UNIVALENCY OF CERTAIN TRANSFORM OF UNIVALENT FUNCTIONS

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Abstract

We consider univalency problem in the unit disc $D$ of the function

$$g(z) = \frac{(z/f(z)) - 1}{-a_2},$$

where $f$ belongs to some classes of univalent functions in $D$ and $a_2 = \frac{f''(0)}{2} \neq 0$.

Key words: analytic, univalent, transform

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1. Introduction. Let $\mathcal{A}$ denote the family of all analytic functions $f$ in the unit disk $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$, i.e., $f$ has the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots.$$ (1)

Let $\mathcal{S}$, $\mathcal{S} \subset \mathcal{A}$, denote the class of univalent functions in $\mathbb{D}$, let $\mathcal{S}^*$ be the subclass of $\mathcal{A}$ (and $\mathcal{S}$ which are starlike in $\mathbb{D}$) and let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$|U_f(z)| < 1 \quad (z \in \mathbb{D}),$$ (2)
where

\[ U_f(z) := \left( \frac{z}{f(z)} \right)^2 f'(z) - 1. \]

In [1], Theorem 4 the authors consider the problem of univalency for the function

\[ g(z) = \frac{(z/f(z)) - 1}{-a_2}, \]

where \( f \in \mathcal{U} \) has the form (1) with \( a_2 \neq 0 \). They proved the following

**Theorem A.** Let \( f \in \mathcal{U} \). Then, for the function \( g \) defined by expression (4) we have

(a) \( |g'(z) - 1| < 1 \) for \( |z| < |a_2|/2 \);

(b) \( g \in \mathcal{S}^* \) in the disk \( |z| < |a_2|/2 \), and even more

\[ \left| \frac{zg'(z)}{g(z)} - 1 \right| < 1 \]

in the same disk;

(c) \( g \in \mathcal{U} \) in the disk \( |z| < |a_2|/2 \) if \( 0 < |a_2| \leq 1 \).

These results are the best possible.

For the proof of the previous theorem the authors used the next representation for the class \( \mathcal{U} \) (see [2] and [3]). Namely, if \( f \in \mathcal{U} \), then

\[ z f'(z) = 1 - a_2 z - z \omega(z), \]

where the function \( \omega \) is analytic in \( \mathbb{D} \) with \( |\omega(z)| \leq |z| < 1 \) for all \( z \in \mathbb{D} \). The appropriate function \( g \) from (4) has the form

\[ g(z) = z + \frac{1}{a_2} z \omega(z). \]

2. **Results.** In this paper we consider other cases of Theorem A(c) and certain related results.

**Theorem 1.** Let \( f \in \mathcal{U} \). Then the function \( g \) defined by equation (4) belongs to \( \mathcal{U} \) in the disc

\[ |z| < \sqrt{\frac{1 - |a_2| + \sqrt{|a_2|^2 + 2|a_2| - 3}}{2}}, \]

i.e., satisfies (2) on this disc, if \( \frac{5}{4} \leq |a_2| \leq 2 \).
Proof. For the first part of the proof we use the same method as in \cite{1}. By the definition of the class \( \mathcal{U} \), i.e., inequality (2), and using the next estimation for the function \( \omega \)

\[
|z\omega'(z) - \omega(z)| \leq \frac{r^2 - |\omega(z)|^2}{1 - r^2},
\]

where \( |z| = r \) and \( |\omega(z)| \leq r \), after some calculations we obtain

\[
|U_g(z)| = \frac{1}{1 - r^2} \left| \frac{z\omega'(z) - \omega(z) - \frac{1}{|a_2|^2} \omega^2(z)}{1 + \frac{1}{|a_2|^2} \omega_1(z)} \right| \leq \frac{|a_2| \cdot |z\omega'(z) - \omega(z)| + |\omega(z)|^2}{(|a_2| - |\omega(z)|)^2}
\]

\[
\leq \frac{|a_2| \cdot \frac{r^2 - |\omega(z)|^2}{1 - r^2} + |\omega(z)|^2}{(|a_2| - |\omega(z)|)^2} =: \frac{1}{1 - r^2} \cdot \varphi(t).
\]

