FINITE GROUPS WHOSE NUMBERS OF REAL-VALUED CHARACTER DEGREES OF ALL PROPER SUBGROUPS ARE AT MOST TWO

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Abstract

Finite groups with real-valued irreducible characters of prime degree are classified by Dolfi, Pacifici and Sanus. In this paper, the structures of finite groups whose all proper subgroups have at most two real-valued-irreducible-character degrees are determined.

Key words: simple group, character value, proper subgroup

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1. Introduction. All groups $G$ that are considered are finite. The set of complex irreducible characters of $G$ is denoted by $\text{Irr}(G)$ and the set of character degrees of $G$ is written by $\text{cd}(G)$. The groups with two irreducible character degrees are classified; see $[1,2]$. As a generalization of $[1,2]$, Noritsch in $[3]$ considered finite groups with three complex irreducible character degrees.

In the following, the degree means the real-valued irreducible character degree.

Finite groups with few character values are classified by some scholars; see $[4-6]$ for example. In particular, Iwasaki in $[7]$ studied the structure of a finite group $G$ concerning the number of real-valued irreducible characters and characterized the structure of finite groups with exactly two real-valued irreducible characters.

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Let $\text{Irr}_{rv}(G)$ be the set of real-valued irreducible characters of a group $G$ and let $\text{cd}_{rv}(G)$ be the set of real-valued-irreducible-character degrees, i.e.

$$\text{cd}_{rv}(G) = \{\chi(1) : \chi \in \text{Irr}_{rv}(G)\}.$$ 

Recently, the author and others considered the relation between the degrees of proper subgroups and group structure; see [8] and [9] for instance. We also consider the impact of the properties of the proper subgroups on the structure of finite group. Let $\sum G$ be the set of all proper subgroups of a group $G$. Then $\text{cd}_{rv}(G)$ is a subset of $\text{cd}(G)$. Obviously an abelian group satisfies $\text{cd}_{rv}(G) = \text{cd}(G)$. In this paper, we first consider the influence of the number of degrees on the structure of finite groups. To say in brief, we introduce the following concept.

**Definition 1.1.** A group $G$ is called a TR-group if $|\text{cd}_{rv}(G)| \leq 2$.

Finite groups with $\text{cd}(G) = \{1, m\}$ are classified ([10], Theorem 12.5). Corresponding to [1, 2, 10], we first show the following result.

**Theorem 1.2.** A finite TR-group is solvable.

In order to argue in short, we introduce the following definition.

**Definition 1.3.** A group $G$ is called a PTR-group if each $H \in \sum G$ is a TR-group.

Now we have the following result.

**Theorem 1.4.** Let $G$ be a non-solvable PTR-group. Then $G$ is isomorphic to either $\text{PSL}_2(q)$, where either $q = 2^p$ with $p$ a prime or $q = 3^p$ with $p$ an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$ or $S_3(2^p)$ with $p$ an odd prime.

2. Basic results. In this section, some elementary results are collected.

**Lemma 2.1.** Let $G$ be a PTR-group.

(1) Let $N \in \sum G$. Then $N$ is a TR-group.

(2) If $1 \neq N \in \sum G$ is normal in $G$, then $G/N$ is a TR-group.

**Proof.** It is easy to get from Definition of a PTR-group and the fact $\text{cd}_{rv}(G/N) \subset \text{cd}_{rv}(G)$. The following result can be used to shorten the proof of main theorem.

**Lemma 2.2** ([9], Corollary 1). Every minimal simple group is isomorphic to one of the following minimal simple groups:

(1) $\text{PSL}_2(2^p)$ for $p$ a prime;

(2) $\text{PSL}_2(3^p)$ for $p$ an odd prime;

(3) $\text{PSL}_2(p)$, for $p$ any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod{5}$;

(4) $S_3(2^p)$ for $p$ an odd prime;

(5) $\text{PSL}_3(3)$. 

