SOME ASPECTS OF DISTANCING

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Abstract

For any metric space, two binary relations are considered: the one consisting of all ordered pairs of a positive real number and a natural number such that some finite subset of the space has cardinality equal to the second number and the distance between any two distinct elements of this subset is not less than the first number, and another defined similarly, but with “greater” instead of “not less”. If the space is totally bounded, then a partial function from natural numbers to positive reals and two functions in the inverse direction can be reasonably defined by using the relations in question. The study of these functions has been initiated by other authors, and it is continued in the paper.

Key words: metric space, ε -separated, bounded, totally bounded, clique number, monotonic function, discontinuity point, left continuous, right continuous, semialgebraic, ordered field, quantifier elimination, computable, algebraic number

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1. Distancing in a metric space. The term distancing became rather familiar due to the coronavirus pandemic. If understood as physical distancing, this term has a geometrical sense which can be an object of mathematical investigation (not necessarily in connection with current practical problems). A general approach going back at least to the fifties of the 20th century makes use of metric spaces.

Let $\mathcal{A} = (A, d)$ be a metric space with $|A| \geq 2$.

Definition 1. For $S \subseteq A$ and $\varepsilon > 0$ we will say that:

S is ε -separated in $\mathcal{A}$ if $d(x, y) \geq \varepsilon$ whenever $x, y \in S$ and $x \neq y$;

S is *strictly* ε -separated in $\mathcal{A}$ if $d(x, y) > \varepsilon$ whenever $x, y \in S$ and $x \neq y$.

In [1, 2], the strict ε -separatedness (with no term for it in [1] and called ε -distinguishability in [2]) is considered in the case of a totally bounded metric space. As to non-strict ε -separatedness, it is considered for instance in [3], pp. 396–397, [4], p. 218, and in [5], Subsection 1.2 (the term “ ε -separated” does not occur in [5], but the inequality $\text{Sep}(S) \geq \varepsilon$ used in one of the definitions actually characterizes the ε -separated subsets S of the considered metric space).

Remark 1. The case of total boundedness of $\mathcal{A}$ is actually the one when, for any $\varepsilon > 0$, all ε -separated subsets of A are finite. For the number of their elements then there is an upper bound (generally depending on ε).

Remark 2. Suppose the carrier A of $\mathcal{A}$ is finite. Given some positive number ε , let us consider the following two graphs whose set of vertices is A : the graph $\Gamma_{\mathcal{A}, \varepsilon}$ with edges connecting x and y iff $d(x, y) \geq \varepsilon$, and the graph $\Gamma_{\mathcal{A}, \varepsilon}^+$ with edges connecting x and y iff $d(x, y) > \varepsilon$. The sets ε -separated in $\mathcal{A}$ are exactly the cliques of $\Gamma_{\mathcal{A}, \varepsilon}$, and the strictly ε -separated ones are exactly the cliques of $\Gamma_{\mathcal{A}, \varepsilon}^+$. For any graph G with at least two vertices and without edges connecting a vertex with itself, a metric space $\mathcal{A}$ exists such that $G = \Gamma_{\mathcal{A}, 2} = \Gamma_{\mathcal{A}, 1}^+$. Indeed, if the set of the vertices of G is A , then we may set $\mathcal{A} = (A, d)$, where $d(x, y) = 2$ if there is an edge of G connecting x and y , $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ otherwise.

Definition 2. We will denote by $\Delta_{\mathcal{A}}$ the set of all pairs $\langle \varepsilon, n \rangle \in (0, +\infty) \times \mathbb{N}$ such that an n -element subset of A ε -separated in $\mathcal{A}$ exists, and by $\Delta_{\mathcal{A}}^+$ – the set of those for which an n -element subset of A strictly ε -separated in $\mathcal{A}$ exists.

Example 1. Let A be a finite interval in $\mathbb{R}$ of length l and let $d(x, y) = |x - y|$ for $x, y \in A$. For $\varepsilon > 0$ and $n \in \mathbb{N}$, $\langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+ \Leftrightarrow (n - 1)\varepsilon < l$. If the interval A is closed, then $\Delta_{\mathcal{A}} = \{\langle \varepsilon, n \rangle \in (0, +\infty) \times \mathbb{N} \mid (n - 1)\varepsilon \leq l\}$ else $\Delta_{\mathcal{A}} = \Delta_{\mathcal{A}}^+$.

