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SOME LACUNARY POWER SERIES AND MAHLER'S U_m-NUMBERS IN p-ADIC DOMAIN

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Abstract

In this paper, we consider some lacunar power series in $\mathbb{Q}_p$, where p is a prime number. We obtain some results about lacunary power series with rational coefficients for U_m-number arguments in $\mathbb{Q}_p$. We determined whether the values of the considered lacunar power series belong to a certain field of algebraic numbers or to the set of transcendental numbers in $\mathbb{Q}_p$. Also, we find similar results for lacunary power series with p-adic algebraic coefficients.
Key words: p-adic number, transcendental number, power series
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1. Introduction. Mahler [1] gave a classification of complex numbers and divided the set of complex numbers into four classes as the A, S, T, U-numbers. The set of A-numbers corresponds to the set of complex algebraic numbers. Later, Mahler [2] introduced a classification of p-adic numbers and he divided the set of p-adic numbers as the p-adic A, S, T, U-numbers. Again, the set of p-adic A-numbers corresponds to the set of p-adic algebraic numbers. Also, a p-adic U-number of degree m is called a p-adic U_m-number. Koksma [3] obtained another classification stand on approximation by algebraic numbers. Moreover, this classification applies to use with p-adic numbers. Koksma's classification also divides the set of p-adic numbers into four disjointed classes called as the p-adic A^*, S^*, T^*, U^*-numbers. These two classifications are equivalent, that is the p-adic A, S, T, U-numbers are the same as the p-adic A^*, S^*, T^*, U^*-numbers. Again, a

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p-adic U^*-number of degree m is called a p-adic U_m^*-number. Furthermore, every p-adic U_m-number is a p-adic U_m^*-number. We refer to [4–6] for more information.

Most of the results obtained on the approximation theory in the field of complex numbers can be transferred to another field, such as the field of p-adic numbers and formal Laurent series [4, 7–9]. For example, many researchers have studied the theory of transcendental numbers in the field of p-adic numbers with the notable papers by Mahler [10, 11]. In the 1930's, Mahler obtained the p-adic analogues of the Hermite–Lindemann and the Gelfond–Schneider Theorems. Then, p-adic analogues of many results obtained in the field of complex numbers were studied [8, 12–14].

In the present paper, we focus on transcendental number theory and consider some lacunary power series in the field of p-adic numbers. We obtain whether the values of some lacunary power series with rational coefficients for U_m-number arguments are p-adic algebraic numbers or p-adic transcendental numbers. Later, we give a generalization of this result and show whether the values of some lacunary power series with p-adic algebraic coefficients for U_m-number arguments are p-adic algebraic numbers or p-adic transcendental numbers. Therefore, the results of this paper are the p-adic analogues of the results obtained in [15] and are generalizations of the results obtained in [8].

The rest of the paper is organized as follows. In Section 2, we recall the basic concepts and some results known on classification of p-adic numbers. In Section 3, we consider some lacunary power series with rational coefficients and prove whether the values of these series for U_m-number arguments belong to a certain field of p-adic algebraic numbers or to the set of p-adic transcendental numbers. In Section 4, we investigate the values of some lacunary power series with p-adic algebraic coefficients for p-adic U_m-number arguments.

2. Preliminaries. Throughout the paper, we assume that p is a prime number. Let $|\cdot|_p$ and $\mathbb{Q}_p$ denote respectively the p-adic valuation on the field of the rational numbers and the field of p-adic numbers. The height and degree of a polynomial F with rational integral coefficients is the maximum of the absolute values of the coefficients of F denoted as $H(F)$ and the degree of maximal degree monomial of F denoted as $\deg(F)$. Also, the height and degree of the minimal polynomial with rational integer coefficients of a p-adic algebraic number α are denoted by $H(\alpha)$ and $\deg(\alpha)$, respectively. The following result proved by İçen [16] gives an upper bound on the height of a p-adic algebraic number.

