

CHAMFER DISTANCE ON THE FOUR-DIMENSIONAL FACE-CENTRED CUBIC GRID

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Abstract

Chamfer distances are step-based distances on various grids. The four-dimensional FCC grid is the extension of the usual face-centred cubic grid into four dimensions. There are two types of steps, and thus, two weights are used. Operational research, namely linear programming and Gomory cut are applied to describe optimal paths and hence, their weighted lengths, the chamfer distance.

Key words: lattices, shortest paths, operational research, nontraditional grids

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1. Introduction. The body-centred cubic (BCC) [5,7] and face-centred cubic (FCC) [4] grids are two of the important non-traditional grids appearing in nature. These grids are in a way 3D extensions of the hexagonal grid, since they are both lattices; actually the FCC grid gives a densest packing of same size spheres similarly as hexagonal grid gives densest circle packing [3]. In this paper, we are interested in four-dimensional generalization of the FCC grid. The points of this grid can be seen as the points of a four-dimensional hypercubic grid and the centre points of its two-dimensional faces, thus, in fact, they form eight hypercubic subgrids. Weighted (aka chamfer) distances [1] are based on paths with minimal

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weights where various steps may have different weights. In our grid, there are two types of steps between neighbour points (type-*a* for the closest neighbours inside a hypercubic subgrid and type-*d* between closest neighbours of different subgrids), thus we use two weights computing optimal paths and distance in a somewhat similar method based on operational research as we have used for the BCC grid in [6]. In fact, we are giving analytical formulae for the distance via analysing how to build shortest (we also refer to those as optimal) paths. We should underline that our approach to compute a shortest path and the distance gives formulae such that it works in constant time, i.e., $\mathcal{O}(1)$, while the general Dijkstra algorithm (to obtain a shortest path) would explore a large fragment of the grid working on $\mathcal{O}(n^4)$, where n is the largest coordinate difference between the points we analyse. In this paper, we show that based on operational research techniques chamfer distance can be computed efficiently on the four-dimensional FCC grid. Analogous results for the three dimension case may be seen in [8].

2. The linear programming problem. Linear programming problem is an optimization problem where the mathematical form of the objective function and the constraints are linear. The problem has the following standard form

$$\begin{aligned} \min f^\top x, \\ Sx = h, \\ x \geq 0, \end{aligned}$$

where S is an $m \times n$ matrix, h is an m -dimensional vector, f is an n -dimensional vector of constants, and x is the n -dimensional vector of variables. Keep in mind that the nonnegativity constraints are linear inequalities and it may be necessary to allow that some constraints are inequalities and some variables are “free”, i.e. these may have positive, negative, and zero values as well.

2.1. The simplex method. Recall that *basic feasible solutions* are the extreme points of the polyhedral set of the linear programming problem. For the reader’s convenience, assume that the rank of matrix S is m . Let B be a subset of the columns of S such that the vectors of B form a basis. The matrix obtained from the elements of B is a square $m \times m$ matrix which is also denoted by B . Assume that matrix S and vector x are partitioned accordingly, i.e. $S = (B, N)$, where N is the submatrix of B containing all non-basic columns of S and $x^\top = (x_B^\top, x_N^\top)$. The basic solution of basis B is obtained if x satisfies the equation system and its x_N part is 0. It can be calculated as follows:

$$Sx = (B, N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = Bx_B + Nx_N = b.$$

Thus,

$$(1) \quad x_B = B^{-1}b - B^{-1}Nx_N.$$

Hence, the x_B part of the basic solution is $B^{-1}b$ as x_N is the zero vector. The basic solution is *feasible* if $B^{-1}b \geq 0$.

The optimality condition of the simplex method is as follows. Let f_B be the vector consisting of the basic components of the objective function. The current basic feasible solution is optimal, if the inequality

$$(2) \quad f_B^\top B^{-1}g_j - f_j \leq 0$$

holds for all non-basic columns of S . In other words, (2) must hold for all columns g_j in N . If the inequality (2) is violated by a column g_j , then g_j can replace a vector of the basis such that the objective function value either improves or remains the same.