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**Lemma 2.3.** Let $G$ be a Frobenius group of the form $E_p^n : C_m$ with kernel $E_p^n$, the elementary abelian $p$-group of order $p^n$ and complement $C_m$, the cyclic group of order $m$. Assume that $|G|$ is even, then $G$ has an irreducible real character of degree $m$.

**Proof.** It follows from Theorem 13.9(b) of [11].

**Lemma 2.4.** Let $G$ be a dihedral group $D_{2m}$. Then $G$ is a TR-group.

**Proof.** It is easy to see from the fact $cd(G) = \{1, 2\}$.

**Lemma 2.5.** Let $G$ be a finite group of even order.

1. If $\exp(G,d)$ is odd, then at least one of the $\chi \in \text{Irr}(G)$ of degree $d$ is real.

   In particular, if $\exp(G,d) = 1$, then the $\chi \in \text{Irr}(G)$ of degree $d$ is real.

2. If $\exp(G,d)$ is even, then for $\chi \in \text{Irr}(G)$ with $\chi(1) = d$, there is a pair $(\chi, \overline{\chi})$ which is either real or non-real.

**Proof.** It follows from [12].

3. Solvability of TR-groups. In this section we will show the proof of Theorem 1.2.

Let $cd_e(G) = \{[\chi_1, 1], [\chi_2, \chi_2(1)], \ldots, [\chi_s, \chi_s(1)]\}$, where $\chi_i \in \text{Irr}_{rv}(G)$.

To prove Theorem 1.2, the following information is taken from [13].

**Lemma 3.1.** (1) Let $s = p^n$, $p > 2$. Then

if $s \equiv -1 \pmod 4$,

$$cd_{e}(\text{PSL}_2) = \{[\chi_1, 1], [\chi_2, s], [\chi(R^a), s + 1], [\chi(S^b), s - 1]\},$$

$$cd_{e}(\text{SL}_2) = \{[\chi_1, 1], [\chi_2, s], [\chi_m, s + 1], [\chi_n, s + 1],$$

$$[\chi(R^a), s - 1], [\chi(S^b), s - 1]\};$$

if $s \equiv 1 \pmod 4$,

$$cd_{e}(\text{PSL}_2) = \{[\chi_1, 1], [\chi_2, s], [\chi_{\mu}, s + 1], [\chi(R^a), s + 1],$$

$$[\chi(S^b), s - 1]\};$$

$$cd_{e}(\text{SL}_2) = \{[\chi_1, 1], [\chi_2, s], [\chi_m, s + 1], [\chi_n, s + 1],$$

$$[\chi_{\mu}, s + 1], [\chi(R^a), s - 1], [\chi(S^b), s - 1]\}.\]$$

(2) Let $s = 2^n$. Then

$$cd_{e}(\text{PSL}_2) = \{[\chi_1, 1], [\chi_2, s], [\chi(R^a), s + 1], [\chi(S^b), s - 1]\}.\]

**Proof.** It follows from ([13], p. 401–403).

Let $\exp(G,d)$ be the number of irreducible characters with the same degree $d$. In order to read at hand, we rewrite Theorem 1.2 here.

**Theorem 3.2.** A finite TR-group is solvable.
Proof. Assume that the result is not true, then we can assume that $G$ is non-sovable, but its maximal subgroups are solvable. Now Lemma 2.2 gives the possibilities for $G$. Thus the following three cases are done with.

Case 1: $\text{PSL}(2,q)$ for $q \geq 4$.

By Lemma 3.1, we have that $\lvert \text{cd}_v(G) \rvert > 2$, a contradiction.

Case 2: $\text{Sz}(2^p)$ for $p$ an odd prime. Let $q = 2^p$. By $[1^4]$, $\exp(\text{Sz}(q), 1) = 1$, $\exp(\text{Sz}(q), q) = 1$ and $\exp(\text{Sz}(q), q^2 + 1) = \frac{q - 2}{2}$. Note that $\frac{q - 2}{2}$ is odd, so by Lemma 2.5, the irreducible characters $\chi_i, i = 1, 2, 3$ are real-valued, where $\chi(1) = 1, \chi_2(1) = q^2, \chi_3(1) = q^2 + 1$. So $\text{Sz}(q)$ is a non-TR-group.