Lemma 2.1. *Let K be a p-adic algebraic number field of degree g. Let $\alpha_1, \dots, \alpha_k$ ($k \geq 1$) be p-adic algebraic numbers in K and $F(y, x_1, \dots, x_k)$ be a polynomial with integral coefficients so that the degree of F in y is at least 1, and ξ be a p-adic algebraic number. If $F(\xi, \alpha_1, \dots, \alpha_k) = 0$, then $\deg(\xi)$ is $\leq dg$, and*

$$H(\xi) \leq 3^{2dg+(l_1+\dots+l_k)g} H(F)^g H(\alpha_1)^{l_1 g} \dots H(\alpha_k)^{l_k g},$$

where l_i is the degree of F in x_i ($i = 1, \dots, k$) and d is the degree of F in y.

ROTH [17] obtained that the exponent for approximation of real algebraic irrationals by rational numbers cannot be greater than 2. The p-adic analogue of this result was proved by RIDOUT [18]. Now, we recall Ridout's result which will be quite useful to obtain our results.

Theorem 2.2. *Let ξ be a number in the field $\mathbb{Q}_p$ and ϵ be a positive real number. If there exists a sequence $\left\{\frac{p_n}{q_n}\right\}$ of rational numbers such that $2 \leq \max(|p_1|,|q_1|) < \max(|p_2|,|q_2|) < \cdots$ and $0 < \left|\xi - \frac{p_n}{q_n}\right|_p < \frac{1}{\max(|p_n|,|q_n|)^{2+\epsilon}}$, where $q_n \neq 0$ and $\gcd(p_n,q_n) = 1$ for $n = 1,2,\ldots$, then ξ is transcendental.*

The result given by Roth [17] was extended to the approximations of algebraic numbers by algebraic numbers [19]. Moreover, the p-adic version of this result was given in [20].

We note that the results obtained in the present paper are true for all sufficiently large n unless otherwise specified.

3. Some lacunary power series with rational coefficients. In this section, we consider some lacunary power series with rational coefficients in the field $\mathbb{Q}_p$ and investigate the values of these series for some p-adic U_m-number arguments. Let us start by introducing the lacunary power series that we consider.

We take the increasing sequence of positive integers $\{k_n\}_{n=0}^{\infty}$. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{b_{k_n}}{a_{k_n}} x^{k_n} \tag{3.1}$$

be a lacunary power series with rational coefficients in $\mathbb{Q}_p$, where $a_{k_n} > 1$, $b_{k_n} \neq 0$ and $\gcd(a_{k_n}, b_{k_n}) = 1$ for sufficiently large n. We assume that $\left|\frac{b_{k_n}}{a_{k_n}}\right|_p = p^{-u_{k_n}}$ and

$$\sigma := \liminf_{n\to+\infty} \frac{u_{k_{n+1}}}{u_{k_n}} > 1, \tag{3.2}$$

$$0 \leq \lambda := \limsup_{n\to+\infty} \frac{\log_p A_{k_n} B_{k_n}}{u_{k_n}} < +\infty, \tag{3.3}$$

$$\lim_{n\to+\infty} \frac{u_{k_n}}{k_n} = +\infty, \tag{3.4}$$

where $A_{k_n} = [a_{k_0}, a_{k_1}, \ldots, a_{k_n}]$ denotes the least common multiple of the numbers $a_{k_0}, a_{k_1}, \ldots, a_{k_n}$ and $B_{k_n} = \max_{0\leq\nu\leq n}\{|b_{k_\nu}|\}$. Also, let us consider the polynomials $f_n(x) = \sum_{\nu=0}^{n} \frac{b_{k_\nu}}{a_{k_\nu}} x^\nu$ $(n = 1,2,\ldots)$.

The following result helps us for the argument selection.