3. The FCC grid and its four-dimensional extension. The three-dimensional FCC grid has four cubic subgrids. One of these cubic subgrids consists of points having all even numbered coordinates and the other three contain two odd and one even coordinates. Geometrically in Fig. 1 (left) we may see a unit cube of the FCC grid having one point at each corner and one at the centre of each of its faces.

Four-dimensional extension of the FCC grid (this latter is abbreviated as FCC4 lattice), can be described as the set of points in $\mathbb{Z}^4$ that have even number of odd coordinates, or equivalently have an even sum of coordinates. Figure 1 (right) gives a visual representation of the hyper unit cube FCC4 lattice. FCC4 consists of eight hypercubic subgrids, where one of this hypercubic subgrid consists of points having four even valued coordinates, here denoted by C_{eeee} , one hypercubic subgrid consists of points having four odd valued coordinates, here denoted by C_{oooo} , and the other six hypercubic subgrids consist of two even and two odd coordinates. Generally, let C_{xyzt} ($x, y, z, t \in \{e, o\}$) denote the subgrid where the 1st/2nd/3rd/4th element is odd or even according to x, y, z, t , respectively. Thus, e.g., C_{eeoo} contains the points where the first two coordinates are even, the other two coordinates are odd. A grid point of C_{xyzt} is referred to as an $xyzt$ -type point or it has parity $xyzt$ where $x, y, z, t \in \{e, o\}$, thus for instance, $eeoo$ -type points are

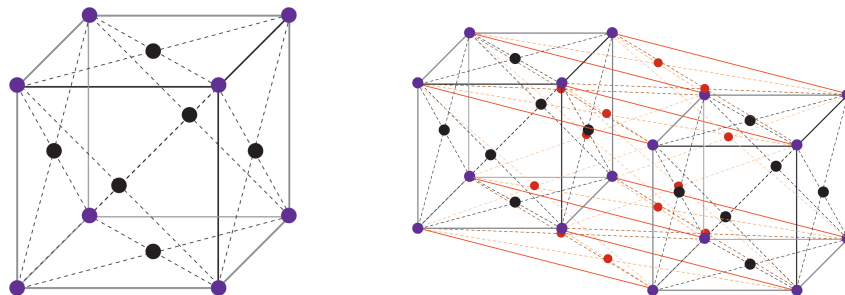


Fig. 1. A unit cube of the FCC grid (left) and a unit hypercube of the 4D extension of the FCC grid (right)

those which belong to C_{eeoo} . The points $(0, 0, 1, 1)^\top$, $(0, 0, 1, -1)^\top$, $(0, 0, -1, 1)^\top$ are some examples of $eeoo$ -type points. We can also use the notation $\bar{x}$ to denote the “complement” of x , i.e., $C_{\bar{x}y\bar{z}\bar{t}}$ denotes C_{oeoe} whenever $x = e$, $y = e$, $z = o$ and $t = o$. One can also see that FCC4 includes the four-dimensional BCC grid points, since it has both C_{eeee} and C_{oooo} . It is also worth to note that the FCC does not contain a subgrid of points having all coordinates odd but FCC4 has.

On the other hand, as we shall see in the next subsection the neighbourhood relation in FCC4 does not include all the neighbourhoods of the four-dimensional BCC grid.

3.1. Steps in the FCC4. In this section we describe the neighbourhood relation defined on FCC4. Since we are interested in paths in this grid the movements from a point to one of its neighbours are called steps.

There are two types (type- a and type- d) of steps between two grid points in the FCC4. Type- a steps are within the same hypercubic subgrid and they can be considered steps on the edges of the unit hypercube of the grid (the size of the unit hypercube is $2 \times 2 \times 2 \times 2$ to support the description by using only integers). Type- d steps are *diagonal* steps between two different hypercubic subgrids. For instance, $(0, 0, 0, 2)^\top$ is a type- a step and $(0, 1, 0, 1)^\top$ is a type- d step. Consider $eeee$ -type point $(2, 2, 2, 2)^\top$. We now see that stepping from the point $(2, 2, 2, 2)^\top$ using the type- a step $(0, 2, 0, 0)^\top$ we arrive to the $eeee$ -type point $(2, 4, 2, 2)^\top$ from which we can deduce that parity of the starting and terminal point is the same. Whereas, stepping from the point $(2, 2, 2, 2)^\top$ using a type- d step $(0, 1, 0, 1)^\top$ we have a terminal point $(2, 3, 2, 3)^\top$ which is $eo eo$ -type and we see that parity (i.e. the type) of the starting and terminal point is not the same.