Example 2. Let $A = \{0, 1\}^m$, where $m \in \mathbb{N}^+$, and let d be the Hamming distance. Then $\langle 2, n \rangle \in \Delta_{\mathcal{A}} \Leftrightarrow n \leq 2^{m-1}$ for $n \in \mathbb{N}$. To prove the implication from right to left, we may consider the set of all elements of A with even number of 1’s in them – it is 2-separated in $\mathcal{A}$ and has 2^{m-1} elements.¹ The converse implication follows from the injectivity of the projection into $\{0, 1\}^{m-1}$ of any subset of A 2-separated in $\mathcal{A}$.

Example 3. As in the second example in [2], §2, let A be the set of the real functions defined in a closed interval $[a, b]$ that satisfy the Lipschitz condition with Lipschitz constant L and vanish at the point a , and let the distance d between two functions from A be the maximum of the modulus of their difference. By [2], (6), $\langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+ \Leftrightarrow n \leq 2^{\lceil 2(b-a)L/\varepsilon \rceil - 1}$.

Clearly $\Delta_{\mathcal{A}}^+ \subseteq \Delta_{\mathcal{A}}$, whatever be the metric space $\mathcal{A}$. If $\Delta = \Delta_{\mathcal{A}}$ or $\Delta = \Delta_{\mathcal{A}}^+$,

¹This set corresponds to a single error detecting code having m binary digits and using even parity check – see [6], pp. 149–150.

then the following holds: $\langle \varepsilon', n' \rangle \in \Delta$ whenever $\langle \varepsilon, n \rangle \in \Delta$, $0 < \varepsilon' \leq \varepsilon$, $n' \in \mathbb{N}$, $n' \leq n$. Therefore $\varepsilon \in \text{pr}_1 \Delta$ implies $\varepsilon' \in \text{pr}_1 \Delta$ for $0 < \varepsilon' < \varepsilon$, and $n \in \text{pr}_2 \Delta$ implies $n' \in \text{pr}_2 \Delta$ for $n' \in \mathbb{N}$, $n' < n$. In fact $\text{pr}_2 \Delta = \{n \in \mathbb{N} \mid n \leq |A|\}$. For trivial reasons, $\langle \varepsilon, n \rangle \in \Delta$ for all $\varepsilon > 0$ when $n = 0$ or $n = 1$.

The equivalence $\langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+ \Leftrightarrow \exists \eta (\eta > \varepsilon \ \& \ \langle \eta, n \rangle \in \Delta_{\mathcal{A}})$ obviously holds, as well as the implication

$$(1) \quad \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}} \Rightarrow \forall \eta (0 < \eta < \varepsilon \Rightarrow \langle \eta, n \rangle \in \Delta_{\mathcal{A}}^+).$$

If the metric space $\mathcal{A}$ is bounded, then $\varepsilon \leq \text{diam}(\mathcal{A})$ whenever $\langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}$ and $n \geq 2$. This allows the next function definition.

Definition 3. If $\mathcal{A}$ is bounded, then we set $\varrho_{\mathcal{A}}(n) = \sup\{\varepsilon \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}\}$ for $n \in \mathbb{N}$, $2 \leq n \leq |A|$.

Remark 3. In the notations from [5], Subsection 1.2, $\varrho_{\mathcal{A}}(n) = R(A, n)$ holds.

Obviously, $\varrho_{\mathcal{A}}(2) = \text{diam}(\mathcal{A})$ and $\varrho_{\mathcal{A}}$ is monotonically decreasing for any bounded metric space $\mathcal{A}$. In the situation from Example 1 we have

$$(2) \quad \varrho_{\mathcal{A}}(n) = \frac{l}{n-1}.$$

Remark 4. In the case of bounded $\mathcal{A}$, we could also define a function $\varrho_{\mathcal{A}}^+$ with the same domain by setting $\varrho_{\mathcal{A}}^+(n) = \sup\{\varepsilon \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+\}$. However, it will coincide with the function $\varrho_{\mathcal{A}}$, because the upper bounds of $\{\varepsilon \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}\}$ and $\{\varepsilon \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+\}$ are the same (this follows from (1) and the inclusion $\Delta_{\mathcal{A}}^+ \subseteq \Delta_{\mathcal{A}}$).

