ANOTHER CLASS OF CR-SLANT WARPED PRODUCT SUBMANIFOLDS IN NEARLY TRANS-SASAKIAN MANIFOLDS

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Abstract

In this paper, we study a CR-slant warped product submanifold of the type $B \times \varphi \, M_{\theta}$, where $M_{\theta}$ is a slant and $B$ is a CR-product submanifold in a nearly trans-Sasakian manifold. We obtain an inequality for the squared norm of the second fundamental form in two cases, depending on the structure vector field affiliation, in such warped products. The equality case is also considered.

Key words: warped products, CR-warped product, CR-slant warped product, slant submanifolds, nearly Sasakian, nearly Kenmotsu, nearly trans-Sasakian manifolds

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1. Introduction Sahin introduced the notion of CR-slant warped products under the name of skew CR-warped products in Kaehler manifolds in [1]. After that, Chen, Uddin and Al-Solamy worked on the pointwise CR-slant warped product of Kaehler manifolds in [2]. Also, the CR-slant warped product was studied in nearly cosymplectic manifolds [3].

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In nearly trans-Sasakian manifold, two types of CR-slant warped product were considered in \cite{1}. Here, we study a new type of a CR-slant warped product of the form \( B \times f M_\theta \), where \( B \) is a CR-product submanifold and \( M_\theta \) is a slant submanifold of nearly trans-Sasakian manifolds. We obtain a geometric inequality which relates the second fundamental form and the warping function in two different cases depending on the structure vector field affiliation. Also, the case where the equality holds is discussed.

2. Preliminaries

Let \( \bar{M} \) be a \((2m+1)\)-dimensional almost contact metric manifold with almost contact structure \((\phi, \xi, \eta)\), where \( \phi \) is a \((1,1)\) tensor field, \( \xi \) is a vector field and \( \eta \) is a 1-form satisfying the following properties \cite{2}

\begin{align}
(2.1) \quad & \phi(\xi) = 0, \quad \phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(\phi) = 0, \\
(2.2) \quad & g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \text{and} \quad \eta(X) = g(X, \xi)
\end{align}

for any \( X, Y \in \Gamma(T\bar{M}) \). The structure \((\phi, \xi, \eta, g)\) is called an almost contact metric structure. An almost contact metric manifold \( \bar{M} \) with almost contact metric structure \((\phi, \xi, \eta, g)\) is called nearly trans-Sasakian manifold if \cite{3}

\begin{equation}
(2.3) \quad (\bar{\nabla}_X \phi) Y + (\bar{\nabla}_Y \phi) X = \alpha (2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) - \beta (\eta(Y)\phi X + \eta(X)\phi Y)
\end{equation}

for some functions \( \alpha \) and \( \beta \) on \( \bar{M} \) and so, the trans-Sasakian manifold \( \bar{M} \) is called of type \((\alpha, \beta)\). On \( \bar{M} \), the covariant derivative of \( \phi \) is defined as

\begin{equation}
(2.4) \quad (\bar{\nabla}_X \phi) Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y,
\end{equation}

for any \( X, Y \in \Gamma(T\bar{M}) \). Let \( M \) be an \( n \)-dimensional submanifold of \( \bar{M} \). Then, for any \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), the Gauss–Weingarten formulas are given by \cite{4}

\begin{equation}
(2.5) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + \nabla^\perp_X N,
\end{equation}

respectively, where \( h \) is the second fundamental form and \( A_N \) is the shape operator of \( M \) which are related as \cite{4}

\begin{equation}
(2.6) \quad g(h(X, Y), N) = g(A_N X, Y),
\end{equation}

where the induced metric on \( M \) from \( g \) is also denoted by \( g \). The gradient of a smooth function \( f \) is denoted by \( \text{grad} \, f \) and is defined by \cite{4}

\begin{equation}
(2.7) \quad g(\text{grad} \, f, X) = X(f) \quad \text{and so} \quad \| \text{grad} \, f \|^2 = \sum_{i=1}^n (E_i(f))^2
\end{equation}

for any \( X \in \Gamma(TM) \), where \( \{E_1, \ldots, E_n\} \) is an orthonormal frame on \( M \). For any point \( p \in M \), let \( \{E_1, \ldots, E_n, \ldots, E_{2m+1}\} \) be an orthonormal frame for the tangent
space $T_pM$, where $E_1, \ldots, E_n$ are tangent to $M$ at $p$. The second fundamental form $h$ is defined as follows [5]

\[(2.8) \quad \|h\|^2 = \sum_{r=n+1}^{2n+1} \sum_{i,j=1}^{n} g(h(E_i, E_j), E_r)^2.\]