Case 3: $\text{PSL}_3(3)$.

From $([15], p. 13)$, we have that $\chi_1, \chi_2, \chi_3$ are real-valued characters with $\chi_1(1) = 1, \chi_2(1) = 12$ and $\chi_3(1) = 13$. So $\text{PSL}_3(3)$ is a non-TR-group.

It follows from the above three cases that $G$ is solvable. □

4. Non-sovable PTR-groups. In this section, the structures of PTR-groups are determined. Denote by $\max G$ the set of all maximal subgroups of $G$ with respect to its subgroup order divisibility.

The following information is used $[1^5, 1^6]$.

Lemma 4.1. $\text{cd}_v(A_4) = \{[x_1, 1], [x_4, 3]\}$; $\text{cd}_v(S_4) = \{[x_1, 1], [x_2, 1], [x_3, 2], [x_4, 3], [x_5, 3]\}$; $\text{cd}_v(A_5) = \{[x_1, 1], [x_2, 3], [x_3, 3], [x_4, 4], [x_5, 5]\}$.

Lemma 4.2. Let $G$ be a non-abelian PTR-group. Then one of the following holds:

1. $G$ is isomorphic to $\text{PSL}_2(q)$, where either $q = 2^p$ with $p$ a prime or $q = 3^p$ with $p$ an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$;

2. $G$ is isomorphic to $\text{Sz}(2^p)$ with $p$ an odd prime.

Proof. By Lemma 2.1, $H \in \sum G$, $H$ is a TR-group, so by Lemma 1.2, $H$ is solvable. It follows that $G$ is a non-sovable group but its subgroups are solvable. So by Lemma 2.2, the following five cases are considered.

| Table 1 |
|-------------------|---------------|
| $\text{max } \text{PSL}_2(q)$ | Condition |
| $C_1$ | $E_q : C_{q-1/1}$ | $k = \gcd(q - 1, 2)$ |
| $C_2$ | $D_{2(q-1)/1}$ | $q \not\in \{5, 7, 9, 11\}$ |
| $C_3$ | $D_{2(q+1)/1}$ | $q \not\in \{7, 9\}$ |
| $C_4$ | $\text{PSL}_2(q_0).\langle k, b \rangle$ | $q = q_0^b, b$ a prime, $q_0 \equiv 2$ |
| $C_5$ | $A_4$ | $q = p \equiv \pm 1 \pmod{3, 5, 13, 27, 37}$ (mod 40) |
| $\mathcal{S}$ | $A_5$ | $q \equiv 1 \pmod{10}, \mathcal{F}_a = F_p[\sqrt{5}]$ |

Case 1: $\text{PSL}_2(2^p)$ for $p$ a prime.
Then $k = 1$ and let $q = 2^p$,

$$\max \text{PSL}_2(q) = \{E_q : C_{q-1}, D_{2(q\pm 1)}\}.$$  

We see from Lemmas 2.3 and 2.4 that $E_q : C_{q-1}$ and $D_{2(q\pm 1)}$ are TR-groups, so PSL$_2(q)$ is a PTR-group and $G$ is isomorphic to PSL$_2(q)$.

**Case 2:** PSL$_2(3^p)$ for $p$ an odd prime.

Then $k = 2$. Let $q = 3^p$. Then max PSL$_2(q)$ contains possibly $E_q : C_{(q-1)/2}$, $D_{q\pm 1}$, PSL$_2(3)$ and $A_5$ as its members. Notice that PSL$_2(3) \cong A_4$, so Lemmas 2.3, 2.4 and 4.1 force that $E_q : C_{(q-1)/2}$, $D_{q\pm 1}$ and PSL$_2(3)$ are TR-groups but $A_5$ is a non-TR-group. It follows from Table 1 that PSL$_2(q)$ is a PTR-group with $q \not\equiv \pm 1 \pmod{10}$.