For any $\varepsilon > 0$ the subsets $\{n \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}\}$ and $\{n \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+\}$ of $\mathbb{N}$ are nonempty (at least 0 and 1 are their elements). If $\mathcal{A}$ is totally bounded, they are bounded, hence each of them has a largest element. This allows two functions from the interval $(0, +\infty)$ to $\mathbb{N}$ to be introduced by means of the next definition.

Definition 4. If $\mathcal{A}$ is totally bounded, then we set

$$\mu_{\mathcal{A}}(\varepsilon) = \max\{n \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}\}, \quad \mu_{\mathcal{A}}^+(\varepsilon) = \max\{n \mid \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+\}.$$

Remark 5. In the notations from [1], [2], §1, [4] and [5] we have, respectively,

$$\mu_{\mathcal{A}}^+(\varepsilon) = N_A^c(\varepsilon), \quad \mu_{\mathcal{A}}^+(\varepsilon) = \mathcal{M}_{\varepsilon}(A), \quad \mu_{\mathcal{A}}(\varepsilon) = \mathcal{M}(\varepsilon, A), \quad \mu_{\mathcal{A}}(\varepsilon) = M(A, \varepsilon).$$

Example 4. In the case from Example 1 we have $\mu_{\mathcal{A}}^+(\varepsilon) = \lceil l/\varepsilon \rceil$,² as well as

$$\mu_{\mathcal{A}}(\varepsilon) = \begin{cases} \lfloor l/\varepsilon \rfloor + 1, & \text{if the interval } A \text{ is closed,} \\ \lfloor l/\varepsilon \rfloor & \text{otherwise.} \end{cases}$$

²The last equality in formula (3) of [2] is essentially the same statement for the case when the interval A is closed.

Remark 6. A counter-example to the inequality $M(E, R(E, n)) \geq n$ stated in [5], (2), can be obtained from Example 4 and equality (2) above. Let the space Ω of [5] be our $\mathcal{A}$ from the case of non-closed interval A , and let $E = A$. Then $M(E, R(E, n)) = \mu_{\mathcal{A}}(\varrho_{\mathcal{A}}(n)) = \lceil l/\varrho_{\mathcal{A}}(n) \rceil = n - 1 < n$ for any $n \in \text{Dom}(\varrho_{\mathcal{A}})$.³

Example 5. In the case from Example 2, $\mu_{\mathcal{A}}(2) = 2^{m-1}$ for all $m \in \mathbb{N}^+$, and for $m = 4$ the graphs of $\varrho_{\mathcal{A}}$ and $\mu_{\mathcal{A}}$ are, respectively,

$$\begin{aligned} & \{ \langle 2, 4 \rangle \} \cup \{ \langle 3, 4, \dots, 8 \rangle \times \{2\} \} \cup \{ \langle 9, 10, \dots, 16 \rangle \times \{1\} \}, \\ & ((0, 1] \times \{16\}) \cup ((1, 2] \times \{8\}) \cup ((2, 4] \times \{2\}) \cup ((4, +\infty) \times \{1\}). \end{aligned}$$

Remark 7. If the set A is finite, then $\mu_{\mathcal{A}}(\varepsilon)$ and $\mu_{\mathcal{A}}^+(\varepsilon)$ are the clique numbers of the graphs $\Gamma_{\mathcal{A}, \varepsilon}$ and $\Gamma_{\mathcal{A}, \varepsilon}^+$ defined in Remark 2.

Definition 5. Let $\varkappa$ be a strictly increasing mapping of the interval $[0, +\infty)$ onto itself. A metric space $\mathcal{A}' = (A', d')$ will be said to be $\varkappa$ -similar to $\mathcal{A}$ if a mapping ι of A onto A' exists such that $d'(\iota(x), \iota(y)) = \varkappa(d(x, y))$ for all $x, y \in A$ (the mapping ι must be obviously injective).

Example 6. Let A and A' be subsets of $\mathbb{R}^m$ for some $m \in \mathbb{N}^+$ that are similar in the usual geometric sense. If d and d' are the metrics on them generated by the Euclidean norm on $\mathbb{R}^m$, then (A', d') is $\varkappa$ -similar to (A, d) for some $\varkappa$ of the form $\varkappa(t) = kt$ with $k > 0$ ($k = 1$ if A and A' are congruent).