The submanifold $M$ is said to be totally geodesic if $h(X, Y) = 0$ and totally umbilical if $h(X, Y) = g(X, Y)H$, where $H$ is the mean curvature on $M$ [6]. For any $X \in \Gamma(TM)$, the vector field $\phi X$ decomposes into tangential and normal components, respectively, as

\[(2.9) \quad \phi X = PX + FX.\]

The submanifold $M$ is called invariant (anti-invariant) if

\[\phi(T_pM) \subseteq T_pM \quad (\phi(T_pM) \subseteq T^\perp_pM)\]

for each $p \in M$. Let $D_T$ and $D_\perp$ denote the invariant and anti-invariant distributions on $M$, respectively. If $TM = D_T \oplus D_\perp \oplus \langle \xi \rangle$, then $M$ is called a CR-submanifold [9]. Now, let $X \in T_pM$ such that it is not proportional to $\xi$. Then, the angle between $\phi X$ and $T_pM$ is called a slant angle and is denoted by $\theta$. If $\theta$ is constant, then $M$ is called a slant submanifold [10]. Moreover, in [11] it is shown that $M$ is slant if and only if there exists a constant $\lambda \in [0, 1]$, which satisfies

\[(2.10) \quad P^2 = \lambda(-Id + \eta \otimes \xi).\]

Furthermore, $\theta$ is the slant angle and satisfies $\lambda = \cos^2 \theta$. Also, the following relations are consequences of (2.10), for any $X, Y \in \Gamma(TM)$,

\[(2.11) \quad g(PX, PY) = \cos^2 \theta (g(X, Y) - \eta(X)\eta(Y)),\]

\[(2.12) \quad g(FX, FY) = \sin^2 \theta (g(X, Y) - \eta(X)\eta(Y)).\]

Now, let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds and $f$ be a positive smooth function on $M_1$. Then the warped product manifold $M = M_1 \times_f M_2$ is a manifold with Riemannian metric $g = g_1 + f^2g_2$. The function $f$ is called the warping function on $M$ [12]. For any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, we have

\[(2.13) \quad \nabla_X Z = \nabla_Z X = (X \ln f)Z.\]

Furthermore, on warped product manifold $M = M_1 \times_f M_2$, $M_1$ is totally geodesic in $M$ and $M_2$ is totally umbilical in $M$ [5].
3. Basic results. Let \( M \) be a submanifold tangent to \( \xi \) of an almost contact metric manifold \( \bar{M} \). Let \( M_T, M_\perp \) and \( M_\theta \) be invariant, anti-invariant and slant submanifolds of \( \bar{M} \), respectively. \( M \) is said to be a CR-slant warped product submanifold if \( M \) is a warped product of one of the form \( B_1 \times f_1 M_T \), where \( B_1 = M_\perp \times M_\theta \), \( B_2 \times f_2 M_\perp \), where \( B_2 = M_T \times M_\theta \), or \( B \times f M_\theta \), where \( B = M_T \times M_\theta \).

In nearly trans-Sasakian manifold, the first two forms of CR-slant warped products were studied in [4]. Here, we study the other type, where \( M = B \times f M_\theta \).

The tangent bundle of the CR-slant warped product submanifold is decomposed as

\[
TM = DT \oplus D_\perp \oplus D_\theta \oplus \langle \xi \rangle.
\]

For the CR-slant warped product \( M = B \times f M_\theta \) of a nearly trans-Sasakian manifold \( \bar{M} \), we have two cases: the first is when \( \xi \) is tangent to \( B \) and the second is when \( \xi \) is tangent to \( M_\theta \). The following theorem deals with the second case.