**Case 3:** PSL$_2(p)$, for $p$ any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod{5}$.

Then $k = 2$ and $q = p$. By Table 1, max PSL$_2(q)$ has possibly $E_q : C_{(q-1)/2}$, $D_{q\pm 1}$, $S_4$ and $A_4$ as its members. By Lemmas 2.3, 2.4 and 4.1, $E_q : C_{(q-1)/2}$ and $D_{q\pm 1}$ are TR-group but $S_4$ is a non-TR-group. By Table 1, we have $q = p \not\equiv \pm 1 \pmod{8}$, so $q = p \equiv 3, 5, 7 \pmod{8}$. Hypothesis $p^2 + 1 \equiv 0 \pmod{5}$ gives that $q = p \equiv \pm 3 \pmod{8}$. Thus similarly as Case 1, we have that PSL$_2(q)$ is a PTR-group with $q = p \equiv \pm 3 \pmod{8}$.

**Case 4:** $Sz(2^p)$ for $p$ an odd prime.

Let $q = 2^p$. Then from ([18], p. 385), we have max $Sz(q) = \{E_q^{1+1} : C_{q-1}, D_{2(q-1)}, (q \pm \sqrt{2q} + 1) : 4\}$. Note that

$$\text{cd}_{rv}(E_q^{1+1} : C_{q-1}) = \{1, q - 1\}, \text{cd}_{rv}(D_{2(q-1)}) = \{1, 2\},$$

$$\text{cd}_{rv}((q \pm \sqrt{2q} + 1) : 4) = \{1, 4\},$$

so for all $H \in \max Sz(q)$, $H$ is a TR-group by Lemmas 2.3 and 2.4. It follows that $Sz(q)$ is a PTR-group, so $G$ is isomorphic to $Sz(q)$ with $q = 2^p$, $p$ an odd prime.

**Case 5:** PSL$_3(3)$.

By ([15], p. 13), $S_4 \in \max$ PSL$_3(3)$. Note from Lemma 4.1 that $\text{cd}_{rv}(S_4) = \{1, 2, 3\}$, so $S_4$ is a non-TR-group. It follows that PSL$_3(3)$ is a non-PTR-group.  

For reading easily, we rewrite Theorem 1.4 here.

**Theorem 4.3.** Let $G$ be a non-solvable PTR-group. Then $G$ is isomorphic to either PSL$_2(q)$, where either $q = 2^p$ with $p$ a prime or $q = 3^p$ with $p$ an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$ or $Sz(2^p)$ with $p$ an odd prime.

**Proof.** The non-solvability of $G$ gives that there is a normal series

$$1 \leq H \leq K \leq G$$

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such that $K/H$ is isomorphic to a direct product of isomorphic non-abelian simple groups and $|\text{Out}(K/H)||G/K|$, where $\text{Out}(M)$ denotes the outer-automorphism group of a group $M$; see [19].

Let $\mathcal{G}$ be the set of groups consisting of the following groups:

(i) $\text{PSL}_2(q)$, where either $q = 2^p$ with $p$ a prime or $q = 3^p$ with $p$ an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$;

(ii) $\text{Sz}(2^p)$ with $p$ an odd prime.