Theorem 1.1. Let $\varkappa$ be a strictly increasing mapping of the interval $[0, +\infty)$ onto itself, and $\mathcal{A}'$ be a metric space $\varkappa$ -similar to $\mathcal{A}$. The following holds:

- (a) If $\mathcal{A}$ is bounded, then $\mathcal{A}'$ is also bounded, $\varrho_{\mathcal{A}'}$ and $\varrho_{\mathcal{A}}$ have the same domain and $\varrho_{\mathcal{A}'}(n) = \varkappa(\varrho_{\mathcal{A}}(n))$ for every n in it.
- (b) If the metric space $\mathcal{A}$ is totally bounded, then $\mathcal{A}'$ is also totally bounded and $\mu_{\mathcal{A}'}(\varkappa(\varepsilon)) = \mu_{\mathcal{A}}(\varepsilon)$, $\mu_{\mathcal{A}'}^+(\varkappa(\varepsilon)) = \mu_{\mathcal{A}}^+(\varepsilon)$ for every $\varepsilon > 0$.

We will further assume in this section that $\mathcal{A}$ is totally bounded.

The functions $\mu_{\mathcal{A}}(\varepsilon)$ and $\mu_{\mathcal{A}}^+(\varepsilon)$ are monotonically decreasing too,⁴ and for every $\varepsilon > 0$ we have the inequality $\mu_{\mathcal{A}}(\varepsilon) \geq \mu_{\mathcal{A}}^+(\varepsilon)$. Let us note that, however, the following holds:

$$(3) \quad 0 < \eta < \varepsilon \Rightarrow \mu_{\mathcal{A}}^+(\eta) \geq \mu_{\mathcal{A}}(\varepsilon).$$

Indeed, if $0 < \eta < \varepsilon$, then we may apply the implication (1) with $n = \mu_{\mathcal{A}}(\varepsilon)$.

Thanks to the equivalences

$$\langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}} \Leftrightarrow \mu_{\mathcal{A}}(\varepsilon) \geq n, \quad \langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}^+ \Leftrightarrow \mu_{\mathcal{A}}^+(\varepsilon) \geq n,$$

³However, $\mu_{\mathcal{A}}(\varepsilon) \geq \mu_{\mathcal{A}}(\varepsilon)^+ \geq n$ holds for $\varepsilon \in (0, \varrho_{\mathcal{A}}(n))$ whatever be the totally bounded metric space $\mathcal{A}$.

⁴The monotonicity of $\mu_{\mathcal{A}}^+$ is indicated in [2], Theorem III.

the function $\varrho_{\mathcal{A}}$ can be expressed through $\mu_{\mathcal{A}}$ and through $\mu_{\mathcal{A}}^+$ as follows:

$$\varrho_{\mathcal{A}}(n) = \sup\{\varepsilon \mid \mu_{\mathcal{A}}(\varepsilon) \geq n\} = \sup\{\varepsilon \mid \mu_{\mathcal{A}}^+(\varepsilon) \geq n\}.$$

Theorem 1.2. *Let $\varepsilon > 0$. If $\mu_{\mathcal{A}}^+$ is left continuous at ε or $\mu_{\mathcal{A}}$ is right continuous at ε , then $\mu_{\mathcal{A}}^+(\varepsilon) = \mu_{\mathcal{A}}(\varepsilon)$.⁵*

Proof. If $\mu_{\mathcal{A}}^+$ is left continuous at ε , we consider some η such that $0 < \eta < \varepsilon$ and $|\mu_{\mathcal{A}}^+(\eta) - \mu_{\mathcal{A}}^+(\varepsilon)| < 1$. Of course $\mu_{\mathcal{A}}^+(\eta) = \mu_{\mathcal{A}}^+(\varepsilon)$. By (3), we have the inequality $\mu_{\mathcal{A}}^+(\eta) \geq \mu_{\mathcal{A}}(\varepsilon)$, hence $\mu_{\mathcal{A}}^+(\varepsilon) \geq \mu_{\mathcal{A}}(\varepsilon)$, and therefore $\mu_{\mathcal{A}}^+(\varepsilon) = \mu_{\mathcal{A}}(\varepsilon)$. If $\mu_{\mathcal{A}}$ is right continuous at ε , we consider some $\eta > \varepsilon$ such that $|\mu_{\mathcal{A}}(\eta) - \mu_{\mathcal{A}}(\varepsilon)| < 1$. Then $\mu_{\mathcal{A}}(\eta) = \mu_{\mathcal{A}}(\varepsilon)$. By (3) (with exchanged ε and η), $\mu_{\mathcal{A}}^+(\varepsilon) \geq \mu_{\mathcal{A}}(\eta)$. We thus again see that $\mu_{\mathcal{A}}^+(\varepsilon) \geq \mu_{\mathcal{A}}(\varepsilon)$, and therefore $\mu_{\mathcal{A}}^+(\varepsilon) = \mu_{\mathcal{A}}(\varepsilon)$. $\square$