**Theorem 3.1.** There does not exist any proper warped product submanifold \( M = B \times f M_\theta \) of a nearly trans-Sasakian manifold \( \bar{M} \) of type \((\alpha, \beta)\) if \( \xi \) is tangent to the proper slant submanifold \( M_\theta \) of \( \bar{M} \), where \( B \) is any Riemannian submanifold of \( \bar{M} \), unless \( \bar{M} \) is nearly \( \alpha \)-Sasakian.

**Proof.** Let \( X \in \Gamma(TB) \). Since \( \xi \in \Gamma(TM_\theta) \), then by (2.3), we have

\[
(\bar{\nabla}_X \phi)\xi + (\bar{\nabla}_\xi \phi)X = -\alpha X - \beta \phi X.
\]

Using (2.4), (2.2), (2.5) and (2.13) in the left hand side of (3.2), we obtain

\[
-2\phi h(X, \xi) + \bar{\nabla}_\xi \phi X = -\alpha X - \beta \phi X.
\]

Taking the Riemannian metric \( g \) with \( \phi X \) in (3.3), we get

\[
g(\bar{\nabla}_\xi \phi X, \phi X) = -\beta g(\phi X, \phi X).
\]

By applying metric compatible property, (2.3), (2.5) and (2.13) in the left hand side of (3.4), we obtain \( g(\bar{\nabla}_\xi \phi X, \phi X) = 0 \), and so, \( \beta \|X\|^2 = 0 \). Hence, \( \bar{M} \) is nearly \( \alpha \)-Sasakian manifold.

If we consider \( \xi \) tangent to \( B \), then we have the following lemmas.

**Lemma 3.1.** Let \( M \) be a nearly trans-Sasakian manifold of type \((\alpha, \beta)\) and \( M = B \times f M_\theta \) be a CR-slant warped product submanifold of \( \bar{M} \), where \( B = M_T \times M_\perp \) is a CR-product submanifold tangent to \( \xi \) and \( M_\theta \) is a proper slant submanifold of \( \bar{M} \). Then, for any \( X \in \Gamma(TM_T) \), \( Z, W \in \Gamma(TM_\perp) \) and \( U \in \Gamma(TM_\theta) \), we have

\[
\xi \ln f = \beta,
\]

\[
g(h(X, Z), FU) = -g(h(X, U), \phi Z),
\]

\[
g(h(Z, W), FU) = g(h(U, W), \phi Z).
\]
Proof. Let $U \in \Gamma(TM_{\theta})$, then, by (2.3) and (2.4), we have

\begin{equation}
\bar{\nabla}_U \phi \xi - \phi \bar{\nabla}_U \xi + \nabla_\xi \phi U - \phi \nabla_\xi U = -\alpha U - \beta \phi U.
\end{equation}

Using (2.5) and (2.13) in (3.8) and since $\phi \xi = 0$, we get

\begin{equation}
-\xi (\ln f) \phi U - 2 \phi h(\xi, U) = -\alpha U - \beta \phi U.
\end{equation}

Taking the Riemannian metric $g$ with $\phi U$ in (3.9), we get

\begin{equation}
\xi (\ln f) = \beta,
\end{equation}

which proves (3.5). Now, let $Z \in \Gamma(TM_{\perp})$, then by (2.3), we have

\begin{equation}
(\bar{\nabla}_Z \phi) U + (\bar{\nabla}_U \phi) Z = -\alpha \eta(Z) U - \beta \eta(Z) \phi U.
\end{equation}

Applying (2.4), in the left hand side of (3.10) and using (2.9), (2.5) and (2.13), the equation (3.10) takes the form

\begin{equation}
-Z (\ln f) P U + h(Z, FU) - A_{FU} Z + \nabla_2 Z FU - 2 Z (\ln f) FU - 2 h(Z, U) - A_{\phi Z} U + \nabla_2 \phi Z = -\alpha \eta(Z) U - \beta \eta(Z) \phi U - \beta \eta(Z) FU.
\end{equation}

Taking the Riemannian metric $g$ with $X \in \Gamma(TM_T)$ in (3.11), we get (3.6). Similarly, by taking the Riemannian metric $g$ with $W \in \Gamma(TM_{\perp})$ in (3.11) and using (2.6), we get

\begin{equation}
g(h(Z, W), FU) + g(h(U, W), \phi Z) = 2 g(h(Z, U), \phi W).
\end{equation}

By using polarization identity in (3.12), we get

\begin{equation}2 g(h(Z, W), FU) - 4 g(h(W, U), \phi Z) = -2 g(h(U, Z), \phi W).
\end{equation}

From (3.12) and (3.13), we prove (3.7).