Here,

\begin{equation}
(7) \quad \varphi(t) = \frac{|a_2| r^2 - (|a_2| - 1 + r^2)t^2}{(|a_2| - t)^2}
\end{equation}

and \( |\omega(z)| = t, 0 \leq t \leq r \). From here we have that

\[
\varphi'(t) = \frac{2|a_2|}{(|a_2| - t)^3} \cdot \left[ r^2 - (|a_2| - 1 + r^2)t \right],
\]

(\( |a_2| - t > 0 \) since \( |a_2| \geq \frac{5}{4} > 1 > t \)). Next, \( \varphi'(t) = 0 \) for

\[
t_0 = \frac{r^2}{|a_2| - 1 + r^2}
\]

and \( 0 \leq t_0 \leq r \) if

\[
\frac{r^2}{|a_2| - 1 + r^2} \leq r,
\]

which is equivalent to

\[
r^2 - r + |a_2| - 1 \geq 0.
\]

The last relation is valid for \( \frac{5}{4} \leq |a_2| \leq 2 \) and every \( 0 \leq t < 1 \). It means that the maximal value of the function \( \varphi \) on \([0, r]\) is

\[
\varphi(t_0) = \frac{(|a_2| - 1 + r^2)r^2}{(|a_2| - 1)(|a_2| + r^2)}.
\]

Finally,

\[
|U_g(z)| \leq \frac{1}{1 - r^2} \cdot \varphi(t_0) = \frac{(|a_2| - 1 + r^2)r^2}{(1 - r^2)(|a_2| - 1)(|a_2| + r^2)} < 1
\]
if
\[ r^4 - (1 - |a_2|)r^2 + (1 - |a_2|) < 0, \]
or if
\[ r < \sqrt{\frac{1 - |a_2| + \sqrt{|a_2|^2 + 2|a_2| - 3}}{2}}. \]
This completes the proof. □

For our next consideration we need the following lemma.

**Lemma 1.** Let \( f \in A \) be of the form (1). If

\[ \sum_{n=2}^\infty n|a_n| \leq 1, \]
then
\[ |f'(z) - 1| < 1 \quad (z \in \mathbb{D}), \]
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{D}) \]
(i.e. \( f \in S^* \), and \( f \in \mathcal{U} \).

For the proof of \( f \in \mathcal{U} \) in the lemma see [3], while the rest easily follows.

Further, let \( S^+ \) denote the class of univalent functions in the unit disc with the representation

\[ \frac{z}{f(z)} = 1 + b_1z + b_2z^2 + \cdots, \quad b_n \geq 0, \quad n = 1, 2, 3, \ldots. \]

For example, the Silverman class (the class with negative coefficients) is included in the class \( S^+ \), as well as the Koebe function \( k(z) = \frac{z}{(1 + z)^2} \in S^+ \). The next characterization is valid for the class \( S^+ \) (for details see [4])

\[ f \in S^+ \iff \sum_{n=2}^\infty (n-1)b_n \leq 1. \]

**Theorem 2.** Let \( f \in S^+ \). Then the function \( g \) defined by (4) belongs to the class \( \mathcal{U} \) in the disc \(|z| < |a_2|/2\) and the result is the best possible.

**Proof.** Using the representation (9), the corresponding function \( g \) has the form
\[ g(z) = \frac{z}{f(z)} - 1 = \frac{z}{b_1} - 1 = z + \sum_{n=2}^\infty \frac{b_n}{b_1}z^n \quad (b_1 \neq 0), \]
and from here
\[ \frac{1}{r}g(rz) = z + \sum_{n=2}^\infty \frac{b_n}{b_1}r^{n-1}z^n \quad (0 < r \leq 1). \]
Then, after applying Lemma 1, we have
\[
\sum_{n=2}^{\infty} n|a_n| = \sum_{n=2}^{\infty} \frac{b_n}{b_1^n} r^{n-1} = \frac{1}{b_1} \sum_{n=2}^{\infty} (n-1)b_n \frac{n}{n-1} r^{n-1} \\
\leq \frac{2r}{b_1} \sum_{n=2}^{\infty} (n-1)b_n \leq \frac{2r}{b_1} \leq 1
\]
if \( r \leq b_1/2 = |a_2|/2 \). It means, by the same lemma, that \( g \in \mathcal{U} \) in the disc \(|z| < |a_2|/2\).