Corollary 1.2.1. *If ε is a continuity point of $\mu_{\mathcal{A}}^+$ or $\mu_{\mathcal{A}}$, then $\mu_{\mathcal{A}}^+(\varepsilon) = \mu_{\mathcal{A}}(\varepsilon)$.*

Theorem 1.3. *The set of the discontinuity points of each of the functions $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{A}}^+$ equals $\text{Range}(\varrho_{\mathcal{A}})$. The two functions coincide on $(0, +\infty) \setminus \text{Range}(\varrho_{\mathcal{A}})$.*

Proof. It is clear from Corollary 1.2.1 that both functions have the same set of discontinuity points, and it remains to prove that the set in question equals $\text{Range}(\varrho_{\mathcal{A}})$. It will be sufficient to prove that $\text{Range}(\varrho_{\mathcal{A}})$ coincides with the set of the discontinuity points of $\mu_{\mathcal{A}}$. Suppose $\varepsilon_0 \in \text{Range}(\varrho_{\mathcal{A}})$. Let n_0 be a natural number such that $\varrho_{\mathcal{A}}(n_0) = \varepsilon_0$. For any $\varepsilon \in (0, \varepsilon_0)$, $\langle \varepsilon, n_0 \rangle \in \Delta_{\mathcal{A}}$, hence $\mu_{\mathcal{A}}(\varepsilon) \geq n_0$. On the other hand, for any $\varepsilon > \varepsilon_0$, $\langle \varepsilon, n_0 \rangle \notin \Delta_{\mathcal{A}}$, hence $\mu_{\mathcal{A}}(\varepsilon) < n_0$, and therefore $\mu_{\mathcal{A}}(\varepsilon) \leq n_0 - 1$. Thus $\mu_{\mathcal{A}}$ has a jump at ε_0 . For the reasoning in the opposite direction, suppose ε_0 is a point of discontinuity of the function $\mu_{\mathcal{A}}$. Let n_0 be the least value of this integer-valued function in the interval $(0, \varepsilon_0)$. We will show that $\varepsilon_0 = \varrho(n_0)$. To prove this, we note that $\mu_{\mathcal{A}}(\varepsilon) < n_0$ for $\varepsilon > \varepsilon_0$ due to the monotone decreasing of $\mu_{\mathcal{A}}$ and its discontinuity at ε_0 . Hence $\langle \varepsilon, n_0 \rangle \in \Delta_{\mathcal{A}}$ implies $\varepsilon \leq \varepsilon_0$. On the other hand, if $\varepsilon \in (0, \varepsilon_0)$, then $\langle \varepsilon, n_0 \rangle \in \Delta_{\mathcal{A}}$ because $n_0 \leq \mu_{\mathcal{A}}(\varepsilon)$. Thus $(0, \varepsilon_0) \subseteq \{\varepsilon \mid \langle \varepsilon, n_0 \rangle \in \Delta_{\mathcal{A}}\} \subseteq (0, \varepsilon_0]$. $\square$

Corollary 1.3.1. *For any $\varepsilon > 0$ the equality $\mu_{\mathcal{A}}^+(\varepsilon) = \lim_{\eta \rightarrow \varepsilon^+} \mu_{\mathcal{A}}(\eta)$ holds.*

Proof. Let $\varepsilon > 0$. By [2], Theorem III, the integer-valued function $\mu_{\mathcal{A}}^+$ is right continuous. Thus some $\delta > 0$ exists such that $\mu_{\mathcal{A}}^+(\eta) = \mu_{\mathcal{A}}^+(\varepsilon)$ for all η in the interval $(\varepsilon, \varepsilon + \delta)$, hence $\mu_{\mathcal{A}}(\eta) = \mu_{\mathcal{A}}^+(\varepsilon)$ for all η in this interval. $\square$

The conclusion of the next statement can be obtained in a similar way.