Then $K/H$ is isomorphic to $S \times S \times \cdots \times S$ with $S \in \mathcal{G}$. If $m \geq 2$, then let $M \in \max \mathcal{S}$, $M \times S \times \cdots \times S$ is a maximal subgroup of $S \times S \times \cdots \times S$. Here two cases are dealt with: if $S = \text{PSL}_2(q)$, then by Lemma 3.1, $|\text{cd}_{rv}(S)| \geq 4$; if $S = \text{Sz}(q)$, then by Lemma 2.5 and [14], $\text{cd}_{rv}(\text{Sz}(q)) \supset \{1, q^2, q^2 + 1\}$ (we also can get this result from Case 2 of the proof of Theorem 1.2). It follows that $\text{cd}_{rv}(M \times S \times \cdots \times S) \geq 4$, so $M \times S \times \cdots \times S$ is a non-TR-group. Now $m = 1$ and

$G$ is a non-almost simple group

as when $S < G$, $|\text{cd}_{rv}(S)| \geq 4$. In what follows, we will divide the proof into two cases in view of $S \in \mathcal{G}$.

Case 1: $\text{PSL}_2(q)$, where either $q = 2^p$ with $p$ a prime or $q = 3^p$ with $p$ an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$.

Based on $q$, two subcases are done with.

Subcase 1: $q = 2^p$ with $p$ a prime.

In this case, $k = 1$ and

$\text{GL}_2(q) \cong \text{SL}_2(q) \cong \text{PSL}_2(q)$;

see [20]. We know from ([15], p. xvi) that the Schur multiplier of $\text{PSL}_2(q)$ with $q \geq 8$ is trivial and the Schur multiplier of $\text{PSL}_2(4)$ is of order 2.

Let $q = 4$, then $G/H$ is isomorphic to $\text{PSL}_2(4)$. Now by ([17], Chap. II, Theorem 6.10), $G'/H \cong \text{PSL}_2(4) \cong \text{PSL}_2(5)$. Note that $[G', H] \leq C_2$, where $C_2 = Z(\text{SL}_2(5))$, so either $G \cong \text{PSL}_2(4) \times H$ when $[G', H] = 1$ or $G \cong \text{SL}_2(5) \times H$ when $[G', H] = C_2$.

Let $G \cong \text{PSL}_2(4) \times H$. If $H > 1$, then $\text{PSL}_2(4) < G$, so $G$ is a non-PTR-group as $|\text{cd}_{rv}(\text{PSL}_2(4))| = 4$. Thus $H = 1$ and $G \cong \text{PSL}_2(4)$.

Let $G \cong \text{SL}_2(5) \times H$. Then $H \geq C_2$. If $H > C_2$, then $\text{SL}_2(5) < G$, so by Lemma 3.1, $[\text{cd}_H(\text{SL}_2(5))] = 5$ and $\text{SL}_2(5)$ is a non-TR-group. Thus $H = C_2$ and $G \cong \text{SL}_2(5) \times H = \text{SL}_2(5)$. By Table 2, $2.A_4 \in \max \text{SL}_2(5)$. By \[16], $\text{cd}_H(2.A_4) = \{1, 2, 3\}$, and so $2.A_4$ is a non-TR-group. It follows that $\text{SL}_2(5)$ is a non-\(\text{PTR}\)-group.

Now let $q \geq 8$, then $G \cong \text{PSL}_2(q) \times H$ as above similar arguments. If $H \neq 1$, then $\text{PSL}_2(q) < G$ is a TR-group, a contradiction to Lemma 3.1. Thus $H = 1$.

So in this subcase, $G$ is isomorphic to $\text{PSL}_2(q)$ with $q = 2^p$, and $p$ a prime.

Subcase 2: $q = 3^p$ with $p$ an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$.

Then by \[15], p. xvi), the Schur multiplier of $\text{PSL}_2(q)$ with $q \neq 9$ odd is of order two. Note that $G$ is not an almost simple group, so $G/H \cong \text{PSL}_2(q)$, where $q = 3^p$ with $p$ an odd prime and $q \equiv \pm 3 \pmod{10}$ or $q = p$ a prime with $q \equiv \pm 3 \pmod{8}$. Similarly as Subcase 1, we can get that $G$ is isomorphic to $\text{PSL}_2(q) \times H$ or $\text{SL}_2(q) \ast H$.