More formally, every step of FCC4 is an integer vector such that the sum of the absolute values of the coordinates is 2. Thus, only one or two coordinates of a step can be non-zero. The type- a steps have one non-zero coordinate. The vectors of the steps are: $a_i = (2\delta_{1i}, 2\delta_{2i}, 2\delta_{3i}, 2\delta_{4i})^\top$ for $1 \leq i \leq 4$, where $\delta_{ij} = 1$ iff $i = j$, otherwise it is 0. Further, $a_i = -a_{i-4}$ for $5 \leq i \leq 8$, the steps in any of the subgrids. Type- d steps have two non-zero coordinates. The steps d_j ($9 \leq j \leq 12$) are between C_{xyzt} and $C_{\bar{x}yzt}$, thus the first two elements are nonzeros. If $13 \leq j \leq 16$, then the steps are between C_{xyzt} and $C_{\bar{x}\bar{y}zt}$. If $17 \leq j \leq 20$, then the steps are between C_{xyzt} and $C_{\bar{x}y\bar{z}\bar{t}}$. If $21 \leq j \leq 24$, then the steps are between C_{xyzt} and $C_{x\bar{y}\bar{z}\bar{t}}$. If $25 \leq j \leq 28$, then the steps are between C_{xyzt} and $C_{x\bar{y}z\bar{t}}$. Finally, if $29 \leq j \leq 32$, then the steps are between C_{xyzt} and $C_{x\bar{y}\bar{z}t}$, thus the last two coordinates are nonzeros. The nonzero elements are both positive if $(j \equiv 1) \pmod{4}$, the first nonzero element is 1, the second is -1 if $(j \equiv 2) \pmod{4}$, the first is -1 , the second is 1 if $(j \equiv 3) \pmod{4}$, and finally, both are -1 if j is a multiple of 4. For instance, $a_7 = (0, 0, -2, 0)^\top$, $d_{11} = (-1, 1, 0, 0)^\top$ and $d_{27} = (0, -1, 0, 1)^\top$.

The vectors $a_1, \dots, a_8$ represent type- a step and the vectors $d_9, \dots, d_{32}$ represent type- d step. It is assumed that type- a and type- d steps have weights (lengths) u and w , respectively where $u, w > 0$.

4. The shortest path problem on FCC4. A path is a sequence of adherent steps from one grid point to another. We investigate shortest path problem between two grid points, where the starting point is the origin, i.e. the point $(0, 0, 0, 0)^\top$. Before introducing optimization model for this problem, we consider the following remarks with some examples. Assume that the cost of each type- a step is u and the cost of each type- d step is w . Assume that $u, w > 0$. Consider a path starts from the point $(0, 0, 0, 0)^\top$ to the point $(2, 0, 0, 2)^\top$. One can see that the starting and the target points are both $eeee$ -types and thus are in the same hypercubic subgrid C_{eeee} . Therefore, if we built a path using the steps $(2, 0, 0, 0)^\top$ and $(0, 0, 0, 2)^\top$ each once, the total cost of this path is $2u$ and can be described as follows

$$(3) \quad (2, 0, 0, 0)^\top + (0, 0, 0, 2)^\top = (2, 0, 0, 2)^\top.$$

We may also have an alternative path from the origin to the point $(2, 0, 0, 2)^\top$ stepping with the step $(1, 0, 0, 1)^\top$ twice. This time the path goes through $oeeo$ -type of point of different hypercubic subgrid. The cost of the path is $2w$ and it can be written as follows

$$(4) \quad 2(1, 0, 0, 1)^\top = (2, 0, 0, 2)^\top.$$

Considering equations (3) and (4) we see that the first path is shorter if $u < w$. Let us now see what happens if we allow fractional steps. For instance,

$$(5) \quad \frac{1}{2}(2, 0, 0, 0)^\top + \frac{1}{2}(0, 2, 0, 0)^\top = (1, 1, 0, 0)^\top.$$