Corollary 1.3.2. *If $\mu_{\mathcal{A}}$ is left continuous at ε , then $\mu_{\mathcal{A}}(\varepsilon) = \lim_{\eta \rightarrow \varepsilon^-} \mu_{\mathcal{A}}^+(\eta)$.*

Remark 8. There are such totally bounded metric spaces $\mathcal{A}$ that $\mu_{\mathcal{A}}$ is both left and right discontinuous at some ε . For instance, let A consist of all points in the interior of an equilateral triangle together with two of its vertices, and let d be the Euclidean metric. If ε is the side length of the triangle in question, then $\mu_{\mathcal{A}}(\varepsilon) = 2$, $\mu_{\mathcal{A}}(\eta) = 1$ for $\eta > \varepsilon$ and $\mu_{\mathcal{A}}(\eta) \geq 3$ for η in $(0, \varepsilon)$.

Theorem 1.4. *If the metric space $\mathcal{A}$ is compact, then $\mu_{\mathcal{A}}$ is left continuous.*

⁵By [2], Theorem III, the function $\mu_{\mathcal{A}}^+$ is left continuous at ε iff it is continuous at ε .

Proof. Let $\mathcal{A}$ be compact, and let $\varepsilon \in (0, +\infty)$. We will prove that $\mu_{\mathcal{A}}$ is left continuous at ε . Let $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots\} \subset (0, \varepsilon)$ and $\lim_{k \rightarrow \infty} \varepsilon_k = \varepsilon$. It will be sufficient to prove that $\mu_{\mathcal{A}}(\varepsilon) \geq n$, where $n = \min\{\mu_{\mathcal{A}}(\varepsilon_0), \mu_{\mathcal{A}}(\varepsilon_1), \mu_{\mathcal{A}}(\varepsilon_2), \dots\}$. Clearly $\langle \varepsilon_k, n \rangle \in \Delta_{\mathcal{A}}$ for all $k \in \mathbb{N}$. Therefore a sequence $S_0, S_1, S_2, \dots$ of n -element subsets of A exists such that S_k is ε_k -separated in $\mathcal{A}$ for any $k \in \mathbb{N}$. Let $S_k = \{x_{k,1}, \dots, x_{k,n}\}$. By the compactness of $\mathcal{A}$, a strictly increasing sequence $k_0, k_1, k_2, \dots$ of natural numbers exists such that $x_{k_0,i}, x_{k_1,i}, x_{k_2,i}, \dots$ converges to some element x_i of A for any $i \in \{1, \dots, n\}$. Clearly $d(x_i, x_j) \geq \varepsilon$ whenever $i, j \in \{1, \dots, n\}$ and $i \neq j$. Thus $\langle \varepsilon, n \rangle \in \Delta_{\mathcal{A}}$ and therefore $\mu_{\mathcal{A}}(\varepsilon) \geq n$. $\square$

Theorem 1.5. *Let A be an open subset of a topological vector space, and let $d(\tau x, \tau y) > d(x, y)$, whenever $\tau > 1$, $x, y, \tau x, \tau y \in A$ and $x \neq y$.⁶ Then $\mu_{\mathcal{A}}$ is right continuous.*

Proof. Suppose $\varepsilon > 0$. We will show that $\mu_{\mathcal{A}}(\varepsilon) \leq \mu_{\mathcal{A}}(\eta)$ for some $\eta > \varepsilon$. Let $S \subseteq A$, $|S| = \mu_{\mathcal{A}}(\varepsilon)$ and S be ε -separated in $\mathcal{A}$. If $\tau > 1$ and τ is sufficiently close to 1, then $\tau S \subseteq A$ and $|\tau S| = \mu_{\mathcal{A}}(\varepsilon)$ again, but τS will be η -separated in $\mathcal{A}$ for some $\eta > \varepsilon$, hence $\mu_{\mathcal{A}}(\varepsilon) \leq \mu_{\mathcal{A}}(\eta)$ for this η . $\square$

2. The case of absolute semialgebraicity.

Definition 6. For $m \in \mathbb{N}^+$, a subset of $\mathbb{R}^m$ will be called *absolutely semialgebraic* if it can be obtained by the operations of intersection and of complement to $\mathbb{R}^m$ from finitely many sets of the form $\{\langle x_1, \dots, x_m \rangle \mid p(x_1, \dots, x_m) \geq 0\}$, where p are polynomials with integer coefficients.⁷

From now on we will assume that $A \subset \mathbb{R}^m$ for some $m \in \mathbb{N}^+$ and the following holds:

1. A is bounded and absolutely semialgebraic.
2. The metric d is generated by a norm in $\mathbb{R}^m$, whose graph is an absolutely semialgebraic subset of $\mathbb{R}^{m+1}$.