Lemma 3.2. Let $\bar{M}$ be a nearly trans-Sasakian manifold of type $(\alpha, \beta)$ and $M = B \times_f M_{\theta}$ be a warped product CR-slant submanifold of $\bar{M}$, where $B = M_T \times M_{\perp}$ is a CR-product submanifold tangent to $\xi$ and $M_{\theta}$ is a proper slant submanifold of $M$. Then, for any $Z \in \Gamma(TM_{\perp})$ and $U, V \in \Gamma(TM_{\theta})$, we have

\begin{equation}
g(h(U, V), \phi Z) = g(h(Z, U), FV) + \alpha \eta(Z) g(U, V) - \frac{1}{3} [Z (\ln f) - \beta \eta(Z)] g(PU, V).
\end{equation}

Proof. Let $Z \in \Gamma(TM_{\perp})$ and $U, V \in \Gamma(TM_{\theta})$. Taking the Riemannian metric $g$ with $V \in \Gamma(TM_{\theta})$ in (3.11) and using (2.6), we get

\begin{equation}-g(h(Z, V), FU) + 2 g(h(Z, U), FV) - g(h(U, V), \phi Z)
= -\alpha \eta(Z) g(U, V) + [Z (\ln f) - \beta \eta(Z)] g(PU, V).
\end{equation}

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By polarization identity of $U$, $V$ in (3.15), we get

\begin{equation}
- g(h(Z, U), FV) + 2g(h(Z, V), FU) - g(h(U, V), \phi Z)
= -\alpha \eta(Z) g(U, V) + [Z(\ln f) - \beta \eta(Z)] g(PV, U).
\end{equation}

From (3.15) and (3.16), we get (3.14), which proves the lemma. \hfill \Box

From Lemma 3.2, one can deduce the following equations: by interchanging $V$ by $PV$ ($U$ by $PU$) in (3.14) and using (2.10) and (2.11), we get, respectively,

\begin{equation}
\begin{aligned}
&g(h(U, PV), \phi Z) = g(h(Z, U), F PV) + \alpha \eta(Z) g(U, PV) \\
&\quad \quad \quad \quad - \frac{1}{3} \cos^2 \theta [Z(\ln f) - \beta \eta(Z)] g(U, V),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&g(h(PU, V), \phi Z) = -g(h(Z, PU), FV) + \alpha \eta(Z) g(PU, V) \\
&\quad \quad \quad \quad + \frac{1}{3} \cos^2 \theta [Z(\ln f) - \beta \eta(Z)] g(U, V).
\end{aligned}
\end{equation}

Also, by interchanging $V$ by $PV$ in (3.18) and using (2.11), we get

\begin{equation}
\begin{aligned}
&g(h(PU, PV), \phi Z) = g(h(Z, PU), FPV) + \alpha \cos^2 \theta \eta(Z) g(U, V) \\
&\quad \quad \quad \quad + \frac{1}{3} \cos^2 \theta [Z(\ln f) - \beta \eta(Z)] g(U, PV).
\end{aligned}
\end{equation}

From (3.17) and (3.18), one can see

\begin{equation}
\begin{aligned}
&g(h(U, PV), \phi Z) + g(h(PU, V), \phi Z) = g(h(Z, U), FPV) + g(h(Z, PU), FV).
\end{aligned}
\end{equation}

**Lemma 3.3.** Let $\bar{M}$ be a nearly trans-Sasakian manifold of type $(\alpha, \beta)$ and $M = B \times_f M_\theta$ be a warped product CR-slant submanifold of $\bar{M}$, where $B = M_T \times M_\perp$ is a CR-product submanifold tangent to $\xi$ and $M_\theta$ is a proper slant submanifold of $\bar{M}$. Then, for any $X \in \Gamma(TM_T)$ and $U, V \in \Gamma(TM_\theta)$, we have

\begin{equation}
\begin{aligned}
g(h(U, X), FV) = (\phi X(\ln f) - \alpha \eta(X)) g(U, V) \\
\quad \quad \quad \quad - \frac{1}{3} (X(\ln f) - \beta \eta(X)) g(PV, U).
\end{aligned}
\end{equation}