Left-hand side and the right-hand side of equation (5) cost u and w , respectively. Thus if fractional steps are used together with the condition $u < w$, then no steps between two hypercubic subgrids can be contained in a minimal path.

Let us now introduce optimization model of the minimal path. FCC4 has a highly symmetric structure. We assume that the minimal path starts from the origin, i.e. $(0, 0, 0, 0)^\top$ to the target point:

$$(6) \quad \mathbf{b} = (p, q, r, s)^\top.$$

Then the optimization model of the minimal path is given in the form as follows:

$$(7) \quad \min u \sum_{j=1}^8 x_j + w \sum_{j=9}^{32} x_j,$$

$$(8) \quad \sum_{j=1}^8 a_j x_j + \sum_{j=9}^{32} d_j x_j = \mathbf{b},$$

$$(9) \quad x_j \geq 0, \quad j = 1, 2, \dots, 32,$$

$$(10) \quad x_j \text{ is integer, } j = 1, 2, \dots, 32.$$

Model (7)–(10) is integer programming model, since constraint (10) claims variables must take integer values. First problem (7)–(9) is solved applying the simplex method which is a well-known algorithm of the theory of linear optimization. If the optimal solution is not integer, then the Gomory method of integer programming is used.

In total we have 32 steps and we may obtain a matrix that consists of the columns of these 32 steps which tells us that there are $\binom{32}{4} = 35\,960$ potential candidates to be a basis which leads to an optimal solution. One can immediately see that some of these candidates are not bases since the columns are not linearly independent. However, since the number of the candidates is too high, we use Theorem 1 of [6] for $m = 4$ to filter out many candidates. In particular, using this filtration we see that only 1960 candidates remain out of 35 960.

5. The optimal solutions of the linear programming relaxation.

Proposition 1. *The basis $\{a_1, a_2, a_3, a_4\}$ is optimal if and only if*

$$(11) \quad u - w \leq 0.$$

Proof. Let B denote the given basis, then the basic solution is $B^{-1}b = (\frac{p}{2}, \frac{q}{2}, \frac{r}{2}, \frac{s}{2})^\top$ and it is always non-negative. Let I_4 denote the 4×4 unit matrix, then one can easily see that $B^{-1} = \frac{1}{2}I_4$. Moreover, basic part of the vector of the objective function, i.e., f_B , is the four-dimensional vector $(u, u, u, u)^\top$. It follows that $f_B^\top B^{-1} = (\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2})$. It is clear that non-basic elements are the vectors $a_5, \dots, a_8, d_9, \dots, d_{32}$. Following this, the optimality conditions (2) have the following forms:

$$(12) \quad f_B^\top B^{-1}a_j - f_j = -u - u = -2u \leq 0, \text{ where } j = 5, \dots, 8$$

and

$$(13) \quad f_B^\top B^{-1}d_j - f_j \leq \max\{u, 0, -u\} - w = u - w \leq 0, \text{ where } j = 9, \dots, 32.$$

We know that u and w are positive. It follows that if $u \leq w$, then (12)–(13) are true for all columns. $\square$

Proposition 2. *The basis $\{a_1, d_9, d_{13}, d_{17}\}$ is optimal if and only if*

$$(14) \quad w \leq u \leq 2w, \quad q + r + s \leq p.$$

Proof. Let B denote the given basis, then the basic solution is $B^{-1}b = (\frac{p-q-r-s}{2}, q, r, s)^\top$ and it is non-negative if and only if $q + r + s \leq p$. Moreover

$$B^{-1} = \begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, $f_B = (u, w, w, w)^\top$. It follows that $f_B^\top B^{-1} = (\frac{u}{2}, w - \frac{u}{2}, w - \frac{u}{2}, w - \frac{u}{2})$. The optimality conditions (2) have the following forms:

$$(15) \quad f_B^\top B^{-1} a_j - f_j \leq \max\{2w - 2u, -2w, -2u\}, \text{ where } j = 2, \dots, 8$$

and

$$(16) \quad f_B^\top B^{-1} d_j - f_j \leq \max\{u - 2w, -u, -2w, w - u, -w, u - 3w\},$$

where $j = 10, 11, 12, 14, 15, 16, 18, \dots, 32$.