For example, the set A could be an open or closed circle region in $\mathbb{R}^2$ whose radius and centre coordinates are rational, or an open or closed convex polygonal region with rational vertex coordinates, and the metric d could be generated by some of the norms $\langle x_1, x_2 \rangle \mapsto \sqrt{x_1^2 + x_2^2}$, $\langle x_1, x_2 \rangle \mapsto \max(|x_1|, |x_2|)$.

Let $\mathcal{R}$ be the ordered field of the real numbers, and $\mathfrak{L}$ be a first-order formal language for ordered field theory. Quantifier elimination algorithms exist that transform any formula Φ of $\mathfrak{L}$ into a quantifier-free formula of $\mathfrak{L}$ equivalent to Φ over $\mathcal{R}$ (for references, see e.g. [7]).

⁶The first half of this assumption can be considerably weakened because of being only partially used in the proof. Of course, $\mathcal{A}$ is supposed to be totally bounded, as assumed before in Definition 4.

⁷For the general notion of semialgebraic set cf. e.g. [7], p. 13.

Lemma 1. *The set $W = \{\langle u, v, n \rangle \in (\mathbb{N}^+)^3 \mid \langle u/v, n \rangle \in \Delta_{\mathcal{A}}\}$ as well as the set $W^+ = \{\langle u, v, n \rangle \in (\mathbb{N}^+)^3 \mid \langle u/v, n \rangle \in \Delta_{\mathcal{A}}^+\}$ are recursive.*

Proof. There is an algorithm that for every triple $\langle u, v, n \rangle \in (\mathbb{N}^+)^3$ gives a closed formula of $\mathfrak{L}$, true in $\mathcal{R}$ just when $\langle u/v, n \rangle \in \Delta_{\mathcal{A}}$. The composition of this algorithm with one eliminating quantifiers shows the recursiveness of W . The recursiveness of W^+ is seen analogously. $\square$

Lemma 2. *The mappings $\langle u, v \rangle \mapsto \mu_{\mathcal{A}}(u/v)$ and $\langle u, v \rangle \mapsto \mu_{\mathcal{A}}^+(u/v)$ of $(\mathbb{N}^+)^2$ into $\mathbb{N}$ are computable.*

Proof. By Lemma 1 and the equalities $\mu_{\mathcal{A}}(\varepsilon) = \min\{n \in \mathbb{N} \mid \langle \varepsilon, n+1 \rangle \notin \Delta_{\mathcal{A}}\}$, $\mu_{\mathcal{A}}^+(\varepsilon) = \min\{n \in \mathbb{N} \mid \langle \varepsilon, n+1 \rangle \notin \Delta_{\mathcal{A}}^+\}$. $\square$

When function values or arguments are real numbers, computability will be understood as in [8].

Theorem 2.1. *The partial function $\varrho_{\mathcal{A}}$ from $\mathbb{N}$ to $\mathbb{R}$ is computable.*

Proof. If we set $\theta(n, i) = \min\{k \in \mathbb{N} \mid \langle (k+1)2^{-i}, n \rangle \notin \Delta_{\mathcal{A}}\}$ for $n \in \text{Dom}(\varrho_{\mathcal{A}})$ and $i \in \mathbb{N}$, then θ is a computable function and $|\theta(n, i)2^{-i} - \varrho_{\mathcal{A}}(n)| < 2^{-i}$. $\square$

Theorem 2.2. *The set $\{\langle n, n' \rangle \in \text{Dom}^2(\varrho_{\mathcal{A}}) \mid \varrho_{\mathcal{A}}(n) = \varrho_{\mathcal{A}}(n')\}$ is recursive.*

Proof. There is an algorithm that for each pair $\langle n, n' \rangle \in \mathbb{N}^2$ gives a closed formula of $\mathfrak{L}$, true in $\mathcal{R}$ just when $\langle n, n' \rangle$ belongs to this set. $\square$