Lemma 3.1. *The radius of convergence of the series $f(x)$ in* (3.1) *is infinite.*

Proof. By using (3.4), we have

$$\frac{1}{\limsup_{k_n\to+\infty} \sqrt[k_n]{\left|\frac{b_{k_n}}{a_{k_n}}\right|_p}} = \liminf_{k_n\to+\infty} p^{u_{k_n}/k_n} = +\infty. \tag{3.5}$$ □

Let us now introduce the p-adic U_m-number arguments that we will investigate the values of the lacunary power series in (3.1). We consider K as a p-adic number field such that $[K:\mathbb{Q}]=m$ and α_n as an algebraic number in K such that $\deg(\alpha)=m$. Let ζ have an approximation with the numbers α_n such that

$$|\zeta-\alpha_n|_p \le \frac{1}{H(\alpha_n)^{n\,w(n)}} \quad (\lim_{n\to+\infty} w(n)=+\infty) \tag{3.6}$$

and

$$p^{u_{k_n}\delta_1} \le H(\alpha_{k_n})^{k_n} \le p^{u_{k_n}\delta_2} \tag{3.7}$$

for sufficiently large n, where δ_1 and δ_2 are two real numbers such that $0<\delta_1\le\delta_2$.

Now, we are ready to settle the main question about the values of the above lacunary power series with rational coefficients for p-adic U_m-number arguments in $\mathbb{Q}_p$.

Theorem 3.2. *Let $\sigma > 2m(1+\lambda+\delta_2)$ for the lacunary power series $f(x)$ in* (3.1). *If $\{f_n(\alpha_{k_n})\}$ is a constant sequence, then $f(\zeta)$ is an algebraic number in K, otherwise $f(\zeta)$ is a transcendental number in $\mathbb{Q}_p$.*

Proof. We handle the polynomials $f_n(x)=\sum_{\nu=0}^{n}\frac{b_{k_\nu}}{a_{k_\nu}}x^\nu$ $(n=1,2,\dots)$, where $\deg(f_n(\alpha_{k_n}))\le m$. If we take $F(z,x)=A_{k_n}z-\sum_{\nu=0}^{n}A_{k_n}\frac{b_{k_\nu}}{a_{k_\nu}}x^{k_\nu}$, then $F(f_n(\alpha_{k_n}),\alpha_{k_n})=0$. Moreover, we find that $H(F)=\max_{\nu=0}^{n}\left\{A_{k_n}, A_{k_n}\left|\frac{b_{k_\nu}}{a_{k_\nu}}\right|\right\}\le A_{k_n}B_{k_n}$, where $B_{k_n}=\max\limits_{0\le\nu\le n}\{|b_{k_\nu}|\}$. Using (3.3), we have a sufficiently small $\varepsilon_1>0$ such that $A_{k_n}B_{k_n}<p^{u_{k_n}(\lambda+\varepsilon_1)}$. Now, using Lemma 2.1 and the inequalities (3.3), (3.4) and (3.7), we obtain the following result

$$H(f_n(\alpha_{k_n})) \le 3^{2m+k_n m}H(F)H(\alpha_{k_n})^{k_n m} \le p^{u_{k_n}m(1+\lambda+\varepsilon_1+\delta_2)}. \tag{3.8}$$

Since there exists a sufficiently small $\varepsilon_2>0$ from (3.4) such that $|\zeta|_p^{k_n}<p^{u_{k_n}\varepsilon_2}$, we see that

$$\begin{aligned}|f(\zeta)-f_n(\zeta)|_p &= \left|\sum_{\nu=n+1}^{\infty}\frac{b_{k_\nu}}{a_{k_\nu}}\zeta^\nu\right|_p\\ &\le \max\left\{\left|\frac{b_{k_{n+1}}}{a_{k_{n+1}}}\right|_p|\zeta|_p^{k_{n+1}}, \left|\frac{b_{k_{n+2}}}{a_{k_{n+2}}}\right|_p|\zeta|_p^{k_{n+2}},\dots\right\}\\ &\le p^{-(1-\varepsilon_2)u_{k_{n+1}}}.\end{aligned}$$