Right-hand side of (15) is non-positive if $w \leq u$ and right-hand side of (16) is non-positive if $w \leq u \leq 2w$. $\square$

Proposition 3. *The basis $\{d_9, d_{10}, d_{13}, d_{17}\}$ is optimal if and only if*

$$(17) \quad 2w \leq u, \quad q + r + s \leq p.$$

Proof. Let B denote the given basis, then the basic solution is $B^{-1}b = (\frac{p-q-r-s}{2}, \frac{p+q-r-s}{2}, r, s)^\top$ and it is non-negative if and only if $q + r + s \leq p$. Moreover

$$B^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, $f_B = (w, w, w, w)^\top$. It follows that $f_B^\top B^{-1} = (w, 0, 0, 0)$. The optimality conditions (2) have the following forms:

$$(18) \quad f_B^\top B^{-1} a_j - f_j \leq \max\{2w - u, -u, -2w - u\}, \text{ where } j = 1, \dots, 8$$

and

$$(19) \quad f_B^\top B^{-1} d_j - f_j \leq \max\{-2w, -w, 0\} = 0,$$

where $j = 11, 12, 14, 15, 16, 18, \dots, 32$.

Right-hand side of (18) is non-positive if $2w \leq u$ and right-hand side of (19) is 0. $\square$

Proposition 4. *The basis $\{d_9, d_{13}, d_{17}, d_{21}\}$ is optimal if and only if*

$$(20) \quad w \leq u, \quad p \leq q + r + s.$$

Proof. Let B denote the given basis, then the basic solution is $B^{-1}b = (\frac{p+q-r-s}{2}, \frac{p-q+r-s}{2}, s, \frac{q+r+s-p}{2})^\top$ and it is non-negative if and only if $p \leq q + r + s$. Moreover

$$B^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 \\ -1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}.$$

Furthermore, $f_B = (w, w, w, w)^\top$. It follows that $f_B^\top B^{-1} = (\frac{w}{2}, \frac{w}{2}, \frac{w}{2}, \frac{w}{2})$. The optimality conditions (2) have the following forms:

$$(21) \quad f_B^\top B^{-1} a_j - f_j \leq \max\{w - u, -w - u\}, \text{ where } j = 1, \dots, 8$$

and

$$(22) \quad f_B^\top B^{-1} d_j - f_j \leq \max\{-2w, -w, 0\} = 0,$$

where $j = 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, \dots, 32$.

Right-hand side of (21) is non-positive if $w \leq u$ and right-hand side of (22) is 0. $\square$

Theorem 1. *The basic feasible solution of one of the bases $\{a_1, a_2, a_3, a_4\}$, $\{a_1, d_9, d_{13}, d_{17}\}$, $\{d_9, d_{10}, d_{13}, d_{17}\}$, and $\{d_9, d_{13}, d_{17}, d_{21}\}$ is optimal.*

In Table 1 we summarize the potential optimal solutions and the objective function values. Note that the values of the not mentioned variables are 0. Actually, Table 1 gives the final results in case all the variables x_i have integer values. In these cases, a shortest path can be given by any order of the steps specified by the variables x_i (number of steps) and by the base vectors (directions of the steps). For example, in the case the first row is giving the optimal solution, we make first $p/2$ steps by vector a_1 , followed by $q/2$ steps by vector a_2 , again followed by $r/2$ steps by vector a_3 , and finished by $s/2$ steps by vector a_4 . Thus, the distance will be given as the formula $u(p + q + r + s)/2$ shows. However, it may happen that not all the values of the variables x_i are integers, thus we need to continue our computation and we may do it by applying Gomory cuts.