Let $G \cong \text{PSL}_2(q) \times H$. Let $H \neq 1$. Then $\text{PSL}_2(q) < G$ and $\text{PSL}_2(q)$ is a TR-group, a contradiction to Lemma 3.1. Now $H = 1$ and $G \cong \text{PSL}_2(q)$ as wanted.

Let $G \cong \text{SL}_2(q) \ast H$. Then $H \geq C_2$, where $C_2 = Z(\text{SL}_2(q))$. If $H > C_2$, then $\text{SL}_2(q) \in \sum G$ and by Lemma 2.1, $\text{SL}_2(q)$ is a TR-group, a contradiction to Lemma 3.1. So $H = C_2$ and $G \cong \text{SL}_2(q) \ast H = \text{SL}_2(q)$.

If $q = 3^p$, then by Table 2, $\text{SL}_2(3) \in \sum \text{SL}_2(3^p)$. We know that $\text{SL}_2(3) \cong S_4$.
so by [16] or Lemma 3.1, \(\text{cd}_{rv}(\text{SL}_2(3)) = \{1, 2, 3\}\). It follows that \(\text{SL}_2(3)\) is a non-TR-group and \(\text{SL}_2(3^p)\) is a non-PTR-group.

If \(q = p\) is a prime, then by Table 2, the groups \(2.A_4\), \(2.S_4\) and \(2.A_5\) are involved possibly in \(\max \text{SL}_2(q)\), so there is no prime satisfying the three inequations

\[
q = p \not\equiv 1 \pmod{8}
q = p \not\equiv 1 \pmod{10}
q = p \not\equiv \pm 3, 5, \pm 13 \pmod{40}.
\]

Thus \(\text{SL}_2(q)\) with \(q\) a prime is a non-PTR-group.

**Case 2:** \(Sz(2^p)\) with \(p\) an odd prime.

Now we get from ([15], p. xvi) that the Schur multiplier of \(Sz(2^p)\) is trivial when \(q > 8\) and the Schur multiplier of \(Sz(q)\) is of order 4 when \(q = 8\). As \(G\) is not an almost simple group, we have that \(G/H \cong Sz(q)\).

If \(q = 8\), then we obtain from ([15], p. 28) that

\[
2.2^{3+3} : 7 \in \sum 2.Sz(8)
2.2^{3+3} : 7 \in \sum 2.Sz(8).
\]

By [16], \(\text{cd}_{rv}(2.2^{3+3} : 7) = \{1, 7, 8\} = \text{cd}_{rv}(2.2^{3+3} : 7)\). It follows that if \(G\) has a subgroup \(2Sz(8)\) or \(2^2Sz(8)\), then \(2Sz(8)\) has a non-TR-subgroup \(2.2^{3+3} : 7\) and \(2^2Sz(8)\) has a non-TR-subgroup \(2.2^{3+3} : 7\), respectively. Thus \(G\) is not a semisimple group. We see that \(G\) is also a non-almost simple group, so \(G\) is isomorphic to \(Sz(8) \times H\). If \(H \neq 1\), then by ([15], p. 28), \(\text{cd}_{rv}(Sz(8)) = \{1, 35, 64, 65, 91\}\), so \(Sz(8)\) is a non-TR-group, a contradiction to Lemma 2.1. So \(H = 1\) and \(G\) is isomorphic to \(Sz(8)\) as needed.

If \(q > 8\), then \(G/H \cong Sz(q)\) as \(G\) is not an almost simple group. We see that the Schur multiplier of \(Sz(q)\) is trivial, so \([G', H] = 1\) and \(G\) is isomorphic to \(Sz(q) \times H\). If \(H \neq 1\), then \(Sz(q) \in \sum G\), so \(Sz(q)\) is a TR-group. We know from [14] and Lemma 2.5 that \(cd_{rv}(Sz(q)) \supseteq \{1, q^2, q^2+1\}\), so \(Sz(q)\) is a non-TR-group. Now \(H = 1\) gives that \(G\) is isomorphic to \(Sz(q)\) as desired. \(\square\)

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