From (3.2), there exists a sufficiently small $\varepsilon_3>0$ such that $(\sigma-\varepsilon_3)u_{k_n}<u_{k_{n+1}}$, where $\sigma-\varepsilon_3>1$. Hence, we obtain that $|f(\zeta)-f_n(\zeta)|_p\le p^{-(\sigma-\varepsilon_3)(1-\varepsilon_2)u_{k_n}}$. Then using (3.8), we have that

$$|f(\zeta)-f_n(\zeta)|_p \le H(f_n(\alpha_{k_n}))^{\frac{-(\sigma-\varepsilon_3)(1-\varepsilon_2)}{m(1+\lambda+\varepsilon_1+\delta_2)}}. \tag{3.9}$$

By the definition of the polynomial $f_n(x)$, it is obvious that

$$|f_n(\zeta) - f_n(\alpha_{k_n})|_p = \left|\sum_{\nu=0}^{n} \frac{b_{k_\nu}}{a_{k_\nu}} \left(\zeta^{k_\nu} - \alpha_{k_n}^{k_\nu}\right)\right|_p$$

$$\leq \max\left\{\left|\frac{b_{k_\nu}}{a_{k_\nu}}\right|_p |\zeta - \alpha_{k_n}|_p \left|\zeta^{k_\nu-1} + \zeta^{k_\nu-2}\alpha_{k_n} + \cdots + \alpha_{k_n}^{k_\nu-1}\right|_p\right\}.$$

We infer that $|\alpha_{k_n}|_p \leq |\zeta|_p + 1$ from (3.6). Since $\{u_{k_n}\}$ is a monotone increasing sequence, we have $p^{-u_{k_\nu}} < C$ $(\nu = 0, 1, \ldots, n)$, where C is a suitable positive number. Combining (3.6) and (3.7), we find that

$$\begin{aligned} |f_n(\zeta) - f_n(\alpha_{k_n})|_p &\leq |\zeta - \alpha_{k_n}|_p \left(|\zeta|_p + 1\right)^{k_n} \max_{\nu=0}^{n} \left\{p^{-u_{k_\nu}}\right\} \\ &\leq |\zeta - \alpha_{k_n}|_p \left(|\zeta|_p + 1\right)^{k_n} C \\ &\leq H(\alpha_{k_n})^{-k_n\, w(k_n)}((|\zeta|_p + 1)C)^{k_n} \\ &\leq p^{-w(k_n)u_{k_n}\delta_1}((|\zeta|_p + 1)C)^{k_n}. \end{aligned}$$

We consider the real number $\frac{\log_p C_1}{\varepsilon_4}$ such that ε_4 is a sufficiently small positive number. From (3.4), we have $C_1^{k_n} < p^{\varepsilon_4 u_{k_n}}$. Therefore, from (3.8), we obtain that

$$(3.10) \qquad |f_n(\zeta) - f_n(\alpha_{k_n})|_p \leq p^{\varepsilon_4 u_{k_n} - w(k_n)u_{k_n}\delta_1} \leq H(f_n(\alpha_{k_n}))^{\frac{-(w(k_n)\delta_1 - \varepsilon_4)}{m(1+\lambda+\varepsilon_1+\delta_2)}}.$$

Since $\lim w(n) = +\infty$ and $\delta_1 > 0$, we see that

$$H(f_n(\alpha_{k_n}))^{\frac{(\sigma-\varepsilon_3)(1-\varepsilon_2)}{m(1+\lambda+\varepsilon_1+\delta_2)}} < H(f_n(\alpha_{k_n}))^{\frac{w(k_n)\delta_1 - \varepsilon_4}{m(1+\lambda+\varepsilon_1+\delta_2)}}.$$