T a b l e 1

The optimal solutions, i.e., distances between the origin and $(p, q, r, s)^\top$

Basis	Variables	The value of the objective function	Optimal condition
$\{a_1, a_2, a_3, a_4\}$	$x_1 = p/2, x_2 = q/2,$ $x_3 = r/2, x_4 = s/2$	$u \frac{p+q+r+s}{2}$	$u \leq w$
$\{a_1, d_9, d_{13}, d_{17}\}$	$x_1 = (p - q - r - s)/2,$ $x_9 = q,$ $x_{13} = r,$ $x_{17} = s$	$u \frac{p}{2} + (2w - u) \frac{q+r+s}{2}$	$w \leq u \leq 2w,$ $q + r + s \leq p$
$\{d_9, d_{10}, d_{13}, d_{17}\}$	$x_9 = (p - q - r - s)/2,$ $x_{10} = (p + q - r - s)/2,$ $x_{13} = r,$ $x_{17} = s$	wp	$2w \leq u,$ $q + r + s \leq p$
$\{d_9, d_{13}, d_{17}, d_{21}\}$	$x_9 = (p + q - r - s)/2,$ $x_{13} = (p - q + r - s)/2,$ $x_{17} = s,$ $x_{21} = (q + r + s - p)/2$	$w(p + q + r + s)/2$	$w \leq u,$ $p \leq q + r + s$

6. Application of the Gomory method. The first exact and still the most general method in integer programming is GOMORY [2]. The method solves the linear programming relaxation of the problem in each iteration. If the optimal solution is not integer, then a cut, i.e. a linear inequality, is introduced. It cuts the non-integer optimal solution. The Gomory method obtains the integer optimal solution in finitely many steps. The details of the complicated method are not discussed here, as the general discussion is too long and the particular steps of the method can be explained directly on the grid.

In the last three rows of Table 1 all values of the optimal solutions are integer, because the number of odd values from $(p, q, r, s)^\top$ are even. However, the solution of the first basis $\{a_1, a_2, a_3, a_4\}$ is integer if all coordinates of b are even.

If two out of p, q, r , and s are odd and the other two are even number, then the optimal solution of the basis $\{a_1, a_2, a_3, a_4\}$ is not integer. (It means that the starting point and the target point are in different lattices. Thus at least one step is needed between the lattices.) As variables $x_9, \dots, x_{32}$ give the numbers of the used steps between the lattices, the inequality

$$(23) \quad x_9 + \dots + x_{32} \geq 1$$

is obtained. The optimal solution of the linear programming relaxation does not satisfy it. The optimality condition of the basis is $u \leq w$. It means that under this condition the left-hand side of inequality (23) cannot be greater than 1 otherwise the solution can be improved. Assume that the coordinates p and q of the target points are odd, then the solution $x_1 = (p-1)/2, x_2 = (q-1)/2, x_3 = r/2, x_4 = s/2, x_9 = 1$ is integer and optimal. The value of the objective function is $u(p+q+r+s-2)/2 + w$.

If all of p, q, r , and s are odd, then the optimal solution of the basis $\{a_1, a_2, a_3, a_4\}$ is also not integer. (It means that the starting point and the target point are in different lattices, but at least two steps are needed between the lattices.) Now the inequality

$$(24) \quad x_9 + \dots + x_{32} \geq 2$$

is obtained. The left-hand side of inequality (24) cannot be greater than 2, again. For example the solution $x_1 = (p-1)/2, x_2 = (q-1)/2, x_3 = (r-1)/2, x_4 = (s-1)/2, x_9 = 1, x_{29} = 1$ is integer and optimal. The value of the objective function is $u(p+q+r+s-4)/2 + 2w$.

Since in each possible case we have computed an optimal solution that uses integer number of steps, both the shortest path problem and the distance computation problems have been solved. Now, depending on the cases, the appropriate formulae for the number of steps and for the distance can be computed in $O(1)$.

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