**Proof.** Let $X \in \Gamma(TM_T)$ and $U, V \in \Gamma(TM_\theta)$. From (2.3), we have

\begin{equation}
(\nabla_X \phi)U + (\nabla_U \phi)X = -\alpha \eta(X) U - \beta \eta(X) PU - \beta \eta(X) FU.
\end{equation}

Applying (2.4) in the left hand side of (3.22) and using (2.9), (2.5) and (2.13), it becomes

\begin{equation}
\begin{aligned}
h(X, PU) - A_{FU} X + \nabla_X^1 FU - \phi X(\ln f) U - X(\ln f) PU - 2 X(\ln f) FU \\
\quad \quad \quad \quad - 2 \phi h(X, U) + h(U, \phi X) = -\alpha \eta(X) U - \beta \eta(X) PU - \beta \eta(X) FU.
\end{aligned}
\end{equation}
Taking the Riemannian metric \( g \) with \( V \in \Gamma(TM) \) in (3.23), we derive

\[
(3.24) \quad 2g(h(U, X), FV) - g(h(X, V), FU) = (\phi X(\ln f) - \alpha \eta(X)) g(U, V) \\
+ (X(\ln f) - \beta \eta(X)) g(PU, V).
\]

By switching \( U \) with \( V \) in (3.24), we have

\[
(3.25) \quad 2g(h(V, X), FU) - g(h(X, U), FV) = (\phi X(\ln f) - \alpha \eta(X)) g(U, V) \\
+ (X(\ln f) - \beta \eta(X)) g(PV, U).
\]

From (3.24) and (3.25), we obtain (3.21), which proves the lemma. \( \square \)

From Lemma 3.3, and by using (2.11), we derive the following equations.

That is, by interchanging \( X \) by \( \phi X \), \( U \) by \( PU \) and \( V \) by \( PV \) in (3.21), we get, respectively,

\[
(3.26) \quad g(h(U, \phi X), FV) = - (X(\ln f) - \alpha \eta(\phi X)) g(U, V) \\
- \frac{1}{3} (\phi X(\ln f) - \beta \eta(\phi X)) g(PV, U),
\]

\[
(3.27) \quad g(h(PU, X), FV) = (\phi X(\ln f) - \alpha \eta(X)) g(PU, V) \\
- \frac{1}{3} (X(\ln f) - \beta \eta(X)) \cos^2 \theta g(V, U),
\]

\[
(3.28) \quad g(h(U, X), FPV) = (\phi X(\ln f) - \alpha \eta(X)) g(U, PV) \\
+ \frac{1}{3} (X(\ln f) - \beta \eta(X)) \cos^2 \theta g(V, U).
\]

Again, by interchanging \( U \) by \( PU \) (\( V \) by \( PV \)) in (3.26), we get, respectively,

\[
(3.29) \quad g(h(PU, \phi X), FV) = - (X(\ln f) - \alpha \eta(\phi X)) g(PU, V) \\
- \frac{1}{3} (\phi X(\ln f) - \beta \eta(\phi X)) \cos^2 \theta g(V, U),
\]

\[
(3.30) \quad g(h(U, \phi X), FPV) = - (X(\ln f) - \alpha \eta(\phi X)) g(U, PV) \\
+ \frac{1}{3} (\phi X(\ln f) - \beta \eta(\phi X)) \cos^2 \theta g(V, U).
\]