Combining (3.9) and (3.10) we conclude that

$$(3.11) \qquad \begin{aligned} |f(\zeta) - f_n(\alpha_{k_n})|_p &= \max\left\{|f(\zeta) - f_n(\zeta)|_p, |f_n(\zeta) - f_n(\alpha_{k_n})|_p\right\} \\ &< \frac{1}{H(f_n(\alpha_{k_n}))^{\frac{(\sigma-\varepsilon_3)(1-\varepsilon_2)}{m(1+\lambda+\varepsilon_1+\delta_2)}}}. \end{aligned}$$

If we choose sufficiently small numbers ε_1, ε_2 and ε_3, then there exists a number $\varepsilon > 0$ such that

$$\frac{(\sigma - \varepsilon_3)(1 - \varepsilon_2)}{m(1 + \lambda + \varepsilon_1 + \delta_2)} > \frac{\sigma}{m(1 + \lambda + \delta_2)} - \varepsilon.$$

Since $\sigma > 2m(1 + \lambda + \delta_2)$, it is possible to choose a number $\varepsilon > 0$ such that $\frac{\sigma}{m(1+\lambda+\delta_2)} - \varepsilon \geq 2 + \varepsilon$. Therefore, we obtain that

$$(3.12) \qquad |f(\zeta) - f_n(\alpha_{k_n})|_p < \frac{1}{H(f_n(\alpha_{k_n}))^{2+\varepsilon}}$$

and this completes the proof. □

4. Some lacunary power series with p-adic algebraic coefficients. Let L be a p-adic algebraic number field such that $[L:\mathbb{Q}] = q$ and $\{k_n\}_{n=0}^{\infty}$ be an increasing sequence of positive integers. In the field $\mathbb{Q}_p$, we deal with the lacunary power series

$$f(x) = \sum_{n=0}^{\infty} \frac{\eta_{k_n}}{a_{k_n}} x^{k_n} \tag{4.1}$$

with the p-adic algebraic coefficients, where η_{k_n} is a p-adic algebraic integer in L and a_{k_n} is a positive rational integer ($a_{k_n} > 1$ for sufficiently large n) that satisfies the following conditions

$$\sigma := \liminf_{n\to+\infty} \frac{u_{k_{n+1}}}{u_{k_n}} > 1, \tag{4.2}$$

$$\lambda := \limsup_{n\to+\infty} \frac{\log_p A_{k_n} H(\eta_{k_n})}{u_{k_n}} < +\infty \tag{4.3}$$

and

$$\lim_{n\to+\infty} \frac{u_{k_n}}{k_n} = +\infty, \tag{4.4}$$

where $\left|\frac{\eta_{k_n}}{a_{k_n}}\right|_p = p^{-u_{k_n}}$, $\lambda \geq 0$ and $A_{k_n} = [a_{k_0}, a_{k_1}, \ldots, a_{k_n}]$ denotes the least common multiple of $a_{k_0}, a_{k_1}, \ldots, a_{k_n}$. Also, let us consider the polynomials $f_n(x) = \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} x^{k_\nu}$ ($n = 1, 2, \ldots$).

Using (4.4), the following results can be proven in a similar way as the proof of Lemma 3.1.

Lemma 4.1. *The radius of convergence of the series $f(x)$ in* (4.1) *is infinite.*

We take M to be the smallest p-adic algebraic number field containing K and L such that $[M:\mathbb{Q}] = t$, where K is the p-adic number field given in Section 3. Also, let ζ be the p-adic U_m-number introduced in Section 3. We are now ready to prove the following theorem.