Corollary 2.2.1. *When ordered in decreasing order without repetitions, the numbers in $\text{Range}(\varrho_{\mathcal{A}})$ form a computable sequence.*

Proof. We use Theorems 2.1 and 2.2, as well as the monotonicity of $\varrho_{\mathcal{A}}$. $\square$

Theorem 2.3. *The numbers in $\text{Range}(\varrho_{\mathcal{A}})$ are algebraic.*

Proof. Let $n \in \text{Dom}(\varrho_{\mathcal{A}})$. There is such a formula Φ of $\mathfrak{L}$ with one free variable that Φ is true in $\mathcal{R}$ just when the value $\varrho_{\mathcal{A}}(n)$ is assigned to the variable in question. Eliminating the quantifiers in Φ , we see that the one-element set $\{\varrho_{\mathcal{A}}(n)\}$ is absolutely semialgebraic, hence $\varrho_{\mathcal{A}}(n)$ is an algebraic number.⁸ $\square$

Theorem 2.4. *The sets $\text{Range}(\mu_{\mathcal{A}})$ and $\text{Range}(\mu_{\mathcal{A}}^+)$ are recursive.*

Proof. There is an algorithm that for every $n \in \mathbb{N}$ gives a closed formula of $\mathfrak{L}$, true in $\mathcal{R}$ just when $n \in \text{Range}(\mu_{\mathcal{A}})$. Similarly for $\text{Range}(\mu_{\mathcal{A}}^+)$. $\square$

Theorem 2.5. *The restriction of $\mu_{\mathcal{A}}$ to the set $(0, +\infty) \setminus \text{Range}(\varrho_{\mathcal{A}})$ (coinciding with the restriction of $\mu_{\mathcal{A}}^+$ to the same set) is computable.*

Proof. Let $\varepsilon \in (0, +\infty) \setminus \text{Range}(\varrho_{\mathcal{A}})$. If $e_0, e_1, e_2, \dots$ is a sequence of rational numbers and $\forall i \in \mathbb{N} (|e_i - \varepsilon| < 2^{-i})$, then the conditions $e_i - 2^{-i} > 0$ and $n = \mu_{\mathcal{A}}(e_i - 2^{-i}) = \mu_{\mathcal{A}}(e_i + 2^{-i})$, obviously implying the equality $\mu_{\mathcal{A}}(\varepsilon) = n$, are satisfied by some i and n in $\mathbb{N}$. Thanks to Lemma 2, finding such i and n is algorithmic relative to the sequence of rational numbers $e_0, e_1, e_2, \dots$. $\square$

Theorem 2.6. *Let $\mathfrak{r}_{<}$ be the representation of real numbers as limits of monotonically increasing sequences of rational numbers, and $\mathfrak{r}_{>}$ be their representation as limits of monotonic decreasing ones. If the set A is closed, then the function $\mu_{\mathcal{A}}$ is $(\mathfrak{r}_{<}, \mathfrak{r}_{>})$ -computable, and if A is open, it is $(\mathfrak{r}_{>}, \mathfrak{r}_{<})$ -computable. The function*

⁸Since this proof is constructive, it can be used for proving Theorem 2.1 in a different way.

$\mu_{\mathcal{A}}^+$ is always $(\mathbf{r}_{>}, \mathbf{r}_{<})$ -computable.

Proof. Suppose A is closed. Theorem 1.1 gives that $\mu_{\mathcal{A}}$ is left continuous. If $\varepsilon > 0$ and $e_0, e_1, e_2, \dots$ is a monotonically increasing sequence of rational numbers converging to ε , then the sequence $\mu_{\mathcal{A}}(e_k), \mu_{\mathcal{A}}(e_{k+1}), \mu_{\mathcal{A}}(e_{k+2}), \dots$, where $k = \min\{i \in \mathbb{N} \mid e_i > 0\}$, is a monotonically decreasing sequence converging to $\mu_{\mathcal{A}}(\varepsilon)$, and Lemma 2 shows that it is obtained algorithmically from the sequence $e_0, e_1, e_2, \dots$. The other two statements of the theorem are proved analogously (using Theorem 2.1 and [2], Theorem 3, respectively), but a little simpler, because $e_i > 0$ for each $i \in \mathbb{N}$ in the considered sequence $e_0, e_1, e_2, \dots$. $\square$

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