From (3.29) and by changing \( V \) to \( PV \), we have

\[
(3.31) \quad g(h(PU, \phi X), FPV) = - (X(\ln f) - \alpha \eta(\phi X)) \cos^2 \theta g(U, V) \\
+ \frac{1}{3} (\phi X(\ln f) - \beta \eta(\phi X)) \cos^2 \theta g(PV, U).
\]
From (3.28) and by changing $U$ to $PU$, we have
\[
g(h(PU, X), FPV) = (\phi X(\ln f) - \alpha \eta(X)) \cos \theta g(U, V) \]
\[+ \frac{1}{3} \left( X(\ln f) - \beta \eta(X) \right) \cos \theta g(PV, U). \]
Clearly, from (3.29) and (3.30), and from (3.27) and (3.28), we have, respectively,
\[
g(h(\phi X, FV) + g(h(U, \phi X), FPV) = 0, \]
\[
g(h(\phi X, FV) + g(h(U, X), FPV) = 0. \]
One can also see by (3.21), if $\xi$ is tangent to $M_T$, we have
\[
g(h(U, \xi), FV) = -\alpha g(U, V). \]

4. Main theorem. Let $\tilde{M}$ be a $(2m+1)$-dimensional nearly trans-Sasakian manifold of type $(\alpha, \beta)$ and $M = B \times_f M_\theta$ be a CR-slant warped product submanifold of dimension $n$, where $B = M_T \times M_\perp$ is a CR-product submanifold tangent to $\xi$ with $\dim(M_T) = 2s$ and $\dim(M_\perp) = t$. The slant submanifold $M_\theta$ is of dimension $2l$. Since $TM = D_T \oplus D_\perp \oplus D_\theta \oplus \langle \xi \rangle$, we consider the following orthonormal bases: \( \{ E_1, \ldots, E_s, E_{s+1} = \phi E_1, \ldots, E_{2s} = \phi E_s, E_{2s+1} = \xi \} \) for $D_T$, \( \{ E_1 = E_{2s+2}, \ldots, E_t = E_{2s+1+t} \} \) for $D_\perp$ and \( \{ E_1^* = E_{2s+t+2}, \ldots, E_t^* = E_{2s+t+1+1}, E_{t+1}^* = \sec \theta PE_{1}^*, \ldots, E_{2t}^* = \sec \theta PE_{t}^* \} \) for $D_\theta$. The orthonormal basis in the normal bundle $T^\perp M$ of $D_\perp$ is \( \{ E_{n+1} = E_1 = \phi E_1, \ldots, E_{n+t} = E_t = \phi E_t \} \), the orthonormal basis $FD_\theta$ is \( \{ E_{n+t+1} = E_{t+1} = \csc \theta PE_{1}^*, \ldots, E_{n+t+t+1} = E_{t+1}^* = \csc \theta PE_{1}^*, \ldots, E_{n+t+2t} = E_{t+2t} = \csc \theta \sec \theta PE_{1}^* \} \) and the invariant normal subbundle $v$ is \( \{ E_{n+t+2t+1}, \ldots, E_{2m+1} \} \). We use these frames in the following theorem.

Theorem 4.1. Let $\tilde{M}$ be a $(2m+1)$-dimensional nearly trans-Sasakian manifold of type $(\alpha, \beta)$ and $M = B \times_f M_\theta$ be a $D_\perp \oplus D_\theta$-mixed geodesic warped product CR-slant submanifold of $M$ of dimension $n$, where $B = M_T \times M_\perp$ is a CR-product submanifold and $M_\theta$ is a slant submanifold of $M$, dim $M_\theta = 2l$. Then, the second fundamental form of $M$ satisfies the following inequalities:

(i) If the structure vector field $\xi$ is tangent to $M_T$, we have
\[
\|h\|^2 \geq \frac{2l}{9} \cos^2 \theta \| \text{grad}_\perp (\ln f) \|^2
\[+ 4l \left( \csc^2 \theta + \frac{1}{9} \cot^2 \theta \right) \| \text{grad}_T (\ln f) \|^2 - \beta^2 \right] + \alpha^2. \]

(ii) If the structure vector field $\xi$ is tangent to $M_\perp$, we have
\[
\|h\|^2 \geq \frac{2l}{9} \cos^2 \theta (\| \text{grad}_\perp (\ln f) \|^2 - \beta^2)
\[+ 4l \left( \csc^2 \theta + \frac{1}{9} \cot^2 \theta \right) \| \text{grad}_T (\ln f) \|^2. \]

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The equality holds identically in (4.1) and (4.2), if $M_T$ and $M_L$ are totally geodesic submanifolds of $M$ and $M_0$ is a totally umbilical submanifold of $M$. Moreover, $M$ is also $D_T \oplus D_L$-mixed geodesic in $M$ but not $D_T \oplus D_0$-mixed geodesic. Hence, $M$ is not minimal of $M$.