Theorem 4.2. *Let $\sigma(\sigma-1) > 4t(\lambda\sigma + (\sigma-1)\delta_2)$ for the lacunary power series* (4.1). *If the sequence $\{f_n(\alpha_{k_n})\}$ is constant, then $f(\zeta)$ is an algebraic number in M, otherwise $f(\zeta)$ is a transcendental number in $\mathbb{Q}_p$.*

Proof. If we take the polynomials $f_n(x) = \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} x^{k_\nu}$, then $\gamma_n := f_n(\alpha_{k_n})$ is an algebraic number in M and $\deg(\gamma_n) \leq t$ for $n = 1, 2, \ldots$. Now, we will obtain an upper bound for $H(\gamma_n)$ by using Lemma 2.1. For this reason, we use the polynomial $F(y, x_0, x_1, \ldots, x_n, x_{n+1}) = A_{k_n} y - A_{k_n} \sum_{\nu=0}^{n} \frac{x_\nu}{a_{k_\nu}} x_{n+1}^{k_\nu}$, where $F(\gamma_n, \eta_{k_0}, \eta_{k_1}, \ldots, \eta_{k_n}, \alpha_{k_n}) = 0$. From Lemma 2.1, we obtain that

$$H(\gamma_n) \leq 3^{2t+(n+1+k_n)t} H^t H(\eta_{k_0})^t \cdots H(\eta_{k_n})^t H(\alpha_{k_n})^{k_n t}.$$

By (4.2), there exists $\sigma_1 := \sigma - \varepsilon_1 > 1$ such that $u_{k_{n+1}} > \sigma_1^{n-N_1} u_{k_{N_1}}$ for a suitable positive number N_1. Therefore, we have $u_{k_\nu} \leq \frac{1}{\sigma_1^{n-k_\nu}} u_{k_n}$ $(n \geq \nu \geq N_1)$ and $u_{k_{N_2}} + \cdots + u_{k_n} \leq \frac{\sigma_1}{\sigma_1 - 1} u_{k_n}$ $(n \geq N_2)$, where N_2 is a suitable positive integer number. By using (4.3), we have a sufficiently small number $\varepsilon_2 > 0$ such that $A_{k_n} H(\eta_{k_n}) < p^{u_{k_n}(\lambda+\varepsilon_2)}$. After these auxiliary results, we can give an upper bound for the height $H(\gamma_n)$. Since $H = A_{k_n}$, from (3.7), we obtain that

$$H(\gamma_n) \leq 3^{5k_n t} p^{u_{k_n}\delta_2 t} D p^{(u_{k_{N_2}}+\cdots+u_{k_n})(\lambda+\varepsilon_2)t} \leq 3^{5k_n t} D^t p^{\left(\frac{\sigma_1(\lambda+\varepsilon_2)}{\sigma_1-1}+\delta_2\right)u_{k_n} t},$$

where $D = H(\eta_{k_0}) \cdots H(\eta_{k_{N-1}})$. By (4.4), we see that there exists a suitable positive number ε_3 such that

$$H(\gamma_n) \leq p^{\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\varepsilon_3+\delta_2\right)u_{k_n} t}. \tag{4.5}$$

Since $|\zeta|_p^{k_n} = p^{k_n \log_p |\zeta|_p}$, we have $\left|\frac{\eta_{k_n}}{a_{k_n}} \zeta^{k_n}\right|_p \leq p^{-u_{k_n}+k_n \log_p |\zeta|_p}$. Using (4.4), we have that $\left|\frac{\eta_{k_n}}{a_{k_n}} \zeta^{k_n}\right|_p \leq p^{\frac{-u_{k_n}}{2}}$. Therefore, combining (4.2) and (4.4), we obtain

$$\begin{aligned}
|f(\zeta) - f_n(\zeta)|_p &= \left|\sum_{\nu=n+1}^{\infty} \frac{\eta_{k_\nu}}{a_{k_\nu}} \zeta^{k_\nu}\right|_p \\
&\leq \max\left\{\left|\frac{\eta_{k_{n+1}}}{a_{k_{n+1}}} \zeta\right|_p^{k_{n+1}}, \left|\frac{\eta_{k_{n+2}}}{a_{k_{n+2}}} \zeta\right|_p^{k_{n+2}}, \ldots\right\} \\
&\leq \max\left\{p^{\frac{-u_{k_{n+1}}}{2}}, p^{\frac{-u_{k_{n+2}}}{2}}, \ldots\right\} \\
&\leq p^{\frac{-\sigma_1}{2} u_{k_n}}.
\end{aligned}$$