In order to show that the result is the best possible, let us consider the function \( f_1 \) defined by
\[
\frac{z}{f_1(z)} = 1 + bz + z^2, \quad 0 < b \leq 2.
\]
Then, \( f_1 \in \mathcal{S}^+ \) is of type \( f_1(z) = z - bz^2 + \cdots \), so the function
\[
g_1(z) = \frac{z}{f_1(z)} - 1 = z + \frac{1}{b} z^2
\]
is such that
\[
\left| \left( \frac{z}{g_1(z)} \right)^2 g_1'(z) - 1 \right| \leq \frac{1}{b} \left| \frac{z}{g_1(z)} \right|^2 < 1
\]
when \(|z| < b/2\). This implies that \( g_1 \) belongs to the class \( \mathcal{U} \) in the disc \(|z| < b/2\).

On the other hand, since \( g_1'(-b/2) = 0 \), the function \( g_1 \) is not univalent in a bigger disc, implying that the result is the best possible.

**Theorem 3.** Let \( f \in \mathcal{S} \). Then the function \( g \) defined by (4) belongs to the class \( \mathcal{U} \) in the disc \(|z| < r_0 \), where \( r_0 \) is the unique real root of equation
\[
\frac{3r^2 - 2r^4}{(1 - r^2)^2} - \ln(1 - r^2) = |a_2|^2
\]
on the interval \((0, 1)\).

**Proof.** We apply the same method as in the proof of the previous theorem. Namely, if \( f \in \mathcal{S} \) has the representation (9), then
\[
\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1
\]
(see [5], Theorem 11, p. 193, Vol. 2). Also, using (4), (9) and (13), we have \( a_2 = -b_1 \), and
\[
\frac{1}{r} g(rz) = z + \sum_{n=2}^{\infty} b_n \frac{r^{n-1}}{b_1^n} z^n, \quad 0 < r \leq 1.
\]
So,
\[
\sum_{n=2}^{\infty} n|a_n| = \sum_{n=2}^{\infty} \frac{|b_n|}{|b_1|} \cdot r^{n-1}
\]
\[
= \frac{1}{|b_1|} \sum_{n=2}^{\infty} \sqrt{n-1} \cdot |b_n| \cdot \frac{n}{\sqrt{n-1}} \cdot r^{n-1}
\]
\[
\leq \frac{1}{|b_1|} \left( \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right)^{1/2} \cdot \left( \sum_{n=2}^{\infty} \frac{n^2}{n-1} r^{2(n-1)} \right)^{1/2}
\]
\[
\leq \frac{1}{|b_1|} \left( r^2 \sum_{n=2}^{\infty} (n-1)(r^2)^{n-2} + 2r^2 \sum_{n=2}^{\infty} (r^2)^{n-2} + \sum_{n=2}^{\infty} \frac{1}{n-1}(r^2)^{n-1} \right)^{1/2}
\]
\[
= \frac{1}{|b_1|} \left[ \frac{3r^2 - 2r^4}{(1-r^2)^2} - \ln(1-r^2) \right]^{1/2} \leq 1
\]
if $|z| < r_0$, where $r_0$ is the root of the equation
\[
\frac{3r^2 - 2r^4}{(1-r^2)^2} - \ln(1-r^2) = |b_1|^2 \left(= |a_2|^2\right).
\]

We note that the function on the left side of this equation is an increasing one on the interval $(0, 1)$, so the equation has a unique root when $0 < |a_2| \leq 2$. \qed

REFERENCES