**Proof.** From the definition of $h$, we have

$$
(4.3) \quad \|h\|^2 = \sum_{r=n+1}^{n+t} \sum_{i,j=1}^{n} g(h(E_i, E_j), E_r)^2 + \sum_{r=n+t+1}^{n+t+2l} \sum_{i,j=1}^{n} g(h(E_i, E_j), E_r)^2 + \sum_{r=n+t+2l+1}^{2n+1} \sum_{i,j=1}^{n} g(h(E_i, E_j), E_r)^2.
$$

By considering the first and second terms in the right hand side of (4.3), one can obtain the following

$$
(4.4) \quad \|h\|^2 \geq \sum_{r=1}^{t} \sum_{i,j=1}^{2s+1} g(h(E_i, E_j), \phi \hat{E}_r)^2 + \sum_{r=1}^{t} \sum_{i,j=1}^{t} g(h(E_i, E_j), \phi \hat{E}_r)^2 + 2 \sum_{r=1}^{t} \sum_{i=1}^{2s+1} \sum_{j=1}^{2l} g(h(E_i, E_j), \phi \hat{E}_r)^2 + 2 \sum_{r=1}^{t} \sum_{i=1}^{t} \sum_{j=1}^{2l} g(h(E_i, E_j), \phi \hat{E}_r)^2
$$

$$
+ \sum_{r=1+t}^{t+2l+1} \sum_{i,j=1}^{2s+1} g(h(E_i, E_j), \hat{E}_r)^2 + \sum_{r=t+1}^{t+2l} \sum_{i,j=1}^{t} g(h(E_i, E_j), \hat{E}_r)^2 + 2 \sum_{r=1+t}^{t+2l} \sum_{i=1}^{2s+1} \sum_{j=1}^{2l} g(h(E_i, E_j), \hat{E}_r)^2 + 2 \sum_{r=1+t}^{t+2l} \sum_{i=1}^{t} \sum_{j=1}^{2l} g(h(E_i, E_j), \hat{E}_r)^2.
$$

First, we deal with the term $2 \sum_{r=t+1}^{t+2l} \sum_{i=1}^{2s+1} \sum_{j=1}^{2l} g(h(E_i, E_j), \hat{E}_r)^2$ in the right hand side of (4.4). By decomposing this term and using (3.33) and (3.34), we obtain that

$$
(4.5) \quad 2 \sum_{r=t+1}^{t+2l} \sum_{i=1}^{2s+1} \sum_{j=1}^{2l} g(h(E_i, E_j), \hat{E}_r)^2 = 2 \sum_{i=1}^{s} \sum_{r,j=1}^{l} g(h(E_i, E_j), \csc \theta FE_r^*)^2 + 2 \sum_{i=1}^{s} \sum_{r,j=1}^{l} g(h(E_i, \sec \theta PE_r^*), \sec \theta \csc \theta FPE_r^*)^2
$$

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By applying (2.7), (3.35) and since $i$ in (4.8) vanish. Using (3.14), (3.19) and since $\eta(E_i) = 0$ for any $i \in \{1, \ldots, 2s\}$, we have

$$
\text{(4.6)} \quad 2 \sum_{r=1}^{t+2l} \left[ \sum_{i,j=1}^{2l} g(h(E_i^r, E_j^r), \hat{E}_r)^2 \right] = 2 \sum_{r=1}^{t+2l} \left[ \sum_{i,j=1}^{2l} g(h(E_i^r, E_j^r), \hat{E}_r)^2 \right] + 4l \left( \csc^2 \theta + \frac{1}{9} \cot^2 \theta \right) \left[ \sum_{i=1}^{2s+1} (E_i(\ln f))^2 - (\xi(\ln f))^2 \right].
$$