Using (4.5), we see that

$$|f(\zeta) - f_n(\zeta)|_p \leq H(\gamma_n)^{\frac{-(\sigma-\varepsilon_1)}{\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\varepsilon_3+\delta_2\right)t}}. \tag{4.6}$$

Since $|\alpha_{k_n}|_p \leq |\zeta|_p + 1$ from (3.6) and $\{u_{k_n}\}$ is a monotone increasing sequence, we see that

$$\begin{aligned}
|f_n(\zeta) - \gamma_n|_p &= \left|\sum_{\nu=0}^{n} \frac{b_{k_\nu}}{a_{k_\nu}} (\zeta^{k_\nu} - \alpha_{k_n})^{k_\nu}\right|_p \\
&\leq \max\left\{\left|\frac{b_{k_\nu}}{a_{k_\nu}}\right|_p |\zeta - \alpha_{k_n}|_p \left|\zeta^{k_\nu-1} + \zeta^{k_\nu-2}\alpha_{k_n} + \cdots + \alpha_{k_n}^{k_\nu-1}\right|_p\right\} \\
&\leq |\zeta - \alpha_{k_n}|_p (|\zeta|_p + 1)^{k_n} \max_{\nu=0}^{n} \left\{p^{-u_{k_\nu}}\right\} \\
&\leq |\zeta - \alpha_{k_n}|_p (|\zeta|_p + 1)^{k_n} M,
\end{aligned}$$

where M is a suitable positive number. By using (3.6), (3.7) and (4.4), we have

$$\begin{aligned}|f_n(\zeta)-\gamma_n|_p &\le H(\alpha_{k_n})^{-k_n w(k_n)}\big((|\zeta|_p+1)M\big)^{k_n}\\ &\le p^{-w(k_n)u_{k_n}\delta_1}\big((|\zeta|_p+1)M\big)^{k_n}\\ &\le p^{(\varepsilon_4-\delta_1 w(k_n))u_{k_n}},\end{aligned}$$

where ε_4 is a sufficiently small positive number. From (4.5), we see that

$$|f_n(\zeta)-\gamma_n|_p \le H(\gamma_n)^{\frac{\delta_1 w(k_n)-\varepsilon_4}{\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\varepsilon_3+\delta_2\right)t}}. \tag{4.7}$$

Since $\lim w(n)=+\infty$ and $\delta_1>0$, combining (4.6) and (4.7), we conclude that

$$|f(\zeta)-f_n(\alpha_{k_n})|_p < \frac{1}{H(\gamma_n)^{\frac{\frac{\sigma-\varepsilon_1}{2}}{\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\varepsilon_3+\delta_2\right)t}}}. \tag{4.8}$$

Again as in the proof of Theorem 2, there exists a suitable positive number ε depending on ε_1, ε_2 and ε_3 such that

$$\frac{\sigma-\varepsilon_1}{2t\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\varepsilon_3+\delta_2\right)} > \frac{\sigma}{2t\left(\frac{\sigma\lambda}{\sigma-1}+\delta_2\right)}-\varepsilon.$$

Since $\sigma(\sigma-1)>4t(\lambda\sigma+(\sigma-1)\delta_2)$, we can choose a positive number ε, such that

$$\frac{\sigma(\sigma-1)}{2t(\lambda\sigma+(\sigma-1)\delta_2)}-\varepsilon \ge 2+\varepsilon.$$

Therefore, we obtain that

$$|f(\zeta)-\gamma_n|_p < \frac{1}{H(\gamma_n)^{2+\varepsilon}} \tag{4.9}$$

and this completes the proof. □

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