)$ lie in the disc

$$|z| < 1 + {}^{n+1}\sqrt{\binom{n+m}{n+1}} e^{\frac{2(m-1)}{(n+1)\left(\sqrt[n]{\binom{n+m}{n}}-1\right)}} < 1 + \frac{(n+m)e^3}{n+1}.$$

In [1] Biernacki also proved the following theorem which is related to the above result:

Theorem 3. *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_{n-m} z^{n-m} + \dots + a_0$ be a complex polynomial of degree n . Suppose that all its zeros lie in the region $|z| \geq r$. Then all zeros of the polynomial $p_1(z) = a_{n-m} z^{n-m} + a_{n-m-1} z^{n-m-1} + \dots + a_0$ lie in the region $|z| \geq r/|z_0|$, where z_0 is a zero of the largest modulus of the polynomial*

$$q_1(z) = \binom{n}{m} + \binom{n}{m+1} z + \dots + \binom{n}{n} z^{n-m}.$$

A generalization of Theorem 3 can be found in [3].

The above polynomial $q_1(z)$ can be obtained from the already introduced polynomial $q(z)$ by replacing n by $n-m$. Since Biernacki proved that the zeros of $q(z)$ lie in the region $|z| \leq m+1$, he stated that the zeros of $p_1(z)$ lie in the region $|z| \geq r/(m+1)$. As we proved, (4) gives us a better estimation for the zeros of $q(z)$, and therefore for $q_1(z)$ also. That means that by using (4) with substitution $n \rightarrow n-m$, we can improve the result of Theorem 3 given by Biernacki:

Theorem 4. *With notation as in Theorem 3, the zeros of $p_1(z)$ lie in the region $|z| \geq r/f(m, n)$, where*

$$f(m, n) = 1 + \sqrt[n-m+1]{\binom{n}{m-1}} e^{\frac{2(m-1)}{(n-m+1)\left(\sqrt[n]{\binom{n}{m}}-1\right)}}.$$

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*Teacher Education Faculty, University of Belgrade, Kraljice Natalije 43,
11000 Belgrade, Serbia*
e-mail: bakicr@gmail.com, <https://orcid.org/0000-0002-5280-011X>