By applying (2.7), (3.35) and since $\xi(\ln f) = \beta$, we can rewrite (4.6) as

$$
\text{(4.7)} \quad 2 \sum_{r=1}^{t+2l} \left[ \sum_{i,j=1}^{2l} g(h(E_i^r, E_j^r), \hat{E}_r)^2 \right] = 4l \left( \csc^2 \theta + \frac{1}{9} \cot^2 \theta \right) \left( \parallel \text{grad}_T(\ln f) \parallel^2 - \beta^2 \right) + \alpha^2.
$$

Now, we decompose the term $\sum_{r=1}^{t} \sum_{i,j=1}^{2l} g(h(E_i^r, E_j^r), \phi \hat{E}_r)^2$ in (4.4) as follows

$$
\text{(4.8)} \quad \sum_{r=1}^{t} \sum_{i,j=1}^{2l} g(h(E_i^r, E_j^r), \phi \hat{E}_r)^2 = \sum_{r=1}^{t} \sum_{i,j=1}^{l} [g(h(E_i^r, E_j^r), \phi E_r)^2 + g(h(\sec \theta P(E_i^r), E_j^r), \phi E_r)^2] + g(h(E_i^r, \sec \theta P(E_i^r), \phi \hat{E}_r)^2 + g(h(\sec \theta P(E_i^r), \phi \hat{E}_r)^2)
$$

Since $M$ is $D_\perp \oplus D_{\theta}$-mixed geodesic and by (3.20), the second and third terms in (4.8) vanish. Using (3.14), (3.19) and since $\eta(E_i) = 0$ for any $r \in \{1, \ldots, t\}$, we obtain

$$
\text{(4.9)} \quad \sum_{r=1}^{t} \sum_{i,j=1}^{2l} g(h(E_i^r, E_j^r), \phi \hat{E}_r)^2 = \frac{2l}{9} \cos^2 \theta \sum_{r=1}^{t} (\hat{E}_r(\ln f))^2 = \frac{2l}{9} \cos^2 \theta \parallel \text{grad}_\perp(\ln f) \parallel^2.
$$
Hence, from (4.7) and (4.9), we get (4.1), which proves the inequality (i).

Now, to prove (ii), let $\xi$ be tangent to $M_{\perp}$. From (4.6) and by using (2.7), we get

\begin{equation}
2 \sum_{r=t+1}^{t+2l} \sum_{i=1}^{2l} 2^{r+1} \sum_{j=1}^{2l} g(h(E_i, E_j^*), \hat{E}_r)^2 = 4l(csc^2 \theta + \frac{1}{9} cot^2 \theta)\| \text{grad}_f (\ln f) \|^2.
\end{equation}

Also, form (4.9) and by (2.7), we get

\begin{equation}
\sum_{r=1}^{t} \sum_{i,j=1}^{2l} g(h(E_i^*, E_j^*), \phi \hat{E}_r)^2 = \frac{2l}{9} \cos^2 \theta \left[ \sum_{r=1}^{t+1} (\hat{E}_r (\ln f))^2 - (\xi (\ln f))^2 \right]
\end{equation}

Hence, from (4.10) and (4.11), we get (4.2), which proves the inequality (ii).

If equality holds in (4.1) and (4.2), then from (4.3) and since $M$ is $D_{\perp} \oplus D_{\theta}$-mixed geodesic in $\bar{M}$, we find

\begin{equation}
\begin{array}{c}
h(D_T, D_T) = 0, \quad h(D_{\perp}, D_{\perp}) = 0, \quad h(D_{\theta}, D_{\theta}) \subseteq D_{\perp}, \\
h(D_T, D_{\perp}) = 0, \quad \text{and} \quad h(D_{\perp}, D_{\theta}) = 0.
\end{array}
\end{equation}

From (3.6) and by (4.12), we vanish the fifth term of the right hand side of (4.4). Thus, from the above facts and since $B$ is totally geodesic and $M_{\theta}$ is totally umbilical in $M$, we deduce that $M_T$ and $M_{\perp}$ are totally geodesic submanifolds of $\bar{M}$, while $M_{\theta}$ is a totally umbilical submanifold of $\bar{M}$. Moreover, from (4.12), $M$ is also $D_T \oplus D_{\perp}$-mixed geodesic. $M$ can never be a $D_T \oplus D_{\theta}$-mixed geodesic, which proves the theorem.

\begin{flushright}
$\square$
\end{flushright}

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